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W. STAHL

Determination of All Primitive Collineation Groups in More than Four Variables which Contain Homologies.

By Howard H. Mitchell.

In a recent paper (*Proc. London Math. Soc.*, Series 2, Vol. X, Part 4, Nov., 1911) Burnside determined all the finite collineation groups in n variables with rational coefficients which contain the symmetric group on those variables, *i. e.*, the group generated by all transformations of the form

$$x'_i = x_{i+1}, \; x'_{i+1} = x_i, \; x'_j = x_j \quad (i = 1, 2, \ldots, n-1; \; j \neq i, \; i+1).$$

The groups of this sort (for $n > 4$) which are primitive (in the sense of Blichfeldt) he found to be groups of order $n+1!$ (for any n), and groups of order $2^7 \cdot 3^4 \cdot 5$, $2^9 \cdot 3^4 \cdot 5 \cdot 7$, and $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ for $n = 6, 7, 8$ respectively.

The group of order $2^7 . 3^4 \cdot 5$ (for $n = 6$) is isomorphic with the group of the equation for the 27 lines on a cubic surface and was first exhibited in explicit form by Burkhardt (*Math. Ann.*, Bd. XLI, pp. 320–326). The groups for $n = 7, 8$ had not been noticed before. That of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ is shown by Burnside to give a representation of the group of the 28 bitangents to a quartic curve. He states that the one of order $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ is isomorphic with Jordan's hypo-abelian group on eight variables.

The author solves a more general problem, which is the determination of of all primitive groups in n (> 4) variables which contain homologies, *i. e.*, transformations all of whose multipliers are equal with the exception of one. A transformation in the symmetric group on the variables which interchanges two of them and leaves the rest unaltered has -1 for one multiplier and $+1$ for each of the others, being therefore an homology of period 2. The determination will be made in such a way as to apply equally well to the groups in a general modular space,* whose orders are not divisible by the modulus.

The results for $n \leqq 4$ are already known. For $n = 3$† all primitive groups contain homologies, that of order 216 containing homologies of period 3 and

* Veblen and Bussey, "Finite Projective Geometries," *Trans. Am. Math. Soc.*, Vol. VII (1906), pp. 241–259.

† See papers by Blichfeldt, *Trans. Am. Math. Soc.*, Vol. V (1904), pp. 310–325; *Math. Ann.*, Bd. LXIII (1907), pp. 552–572; also one by the author, *Trans. Am. Math. Soc.*, Vol. XII (1911), pp. 207–242.

of period 2, and those of order 72, 36, 168, 60, 360 homologies of period 2. For $n=4$* the primitive groups of this sort are of order 25920, 120, 288, 576, 1152, 1920, 11520, 7200. That of order 25920 contains homologies of period 3 and the others homologies of period 2.

For $n>4$ there are no primitive groups containing homologies of period greater than 2. In addition to the primitive groups in Burnside's list there are two others which contain homologies of period 2. These are of order $2^6 \cdot 3^4 \cdot 5$ and $2^8 \cdot 3^6 \cdot 5 \cdot 7$ for $n=5, 6$ respectively. The first of these has been discussed by Burkhardt (*Math. Ann.*, Bd. XXXVIII, pp. 185–224), whereas the latter does not seem to have been noticed. The former is isomorphic with the well-known simple group of that order, and the latter is (1, 2) isomorphic with the first orthogonal group on six indices with modulus 3.†

§ 1. THEOREM 1. *No primitive group in more than four variables contains homologies of higher period than* 2.

Two homologies which are not commutative and a third not commutative with the group generated by the first two and with center not on the line of centers must generate a group which permutes the points of the plane containing the centers. There is no possible group in the plane if the homologies are of higher period than 3. If they are of period 3, the group generated must be of order $3 \cdot 216$.

An homology not commutative with this group of order $3 \cdot 216$ must generate with it a group of order $6 \cdot 25920$. The invariant C_3 is in the group generated by any four mutually commutative homologies. The invariant C_2 may be obtained as the product of two commutative C_2, such as $x_i' = -x_i$, $x_j' = x_j$ $(i=1, 2;\ j>2)$, and $x_i' = -x_i$, $x_j' = x_j$ $(i=3, 4;\ j \neq 3, 4)$.

All other homologies of period 3 must then be commutative with this group. For otherwise there would be on the points of a line a C_6 and a C_3 not in the C_6. But no group on the line can contain such cyclic groups, Hence no primitive group in more than four variables can contain homologies of period 3.

THEOREM 2. *No primitive group in more than four variables contains two homologies of period* 2 *whose product is of period greater than* 3.

Two homologies of period 2 which are not commutative and a third which is not commutative with the group generated by the first two and whose center

* See Bagnera, *Rendiconti del Circolo Matematico di Palermo*, T. XIX (1905), pp 1–56; Blichfeldt, *Math. Ann.*, Bd. LX (1905), pp. 204–231.

† Jordan, *Traité des Substitutions*, pp. 161–170; Dickson, *Linear Groups*, Chap. VII.

is not on the line of centers must generate a group permuting the points of the plane containing the centers. The group in this plane may be primitive or it may permute in all six ways the vertices of a triangle. In the latter case only one triangle is left invariant in this plane provided there are homologies present whose product is of period greater than 3.

Consider now an homology not commutative with this group and with center not in the plane of centers. The group then generated in the S_3 which contains the centers may be either primitive or imprimitive. In the latter case it must permute the vertices of a tetrahedron, three of the vertices of which are vertices of the invariant triangle in the plane. For the latter group carries every point which does not remain invariant under it and which does not lie in the plane of centers into more than four positions, provided there are homologies present whose product is of period greater than 3. Hence any homology with center in the S_3 which is not commutative with the group and whose center does not lie in the plane of centers must leave fixed two of the vertices of the triangle and carry the third into a point which remains invariant under the group.

In case there is no primitive group in an S_3 whatever, every homology must be of this sort. Hence an homology with center not in the S_3 and not commutative with the group permuting the points of the S_3 must leave fixed at least two of the vertices of each of the four triangles of the tetrahedron and hence must leave fixed three of the four. It must carry the fourth into a point which remains invariant under each of the four subgroups and hence into a point which remains invariant under the whole group permuting the points of the S_3. Hence the group generated by homologies with centers in an S_4 must permute five points in that S_4.

We may then prove readily by induction that the only possible group generated by homologies with centers in an S_{n-1} must permute n points in that S_{n-1}.

It follows therefore that there must be a primitive group generated by homologies with centers in an S_3. Those which contain homologies whose product is of period greater than 3 are of order 576, 1152, 1920, 11520, 7200.* In every case there is an invariant C_2 obtained as the product of four mutually commutative homologies. The first three of these groups contain homologies each commutative with a group of order $2 \cdot 24$, that of order 11520 contains homologies each commutative with a group of order $2 \cdot 96$, and the one of order 7200 contains homologies each commutative with a group of order $2 \cdot 60$.

* *Loc. cit.*

In any larger space containing the S_3 all these groups are of double the order. We denote the whole group by G and by H a subgroup of order $2 \cdot 24$, $2 \cdot 96$, or $2 \cdot 60$ generated by the homologies with centers in a plane.

An homology not commutative with G and with center not in the S_3 must generate with G a group permuting the points of an S_4. In this S_4 the C_2 which is invariant under G is an homology, which we denote by h_g. The homology which is invariant under H but not under G we denote by h_h.

In the S_4 there must certainly be centers which lie in neither of the two S_3 determined by the plane of centers of H and the center of h_g or h_h. An homology with such a center must generate with H either a primitive group in the S_3 containing the centers or else a group permuting the vertices of a tetrahedron. In either case it must contain an homology commutative with H with center at the point where this S_3 meets the line joining the centers of h_g and h_h.

But there can be no other centers of homologies on this line, for it is contained by planes in which the group of the points is a dihedral G_{12}, whereas no primitive group of the points in a plane contains a dihedral group of higher order than 10. Such a G_{12} is generated by h_g, h_h, and an homology which is commutative with h_g and whose product by h_h is of period 3. Hence no primitive group in more than four variables contains homologies whose product is of period greater than 3.

§ 2. *Groups in which every homology is commutative with at least one homology in each dihedral* G_6.

A dihedral G_6 generated by two homologies and an homology not commutative with this G_6 must generate a group permuting the points of the plane containing the centers. The only possible groups in a plane in which the product of no two homologies is of higher period than 3 are of order 24 and 18. The former permutes the vertices of one triangle in that plane and the latter permutes the vertices of each of four triangles. If the group in the plane is of order 18, the group in any larger space containing this plane is of order $3 \cdot 18$. We first consider groups which do not contain such groups as subgroups. Hence we may suppose that every homology is commutative with at least one in each dihedral G_6, since this is the case in a G_{24}.

THEOREM 3. *The only primitive collineation groups in* n (>4) *variables, which contain homologies and in which every homology is commutative with at least one in each dihedral* G_6, *are groups of order* $n+1!$ *for any* n, *and groups of order* $2^7 \cdot 3^4 \cdot 5$, $2^9 \cdot 3^4 \cdot 5 \cdot 7$ *and* $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ *for* $n = 6, 7, 8$ *respectively.*

Consider a G_{24}, which we represent as the symmetric group in the first four variables. In a primitive group there must be an homology not commutative with this group. Since it must be commutative with at least one homology in each dihedral G_6, its center must be conjugate under the G_{24} with a point such as $(a\,a\,a\,b\,c\,\ldots.)$ or $(a\,a\,b\,b\,c\,\ldots.)$, where $a \neq b$.

In case the center of every homology which is not commutative with the G_{24} is of the first type, the only primitive groups in n variables which exist are of order $n+1$! This may be proved by induction. Consider the symmetric group in n variables. The center of an homology not commutative with it must be conjugate under the symmetric group with a point such as

$$(\xi_1 + \xi_2 + \ldots. + \xi_{n-1})\,a + \xi_n b + \ldots. = 0,^*$$

since three of any four of the first n coordinates must be equal.

We first suppose that it is linearly dependent on the centers in the symmetric group. It is then

$$(\xi_1 + \xi_2 + \ldots. + \xi_{n-1}) + (1-n)\,\xi_n = 0.$$

We may suppose that the invariant linear system of this homology is

$$(x_1 + x_2 + \ldots. + x_{n-1}) + (1-n)\,x_n = 0,$$

since we are concerned only with the group in the S_{n-2} which contains the centers. But the product of this homology by

$$x'_{n-1} = x_n,\ x'_n = x_{n-1},\ x'_j = x_j \quad (j \neq n-1,\, n)$$

is not of period 3 since the centers and invariant linear systems do not cut out the proper cross-ratio on the line joining the centers for any positive integral value of n.†

We now suppose that the center is linearly independent of the centers in the symmetric group. We may then by a transformation commutative with the symmetric group transform it into $\xi_n - \xi_{n+1} = 0$. We find that the invariant linear system of points may be chosen as $x_n - x_{n+1} = 0$. The homology is then

$$x'_n = x_{n+1},\ x'_{n+1} = x_n,\ x'_j = x_j \quad (j \neq n,\, n+1).$$

We then have the symmetric group on $n+1$ variables. It is of course primitive only if represented on n variables, which is possible by reason of the invariant linear relation $x_1 + x_2 + \ldots. + x_{n+1} = 0$.

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* We shall find it convenient to use a dual coordinate system.

† It is perhaps of interest to note that if the coefficients be reduced by any odd prime as a modulus that is contained in $n+1$, the product of these transformations is of period 3. In such a case a group of order $n+1$! exists for which only $n-1$ independent variables are necessary. For example, we may exhibit the $G_{5!}$ in three variables by using the modulus 5, and the $G_{6!}$ in four variables by using the modulus 3 (cf. Dickson, *Trans. Am. Math. Soc.*, Vol. IX, pp. 121–148).

We may suppose therefore that any other primitive group in which every homology is commutative with at least one in each dihedral G_6 contains the symmetric group on the first four variables and an homology with center $(aabbc\ldots.)$. We may take this center to be $\xi_1+\xi_2=0$. The invariant linear system of the homology we may take to be $x_1+x_2=0$. The homology is then $x_1'=-x_2$, $x_2'=-x_1$, $x_j'=x_j$ $(j>2)$. The group generated is of order $2^3\cdot 4!$ in the $n(>4)$ variables. It permutes the vertices of each of three tetrahedra in $x_5=x_6=\ldots.=0$, one vertex of each being

$$\xi_1=0,\ \xi_1+\xi_2+\xi_3+\xi_4=0,\ -\xi_1+\xi_2+\xi_3+\xi_4=0.$$

Consider an homology not commutative with this group. In order that it be commutative with at least one in each dihedral G_6 we find that of the first four coordinates of its center either three must be 0 or else the ratios of the four must be ± 1. There can be no center of either sort linearly dependent on the centers in the above group without involving homologies whose product is of period 4. Hence we suppose the existence of a center linearly independent of them. Since the group of order $2\cdot 96$ is invariant under groups of order $2\cdot 288$ and $2\cdot 576$ permuting the three tetrahedra mentioned above, we may take a center to be of the first type, *e. g.*, $\xi_4-\xi_5=0$. We find that the invariant linear system may be taken to be $x_4-x_5=0$. The homology is then $x_4'=x_5$, $x_5'=x_4$, $x_j'=x_j$ $(j\neq 4,5)$. A group is generated of order $2^4\cdot 5!$.

Consider an homology not commutative with this group. Of the first five coordinates of its center either four must be 0 or else the ratios of the five must be ± 1. Since the group is invariant under a transformation which changes the sign of an arbitrary number of variables and leaves the rest unaltered, we may in the latter case choose the first five coordinates all as $+1$.

We find readily that there can be no more centers linearly dependent on the centers in the above group. We therefore suppose the existence of an homology with center linearly independent of them. If all the centers of this sort are such that of the first five coordinates four are 0, it may readily be shown that no primitive group in n variables is possible. We therefore take an homology with center $\xi_1+\xi_2+\ldots.+\xi_8=0$. We find then that its invariant system of points may be taken to be $x_1+x_2+\ldots.+x_8=0$. This homology may be written:

$$x_1'+x_2'+\ldots.+x_8'=-(x_1+x_2+\ldots.+x_8),$$
$$x_{i+1}'-x_i'=x_{i+1}-x_i,\ x_j'=x_j\ (i<8,\ j>8).$$

A group is then generated, permuting 36 centers of homologies. The group commutative with one of the homologies is of order $2\cdot 6!$ and the whole

group of order $2\cdot 6!\cdot 36 = 2^7\cdot 3^4\cdot 5$. It is the well-known group which gives a representation of the group of the 27 lines on a cubic surface. It requires of course only six independent variables for its representation, the above choice of coordinates having been made for convenience in exhibiting the larger groups.

Under this group the subgroup commutative with $T: x_i' = -x_i,\ x_j' = x_j$ $(i \leqq 4,\ j > 4)$ is of order $2\cdot 576$. This subgroup permutes the three invariant tetrahedra of the group of order $2\cdot 96$. On the points in $x_1 = x_2 = x_3 = x_4 = 0$ it is of order 6. Each of the six transformations on these points corresponds to a particular permutation of the three tetrahedra. To one which leaves fixed the tetrahedron $\xi_1 = 0,\ \xi_2 = 0,\ \xi_3 = 0,\ \xi_4 = 0$ and interchanges the other two, there corresponds the transformation $U: x_5' = -x_5,\ x_j' = x_j\ (j > 5)$, and to one which leaves fixed the tetrahedron $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0,\ \xi_1 + \xi_2 - \xi_3 - \xi_4 = 0$, $\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0,\ \xi_1 - \xi_2 - \xi_3 + \xi_4 = 0$ and interchanges the other two, there corresponds the transformation $V: x_5' + x_6' + x_7' + x_8' = -(x_5 + x_6 + x_7 + x_8)$, $x_{i+1}' - x_i' = x_{i+1} - x_i,\ x_j' = x_j\ (i = 5, 6, 7;\ j > 8)$. Of the 36 centers of homologies 12 are in $x_5 = x_6 = \ldots\ldots = 0$ and the other 24 lie by pairs on the 12 lines joining the centers of U, V, UVU in $x_1 = x_2 = x_3 = x_4 = 0$ to the vertices of the corresponding tetrahedra in $x_5 = x_6 = \ldots\ldots = 0$.

Consider now an homology not commutative with the whole group. Of the first five coordinates of its center either four must be 0 or else the ratios of the five must be ± 1. Under the group of order $2\cdot 576$ commutative with T a center of the latter type will be conjugate with a center of the former type. Since the symmetric group on the first five variables is present, it will be conjugate with a center in $x_1 = x_2 = x_3 = x_4 = 0$. An homology of this sort must be commutative with all those with centers in $x_5 = x_6 = \ldots\ldots = 0$. Hence under the group generated by any such homologies no two of the three C_2, U, V, UVU can be conjugate. For in that case there would be centers on lines joining $\xi_5 = 0$ to points in $x_5 = x_6 = \ldots\ldots = 0$ other than the four vertices of the corresponding tetrahedron. But there can be no more centers in the S_4 determined by $\xi_5 = 0$ and that tetrahedron.

There can be no homology with center on the line containing the centers of U, V, UVU in $x_1 = x_2 = x_3 = x_4 = 0$. For the only such homologies which could generate a group under which these three centers were not conjugate would be two with centers at the two points interchanged by the dihedral G_6. But neither of these points lies in

$$x_4 + x_5 = 0,\ x_1 + x_2 + \ldots\ldots + x_8 = 0,\ x_1 + x_2 + x_3 - x_4 - x_5 + x_6 + x_7 + x_8 = 0,$$

and hence neither homology could be commutative with any one of the three in the dihedral G_6 in which these three are the invariant linear systems.

Hence we may suppose the existence of centers of homologies in $x_1 = x_2 = x_3 = x_4 = 0$, but not on the line containing the centers of U, V, UVU. The group generated by one such homology together with U, V, UVU must have a self-conjugate subgroup containing the homologies under which U, V, UVU are not conjugate. Hence the whole group must be of order $2 \cdot 24$ and the homologies must generate its invariant subgroup of order $2 \cdot 4$. One of the three homologies must be commutative with U and we may take its center to be $\xi_7 + \xi_8 = 0$. The centers of the other two are then $\xi_5 + \xi_6 = 0$ and $\xi_5 - \xi_6 = 0$. The invariant linear system of the homology with center $\xi_7 + \xi_8 = 0$ may then be chosen as $x_7 + x_8 = 0$. It is therefore $x'_7 = -x_8$, $x'_8 = -x_7$, $x'_j = x_j$ $(j \neq 7, 8)$.

A group is then generated permuting 63 centers of homologies. Of these 63 twelve are in $x_5 = x_6 = \ldots\ldots = 0$ and three in $x_1 = x_2 = x_3 = x_4 = 0$, and the other 48 lie by pairs on 24 lines joining the vertices of the three tetrahedra in $x_5 = x_6 = \ldots\ldots = 0$ with three corresponding pairs of points in $x_1 = x_2 = x_3 = x_4 = 0$. The group commutative with one of the 63 homologies is of order $2^6 \cdot 6!$. It contains, however, the C_2, $x'_i = -x_i$, $x'_7 = -x_8$, $x'_8 = -x_7$, $x'_j = x_j$ $(i < 7,\ j > 8)$, which is invariant under the whole group. The group of collineations in the S_6 which contains the centers of homologies is then of order $2^5 \cdot 6! \cdot 63 = 2^9 \cdot 3^4 \cdot 5 \cdot 7$.

This is one of the two groups found by Burnside (*loc. cit.*), and it is shown by him to give a representation of the group of the 28 bitangents to a quartic curve.

If this group is contained by a larger primitive group there must be homologies not commutative with the above group and with centers in $x_1 = x_2 = x_3 = x_4 = 0$. The group on this system of points must then be of order $2 \cdot 576$ and the homologies must generate its invariant subgroup of order $2 \cdot 96$. In the S_3 containing the centers of these homologies their centers and axial planes must form the vertices and opposite faces of three tetrahedra. One vertex of each of these three tetrahedra must be left invariant by both U and V. We may choose one of these as $\xi_6 - \xi_7 = 0$. The other three vertices of this tetrahedron must then be $\xi_6 + \xi_7 = 0$, $\xi_5 + \xi_8 = 0$, $\xi_5 - \xi_8 = 0$. The invariant linear system of the homology with center $\xi_6 - \xi_7 = 0$ may be taken as $x_6 - x_7 = 0$. This homology is $x'_6 = x_7$, $x'_7 = x_6$, $x'_j = x_j$ $(j \neq 6, 7)$.

A group is then generated which permutes 120 centers of homologies. Of the 120 twelve are in $x_5 = x_6 = \ldots\ldots = 0$ and twelve in $x_1 = x_2 = x_3 = x_4 = 0$, and the other 96 lie by pairs on the 48 lines joining the vertices of the three tetrahedra in $x_5 = x_6 = \ldots\ldots = 0$ with the vertices of three corresponding tetrahedra in $x_1 = x_2 = x_3 = x_4 = 0$. The order of the group commutative

with one of the homologies is $2^{11}\cdot 3^4\cdot 5\cdot 7$. It contains, however, the C_2 $x_i'=-x_i$, $x_j'=x_j$ $(i\leqq 8,\ j>8)$, which is invariant under the whole group. Hence the order of the whole group as a collineation group in the S_7 containing the centers of the homologies is $2^{10}\cdot 3^4\cdot 5\cdot 7\cdot 120=2^{13}\cdot 3^5\cdot 5^2\cdot 7$.

This is the other group found by Burnside (*loc. cit.*). To put it in the form in which it is exhibited by him, we first transform it by $x_i'=-x_i$, $x_j'=x_j$ $(i\leqq 3,\ j>3)$, and then by $x_{i+1}'-x_i'=x_{i+1}-x_i$, $x_1'+x_2'+\ldots\ldots+x_8'=-3(x_1+x_2+\ldots\ldots+x_8)$, $x_j'=x_j$ $(i<8,\ j>8)$. The group then contains and may be generated by the symmetric group in the first eight variables and the homology with center $\xi_1+\xi_2+\xi_3=0$ and invariant linear system $2(x_1+x_2+x_3)-(x_4+x_5+x_6+x_7+x_8)=0$. The latter is obtained by the two transformations from that with center $\xi_1+\xi_2+\ldots\ldots+\xi_8$ and invariant linear system $x_1+x_2+\ldots\ldots+x_8=0$. It may be written:

$$x_1'=\frac{1}{3}s_1-(x_2+x_3),\ x_2'=\frac{1}{3}s_1-(x_1+x_3),\ x_3'=\frac{1}{3}s_1-(x_1+x_2),\ x_i'=x_i\ (i>3),$$

where $s_1=x_1+x_2+\ldots\ldots+x_8$.

If when the group is exhibited in this form the coefficients of the transformations and the coordinates of the points be reduced modulo 2, the group remains the same as a permutation group on the 120 points, but it has then the quadratic invariant $\Sigma x_i x_j\equiv 0$ $(i,j=1,2,\ldots,8;\ i\neq j)$. The group is therefore simply isomorphic with the first hypo-abelian group on eight variables.* Burnside states this result without proof.

Returning now to the original choice of coordinates, we inquire whether any larger primitive group can contain this group. If so, there must be more homologies not commutative with it having their centers in $x_1=x_2=x_3=x_4=0$. Hence there must be a group in an S_4 containing a subgroup of order $2\cdot 576$. But there is no such group. Hence we have found all groups in which every homology is commutative with at least one homology in each dihedral G_6.

§ 3. THEOREM 4. *The only primitive collineation groups in $n\,(>4)$ variables, which contain homologies not commutative with any one of the three in a dihedral G_6, are of order $2^6\cdot 3^4\cdot 5$ and $2^8\cdot 3^6\cdot 5\cdot 7$ for $n=5,6$ respectively.*

An homology which is not commutative with any one of the three in a dihedral G_6 must generate with it a group of order $3\cdot 18$. Such a group is generated by

$$\begin{aligned} &x_1'=x_2, && x_2'=x_1, && x_j'=x_j\ (j>2);\\ &x_1'=\omega x_2, && x_2'=\omega^2 x_1, && x_j'=x_j\ (j>2);\\ &x_2'=x_3, && x_3'=x_2, && x_j'=x_j\ (j\neq 2,3),\end{aligned}$$

where ω is a cube root of unity.

* Jordan, *Traité des Substitutions*, pp. 195–206; Dickson, *Linear Groups*, Chap. VIII.

Consider an homology which is not commutative with this group. It must generate with it a group permuting the vertices of a tetrahedron. The group of order $3\cdot 18$ leaves invariant four triangles in $x_4 = x_5 = \ldots . = 0$, one vertex of each being $\xi_1 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$, $\omega\xi_1 + \xi_2 + \xi_3 = 0$, $\omega^2\xi_1 + \xi_2 + \xi_3 = 0$. We suppose that the vertices of the first triangle are vertices of the invariant tetrahedron, and we choose the fourth to be $\xi_4 = 0$. If the center of an homology be chosen as $\xi_3 - \xi_4 = 0$, the invariant system of points may be taken to be $x_3 - x_4 = 0$. This homology is then $x_3' = x_4$, $x_4' = x_3$, $x_j' = x_j$ $(j \neq 3, 4)$. The group generated is of order $3^3\cdot 4!$.

As we have seen, every homology which is not commutative with the group of order $3\cdot 18$ must permute in common with it the vertices of a tetrahedron, and the same must be true for the conjugate groups. Hence an homology which is not commutative with the group of order $3^3\cdot 4!$ must be such that of the first four coordinates of its center either three are 0 or else the ratios are 1, ω, ω^2. If the only centers present are of the former type, it is easy to show that no primitive group can exist. Since the group of order $3^3\cdot 4!$ is invariant under any transformation which multiplies each of the first four variables by 1, ω, ω^2 and makes an arbitrary substitution on the others, we may take a center of the latter type to be $\xi_1 + \xi_2 + \ldots . + \xi_6 = 0$. The invariant linear system of this homology may be taken as $x_1 + x_2 + \ldots . + x_6 = 0$. The homology may be written:

$$x_1' + x_2' + \ldots . + x_6' = -(x_1 + x_2 + \ldots . + x_6),$$
$$x_{i+1}' - x_i' = x_{i+1} - x_i,\ x_j' = x_j \quad (i < 6,\ j > 6).$$

We find that a group is generated which permutes 45 centers of homologies. Of these 45, nine are in $x_4 = x_5 = \ldots . = 0$, and the other 36 lie by threes on the 12 lines joining the vertices of the four triangles in $x_4 = x_5 = \ldots . = 0$ with four corresponding points in $x_1 = x_2 = x_3 = 0$, *i. e.*, $\xi_4 = 0$, $\xi_4 + \xi_5 + \xi_6 = 0$, $\omega\xi_4 + \xi_5 + \xi_6 = 0$, $\omega^2\xi_4 + \xi_5 + \xi_6 = 0$. To determine the order of the group as a collineation group in the S_4 which contains the centers of the homologies, we find that there are 40 planes, each of which contains the centers of nine homologies, and hence 40 cyclic groups conjugate with that generated by $T : x_i' = \omega x_i$, $x_j' = x_j$ $(i \leq 3,\ j > 3)$. The order of the subgroup (in the S_4) which is commutative with T is $3\cdot 216$. It permutes the four triangles in $x_4 = x_5 = \ldots . = 0$, and on the points in $x_1 = x_2 = x_3 = 0$ it is of order 12. Each permutation of the four triangles corresponds to a particular transformation on these points. To one which leaves fixed the triangle $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$ and is of period 3 on the others, there corresponds the transformation $U : x_4' = \omega x_4$, $x_j' = x_j$ $(j > 4)$, and to one which leaves fixed the triangle

$\xi_1+\xi_2+\xi_3=0$, $\xi_1+\omega\xi_2+\omega^2\xi_3=0$, $\xi_1+\omega^2\xi_2+\omega\xi_3=0$ and is of period 3 on the others, there corresponds the transformation

$$V: x_4'+x_5'+x_6'=\omega\,(x_4+x_5+x_6),$$
$$x_{i+1}'-x_i'=x_{i+1}-x_i,\ x_j'=x_j\ \ (i=4,5;\ j>6).$$

The order of the group as a collineation group in the S_4 is then $3\cdot 216\cdot 40 = 2^6\cdot 3^4\cdot 5$. The order of the group in any larger space containing this S_4 is, however, of order $2^7\cdot 3^4\cdot 5$, since it contains the invariant C_2: $x_i'=-x_i$, $x_5'=-x_6$, $x_6'=-x_5$, $x_j'=x_j$ $(i<5,\ j>6)$. This C_2 is the product of any five mutually commutative homologies. The group in the S_4 is readily shown to contain primitive subgroups of order $6!$.

If this group is contained by a larger primitive group, there must be homologies not commutative with it. Of the first four coordinates of the center of such an homology either three must be 0 or else the ratios must be $1, \omega, \omega^2$. Under the group commutative with T a center of the latter sort will be conjugate with centers of the former sort. Since the symmetric group on the first four variables is present, there must be centers whose first three coordinates are 0. Such homologies must be commutative with all those with centers in $x_4=x_5=\ldots\ldots=0$.

There can be no centers of homologies on the line containing the centers of the four C_3 generated by U, V, UVU^2, U^2VU. For under any group generated by homologies with centers on that line the four centers would be conjugate. Hence there would be additional centers in an S_3 such as $x_5=x_6=\ldots\ldots=0$, which is impossible.

An homology with center in $x_1=x_2=x_3=0$, but not on this line and not commutative with the above group, must generate with the four C_3 a group of order $3\cdot 216$. An homology which is commutative with V we may take to be $x_4'=x_5$, $x_5'=x_4$, $x_j'=x_j$ $(j\neq 4,5)$.

We find that a group is generated permuting 126 centers of homologies. Of the 126 centers nine are in $x_4=x_5=\ldots\ldots=0$ and nine in $x_1=x_2=x_3=0$, and the other 108 lie by threes on the 36 lines joining the vertices of the four triangles in $x_4=x_5=\ldots\ldots=0$ with the vertices of four corresponding triangles in $x_1=x_2=x_3=0$. The order of the group as a collineation group in the S_5 containing the centers of homologies is $2^7\cdot 3^4\cdot 5\cdot 126=2^8\cdot 3^6\cdot 5\cdot 7$.

This group must evidently contain a self-conjugate subgroup of index 2, consisting of all the transformations which may be written with determinant unity without introducing any irrationality other than ω. It will be shown later that the latter group is $(1,1)$ isomorphic with a known simple group. In any larger space containing the S_5 each of these groups contains an invariant C_6 generated by $x_i'=-\omega x_i$, $x_j'=x_j$ $(i\leqq 6,\ j>6)$.

If there is a larger primitive group containing the group of order $2^8 \cdot 3^6 \cdot 5 \cdot 7$, there must be homologies not commutative with it. The centers of some of these must lie in $x_1 = x_2 = x_3 = 0$. But the group on these points is of order $6 \cdot 216$ and no larger group can contain it. Hence we have found all collineation groups in more than four variables which contain homologies not commutative with any one of the three in a dihedral G_6.

THEOREM 5. *The primitive group of order* $2^8 \cdot 3^6 \cdot 5 \cdot 7$ *in six variables is* (1, 2) *isomorphic with the first orthogonal group on six indices with modulus* 3.*

To prove this theorem we first transform the group as written above by $x'_i = \omega x_i$, $x'_j = x_j$ $(i = 1, 2;\ j > 2)$, and then by

$$x'_1 + x'_2 + \ldots + x'_6 = \omega^2 (x_1 + x_2 + \ldots + x_6),$$
$$x'_{i+1} - x'_i = x_{i+1} - x_i,\ x'_j = x_j\ (i < 6,\ j > 6).$$

The group as transformed contains the symmetric group in the six variables, the homology $x'_1 = -x_2$, $x'_2 = -x_1$, $x'_j = x_j$ $(j > 2)$ and the homology with center $-\sqrt{-3}\,\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0$ and invariant linear system $\sqrt{-3}\,x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$. These two homologies are obtained from those with centers $\xi_1 + \xi_2 + \ldots + \xi_6 = 0$ and $\omega^2 \xi_1 + \omega \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0$ respectively. The symmetric group on the six variables together with the first of these homologies generates a group of order $2^4 \cdot 6!$ under which $\xi_1 + \xi_2 = 0$ goes into 30 positions and $-\sqrt{-3}\,\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0$ into 96 positions. These are the 126 points permuted by the group.

If now the coefficients of the transformations and the coordinates of the points be reduced modulo 3, the group remains the same as a permutation group on these 126 points, but it then has the quadratic invariant $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \equiv 0$. The self-conjugate subgroup of index 2 is therefore (1, 1) isomorphic with the simple group of the same order.

The whole group has primitive subgroups in the six variables. The symmetric group on the first five variables together with the two homologies with centers $\xi_1 + \xi_2 = 0$ and $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 - \sqrt{-3}\,\xi_6 = 0$ generates a group of order $2^7 \cdot 3^4 \cdot 5$, having the invariant quadratic spread $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0$. This is the well-known group of this order and may, by a slight change of variables, be exhibited with rational coefficients. The symmetric group on the six variables together with the homology with center $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 - \sqrt{-3}\,\xi_6 = 0$ generates a group of order $7!$.

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* Jordan, *Traité des Substitutions*, pp. 161–170; Dickson, *Linear Groups*, Chap. VII.

On Non-Homogeneous Equations with an Infinite Number of Variables.[*]

By R. D. Carmichael.

1. *Introduction.*

In recent years important contributions to the problem of solving linear equations with an infinite number of variables have been made by Hill,[†] Poincaré,[‡] von Koch,[§] Hilbert,[||] Toeplitz,[¶] Schmidt,[**] and Bôcher and Brand.[††] The most interesting and far-reaching developments of the theory which have been given up to the present time are those of the last two papers mentioned.

Two memoirs by Kötteritzsch[‡‡] appeared earlier than any of those just referred to. They are formal in their treatment; that is to say, the requisite convergence proofs are not given, so that one is without information as to the range of validity of the results. These researches have usually been passed over in silence by those who have written on the subject of infinite systems of equations. To be sure they are incomplete in several respects and contain actual errors; but they do not deserve the neglect with which they have met.[§§]

The purpose of the present paper is to indicate an important range of validity for the results of Kötteritzsch. His formal solutions are derived directly and convergence proofs are supplied for two important classes of cases. The results are stated in a theorem at the close of the paper.

In connection with the results of § 5 I point out that *the methods of the present paper have a valuable range of applicability different from that of previously developed theories.*

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* Presented to the American Mathematical Society, December 31, 1912.

† *Acta Mathematica,* VIII (1886), pp. 1–36. Previously published at Cambridge, U. S. A., in 1877.

‡ *Bulletin de la Société mathématique de France,* XIV (1886), pp. 77–90.

§ *Rendiconti del Circolo matematico di Palermo,* XXVIII (1909), pp. 255–266. See also the numerous references in this paper.

|| *Göttinger Nachrichten,* 1906, pp. 157–227; see especially pp. 218–227.

¶ *Rendiconti del Circolo matematico di Palermo,* XXVIII (1909), pp. 88–96. See also the references in this paper.

** *Rendiconti del Circolo matematico di Palermo,* XXV (1908), pp. 53–77.

†† *Annals of Mathematics,* XIII (1912), pp. 167–186.

‡‡ *Zeitschrift für Mathematik und Physik,* XV (1870), pp. 1–15, 229–268.

§§ Compare *Encyclopédie des Sciences Mathématiques,* I, pp. 319–321.

2. *Normal Form of the System.*

Let us consider the system of linear equations

$$\sum_{j=1}^{\infty} \bar{a}_{ij}\bar{u}_j=\bar{c}_i, \quad i=1, 2, \ldots., \tag{1}$$

having the property that no linear relation exists among any finite number of the first members. It is easy to see that the variables $\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots.$ can be rearranged into a sequence $u_1, u_2, u_3, \ldots.$ so that the system (1) can be reduced to an equivalent system of the form

$$\sum_{j=i}^{\infty} a_{ij}u_j=c_i, \quad a_{ii}=1, \quad i=1, 2, 3, \ldots., \tag{2}$$

the k-th equation in the new system being obtained by means of a linear combination of the first k equations of the old. We shall study the equations in the normal form (2). It is obvious that the results for this system can be carried over to system (1) without difficulty.

3. *The Formal Solution of Kötteritzsch.*

It is natural to expect that a solution of (2) is linear in the c's. Accordingly let us seek a solution of the form

$$u_k=\sum_{l=1}^{\infty} s_{kl}c_l, \quad k=1, 2, \ldots.. \tag{3}$$

If we substitute this value of u_k from (3) into (2) we have formally

$$\sum_{j=i}^{\infty} a_{ij}\sum_{l=1}^{\infty} s_{jl}c_l=\sum_{l=1}^{\infty}\sum_{j=i}^{\infty} a_{ij}s_{jl}c_l=c_i, \quad i=1, 2, \ldots..$$

This is a formal identity in the c's provided that

$$\sum_{j=i}^{\infty} a_{ij}s_{jl}=\delta_{il}, \quad i, l=1, 2, 3, \ldots., \tag{4}$$

where δ_{il} is equal to 1 or zero according as i is or is not equal to l.

If we consider any fixed value of l, we have in (4) a singly infinite system of equations for determining those s's which have the given second subscript l. The l-th equation of the set has the second member 1; all the other equations of the set have the second member 0. It is obvious that there exists a particular solution $s_{kl}=\bar{s}_{kl}$ where $\bar{s}_{kl}=0$ when $k>l$ and $\bar{s}_{kl}=\Delta_{kl}$ when $k\leqq l$, Δ_{kl} being the determinant formed from

$$\Delta_l=\begin{vmatrix} 1 & a_{12} & a_{13} & a_{14} & \ldots. & a_{1l} \\ 0 & 1 & a_{23} & a_{24} & \ldots. & a_{2l} \\ 0 & 0 & 1 & a_{34} & \ldots. & a_{3l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots. & 1 \end{vmatrix}$$

by replacing the last element in the k-th column by 1 and all the other elements in that column by 0. If now we let l vary over the range 1, 2, 3, ..., we obtain a particular solution of (4).

Consider now the homogeneous system corresponding to (2):

$$\sum_{j=i}^{\infty} a_{ij}v_j=0, \quad i=1, 2, \ldots\ldots \tag{5}$$

This is also the homogeneous system corresponding to (4) for a fixed l and varying i. Consider the infinite set of solutions

$$v_j=v_{jk}, \quad k=1, 2, \ldots\ldots,$$

of (5). These are assumed to be any solutions whatever of (5), whether the same or different; in particular we may have $v_{jk}=0$ for all values of j and k.

Now, since $s_{kl}=\bar{s}_{kl}$ is a solution of (4), it is obvious that $s_{kl}=\bar{s}_{kl}+v_{kl}$ is also a solution of (4). This may be written in the form

$$s_{kl}=(k, l)\Delta_{kl}+v_{kl}, \quad k, l=1, 2, \ldots\ldots,$$

where

$$(k, l)=\begin{cases}0 \text{ when } k>l, \\ 1 \text{ when } k\leqq l.\end{cases}$$

Substituting these values of s_{kl} in (3) we have formally:

$$u_k=\sum_{l=1}^{\infty}\{(k, l)\Delta_{kl}+v_{kl}\}c_l, \quad k=1, 2, \ldots\ldots \tag{6}$$

If this value for u_k is substituted for u_k in (2), a formal identity in the c's will be obtained. Thus we have a formal solution of (2). If we put $v_{kl}=0$, we have the simpler formal solution

$$u_k=\sum_{l=1}^{\infty}(k, l)\Delta_{kl}c_l=\sum_{l=k}^{\infty}\Delta_{kl}c_l, \quad k=1, 2, \ldots\ldots \tag{7}$$

This latter is essentially the formal solution of Kötteritzsch. It is the one which we shall study in the following sections of this paper.

In a previous paper* I have had occasion to solve a particular system of equations of the form (2). This example illustrates the method by which appropriate determinations of the quantities v_{kl} may be made, so as to ensure the convergence of (6) when the condition $v_{kl}=0$ for every k and l gives rise to divergent series. The example also brings to notice the usefulness of this method when it is desired to select a particular solution (6) of (2) which has important properties in addition to satisfying the system of equations.

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* American Journal of Mathematics, XXXV (1913), pp. 164–175.

4. *Validity of the Formal Solution* (7) *as an Actual Solution.*

In the first place it is obvious that (7) cannot afford a solution of (2) unless every infinite series in (7) is convergent. But there are other conditions which must also be satisfied. Substituting from (7) into the first member of (2) we have

$$\sum_{j=i}^{\infty}\sum_{l=1}^{\infty} a_{ij}(j, l)\Delta_{jl}c_l. \tag{8}$$

Assuming for a moment the validity of an interchange in the order of summations in (8), we should have for the value of (8)

$$\sum_{j=i}^{\infty}\sum_{l=1}^{\infty} a_{ij}(j, l)\Delta_{jl}c_l = \sum_{l=1}^{\infty}\sum_{j=i}^{\infty} a_{ij}(j, l)\Delta_{jl}c_l = c_i,$$

since

$$\sum_{j=i}^{\infty} a_{ij}(j, l)\Delta_{jl} = \sum_{j=i}^{\infty} a_{ij}\delta_{jl} = \delta_{il}.$$

Hence we conclude that (7) will afford a valid solution of (2) when and only when the series in (7) converges for every value of k and the interchange of the order of summations in (8) is legitimate.

Now we may think of (8) arranged as a double series:

$$\begin{array}{l}
a_{ii}(i, 1)\Delta_{i1}c_1 + a_{ii}(i, 2)\Delta_{i2}c_2 + a_{ii}(i, 3)\Delta_{i3}c_3 + \ldots,\\
a_{i,i+1}(i+1, 1)\Delta_{i+1,1}c_1 + a_{i,i+1}(i+1, 2)\Delta_{i+1,2}c_2 + a_{i,i+1}(i+1, 3)\Delta_{i+1,3}c_3 + \ldots,\\
a_{i,i+2}(i+2, 1)\Delta_{i+2,1}c_1 + a_{i,i+2}(i+2, 2)\Delta_{i+2,2}c_2 + a_{i,i+2}(i+2, 3)\Delta_{i+2,3}c_3 + \ldots,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

The necessary convergence in (7) implies the convergence of every row of this double series. Then for the convergence of this double series it is necessary and sufficient* that the convergence of the rows be uniform and that the series

$$S_1 + S_2 + S_3 + \ldots.$$

shall be convergent, where S_m is the sum of the m-th row of the double series.

Clearly the convergence of this double series is sufficient to ensure the legitimacy of the interchange in the order of summations in (8). Hence we conclude that (7) will be a valid solution of (2) in all cases when the convergence of the first series in (7) is uniform with respect to k and the series $\sum_{j=i}^{\infty} a_{ij}u_j$ is convergent, where u_j is the sum of the series (7) for $k=j$. We shall determine two important cases when these convergence conditions are satisfied.

* See Hobson's *Theory of Functions of a Real Variable*, p. 466.

In developing each of these two conditions we shall have need of Hadamard's fundamental theorem* concerning an upper bound to the absolute value of a determinant. This theorem may be stated as follows:

If Δ is an n-th order determinant in which β_{ij} is the element in the i-th row and the j-th column, then

$$|\Delta| \leqq \sqrt{r_1 r_2 \ldots . r_n}, \qquad |\Delta| \leqq \sqrt{\rho_1 \rho_2 \ldots . \rho_n},$$

where

$$r_i = \sum_{j=1}^{n} |\beta_{ij}|^2, \qquad \rho_i = \sum_{j=1}^{n} |\beta_{ji}|^2.$$

5. *First Class of Cases.*

To obtain our first condition implying the validity of (7) as a solution of (2) we proceed thus: Denote by σ_i the sum

$$\sigma_i = \sum_{j=1}^{i} |a_{ji}|^2.$$

From the definition of Δ_{kl} and the second inequality for $|\Delta|$ in Hadamard's theorem we see that

$$|\Delta_{kl}|^2 \leqq \frac{1}{\sigma_k} \prod_{j=1}^{l} \sigma_j \leqq \sigma_1 \sigma_2 \ldots . \sigma_l .$$

From this it follows that the first series in (7) for u_k is term by term not greater in absolute value than the series

$$\sum_{l=1}^{\infty} \sqrt{\sigma_1 \sigma_2 \ldots . \sigma_l}\, |c_l|. \tag{9}$$

If this series converges, it follows that the first series for u_k is uniformly convergent with respect to k. Also, it is obvious that

$$|u_k| \leqq \sum_{l=k}^{\infty} \frac{\sqrt{\sigma_1 \sigma_2 \ldots . \sigma_l}}{\sqrt{\sigma_k}} |c_l| = A_k,$$

where A_k is defined by this relation. Consider the series

$$\sum_{j=i}^{\infty} |a_{ij}| A_j, \quad i = 1, 2, \ldots . . \tag{10}$$

From our test in § 4 it follows that the convergence of series (9) and (10) is sufficient to ensure the validity of (7) as a solution of (2).

Since $|u_k|$ is evidently not greater than the sum of (9), it follows from our test in § 4 that the convergence of (9) and of the series

$$\sum_{j=i}^{\infty} |a_{ij}|, \quad i = 1, 2, \ldots ., \tag{11}$$

is also sufficient for the validity of (7) as a solution of (2).

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* *Bulletin des Sciences Mathématiques* (Darboux), XVII (1893), pp. 240–246.

A comparison of these results with the theory of von Koch and also with that of Schmidt will be sufficient to show that *the methods of the present paper have an important range of applicability different from that of previously developed theories.* To take a fairly obvious case, let us consider a system of equations (2) in which

$$|a_{ij}| \leqq M^{i+j},$$

where M is independent of i and j. It is easy to see that the numbers $c_1, c_2, c_3, \ldots.$ can be chosen in an infinity of ways so as to ensure that the requisite convergence conditions are satisfied. A comparison of this case with those treated by von Koch and Schmidt is sufficient to justify the statement made above.

6. *Second Class of Cases.*

To obtain a second condition under which (7) affords an actual solution of (2) let us assume with Schmidt the convergence of the series $|a_{i1}|^2 + |a_{i2}|^2 + \ldots.$ for every value of i. We write

$$s_i = \sum_{j=1}^{\infty} |a_{ij}|^2, \quad i=1, 2, \ldots..$$

Applying the first inequality for $|\Delta|$ in Hadamard's theorem we see that

$$|\Delta_{kl}|^2 \leqq s_0 s_1 s_2 \ldots. s_{l-1}, \quad s_0 = 2,$$

since the sum of the squares of the absolute values of the elements in the i-th row of Δ_{kl}, $i \neq l$, is equal to or less than s_i, while the corresponding sum for the l-th row is 2. From this it follows that the first series in (7) for u_k is term by term not greater in absolute value than the series

$$\sum_{l=1}^{\infty} \sqrt{s_0 s_1 \ldots. s_{l-1}}\, |c_l|. \tag{12}$$

If this series converges, it follows that the first series for u_k is uniformly convergent with respect to k. Also, it is obvious that

$$|u_k| \leqq \sum_{l=k}^{\infty} \sqrt{s_0 s_1 \ldots. s_{l-1}}\, |c_l| = B_k,$$

where B_k is defined by this relation. Consider the series

$$\sum_{j=i}^{\infty} |a_{ij}| B_j, \quad i=1, 2, \ldots.. \tag{13}$$

From our test in § 4 it follows that the convergence of series (12) and (13) is sufficient to ensure the validity of (7) as a solution of (2).

Since B_k obviously decreases with increase of k, it follows from the above result that the convergence of (12) and (11) is also sufficient for the validity of (7) as a solution of (2).

7. *Statement of the Principal Results.*

The principal results which we have obtained may be stated as follows:

Theorem. *If the system of equations*

$$\sum_{j=1}^{\infty} \bar{a}_{ij}\bar{u}_j=\bar{c}_i, \quad i=1, 2, \ldots, \tag{$\bar{\mathrm{A}}$}$$

has the property that no linear relation exists among any finite number of the first members, it can readily be reduced to an equivalent system

$$\sum_{j=i}^{\infty} a_{ij}u_j=c_i, \quad a_{ii}=1, \quad i=1, 2, \ldots. \tag{A}$$

Denote by Δ_{kl} the determinant formed from

$$\begin{vmatrix} 1 & a_{12} & a_{13} & a_{14} & \ldots & a_{1l} \\ 0 & 1 & a_{23} & a_{24} & \ldots & a_{2l} \\ 0 & 0 & 1 & a_{34} & \ldots & a_{3l} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{vmatrix}$$

by replacing the last element in the k-th column by 1 *and all the other elements in that column by zero. Denote by v_{kl}, $k=1, 2, \ldots$, any solution $v_k=v_{kl}$ of the system*

$$\sum_{j=i}^{\infty} a_{ij}v_j=0, \quad i=1, 2, \ldots.$$

Then a formal solution of equations (A) *is*

$$u_k=\sum_{l=1}^{\infty} \{(k, l)\Delta_{kl}+v_{kl}\}c_l, \quad k=1, 2, \ldots, \tag{B}$$

where (k, l) is unity when $l \geqq k$, and is otherwise zero. If we take $v_{kl}=0$, this reduces to

$$u_k=\sum_{l=k}^{\infty} \Delta_{kl}c_l, \quad k=1, 2, \ldots. \tag{C}$$

Among the cases in which (C) *certainly affords an actual solution of* (A), *are the two general classes in which the one or the other of the two following conditions are satisfied:*

1. *If*

$$\sigma_i=\sum_{j=1}^{i} |a_{ji}|^2,$$

then the series

$$\sum_{l=1}^{\infty} \sqrt{\sigma_1\sigma_2\ldots\sigma_l}\,|c_l|\,; \qquad \sum_{j=i}^{\infty} |a_{ij}|A_j, \quad i=1, 2, \ldots,$$

converge, where A_j *has either of the two values* $A_j=1$, $j=1, 2, \ldots.,$ *or*

$$A_j=\sum_{l=j}^{\infty}\frac{\sqrt{\sigma_1\sigma_2\ldots.\sigma_l}}{\sqrt{\sigma_j}}|c_l|,\quad j=1, 2, \ldots..$$

2. *The series* $|a_{i1}|^2+|a_{i2}|^2+|a_{i3}|^2+\ldots.$ *converges to the sum* s_i. *The series*

$$\sum_{l=1}^{\infty}\sqrt{s_0s_1\ldots.s_{l-1}}|c_l|,\quad s_0=2;\qquad \sum_{j=i}^{\infty}|a_{ij}|B_j,\quad i=1, 2, \ldots.,$$

converge, where B_j *has either one of the two values* $B_j=1$, $j=1, 2, \ldots.,$ *or*

$$B_j=\sum_{l=j}^{\infty}\sqrt{s_0s_1\ldots.s_{l-1}}|c_l|,\quad j=1, 2, \ldots..$$

From a solution of (A) *a solution of* ($\bar{\mathrm{A}}$) *may readily be found.*

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On Constrained Motion.

BY PETER FIELD.

Introduction.

The traditional problem in the mechanics of a particle supposes given a system of particles with a certain number of degrees of freedom and a system of forces; the solution of the problem consists in determining the motion of each particle. Lagrange's equations enable us to do this provided it is possible to overcome the difficulties incident to the integration of the equations of motion.

Painlevé* has shown that in discussing problems of this kind, if we assume Coulomb's laws of friction to hold true, we must proceed with caution, as the supposition that motion takes place may lead either to conditions which are incompatible or to the possibility of more than one solution.† In some of the problems discussed by Painlevé, our interest is not primarily in the whole history of the subsequent motion but rather in knowing how many values of the acceleration are possible at the given moment with the given initial conditions. This is the point of view which is adopted in this paper.

Efforts have been made to dispose of the cases where more than one motion is possible by taking account of the elasticity of the material and also by supposing that it must be possible for the frictional force to increase from zero up to its maximum value,‡ but the fact nevertheless remains that when we suppose that we are dealing with a rigid system and suppose the frictional force at any point is proportional to the normal pressure at that point, and make no additional assumptions, we are led to cases where there may be more than one solution.§

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* "Leçons sur le frottement."

† See my paper on Coulomb's laws of friction, *Zeitschrift für Mathematik und Physik*, Vol. LXI, p. 68.

‡ See Lecornu, *Comptes rendus*, Vol. CXL (1905), p. 635, and De Sparre, Vol. CXLI (1905), p. 310; also *Zeitschrift für Mathematik und Physik*, Vol. LVIII (1910), articles by Klein, Mises, Hamel, Prandtl, Pfeiffer.

§ See Appell, *Traité de mécanique rationnelle*, Vol. II, p. 127, 3d edition.

Statement of the Problem.

The purpose of this paper is to investigate what conditions must be satisfied in order that the supposition that motion takes place may lead to conditions which are compatible, when there are no external forces and we have two particles m_1 and m_2 connected by a weightless rod of length l and one or both are constrained. The different cases may be tabulated as follows:*

(a) m_1 is constrained to a surface and m_2 is free,

(b) m_1 is constrained to a curve and m_2 is free,

(c) m_1 and m_2 are both constrained to surfaces,

(d) m_1 is constrained to a surface, m_2 to a curve,

(e) m_1 and m_2 are both constrained to curves.

The data are chosen as follows: The particle of mass m_1 is at P_1 (Fig. 1) and the second particle is at P_2. The velocities of m_1 and m_2 are u_1 and u_2.

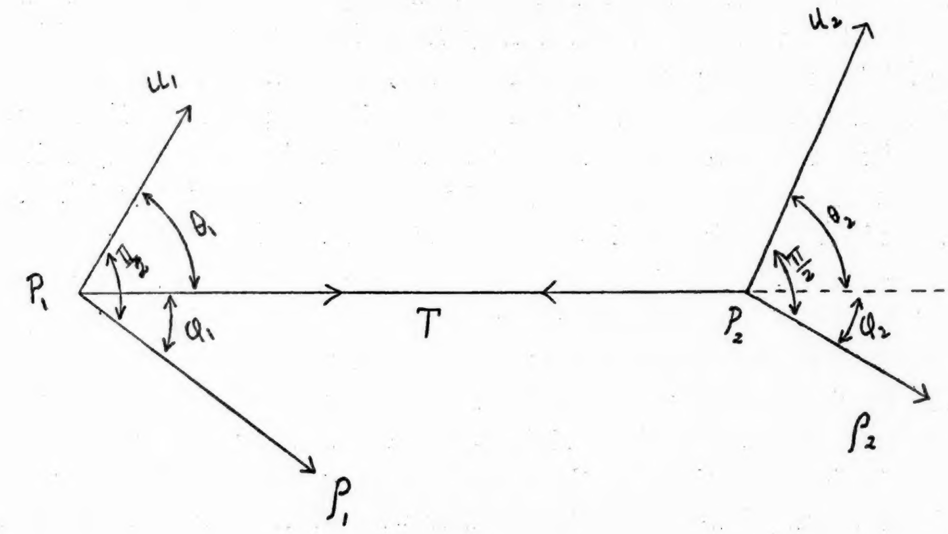

FIG. 1.

The projections of these velocities along l and at right angles to l are (v, v_1), (v, v_2) respectively, the projections along l being equal as P_1P_2 is of constant

* Some special cases have been studied by Painlevé and others. A study of the case where the two particles are constrained to parallel plane curves will appear in one of the forthcoming numbers of the *Zeitschrift für Mathematik und Physik.*

length. The angle between v_1 and v_2 will be called θ. In case the particles are constrained to curves, ρ_1 and ρ_2 are the radii of curvature of the paths; while if the particles are constrained to surfaces, ρ_1 and ρ_2 are the radii of curvature of the plane sections determined by the velocity and the normal to the surface. The coefficients of friction will be denoted by μ_1 and μ_2.

The angles θ_1 and θ_2 shown in Fig. 1 can be expressed in terms of v, v_1, v_2; but ϕ_1 and ϕ_2, also shown in the figure, are independent of the preceding parameters. We therefore have the following thirteen independent parameters: $m_1, m_2, \mu_1, \mu_2, l, v, v_1, v_2, \phi_1, \phi_2, \theta, \rho_1, \rho_2$. The tension in the rod is called T.

Case (a): *m_1 is Constrained to a Surface and m_2 is Free.*

The projections of the acceleration of m_1 are $\frac{u_1^2}{\rho_1}$ along ρ_1, $\frac{du_1}{dt}=j_1$ in the direction of u_1, and $\frac{T}{m_1}$ $(1-\cos^2\phi_1-\cos^2\theta_1)^{\frac{1}{2}}$ along a line perpendicular to u_1 and ρ_1 (Fig. 2).

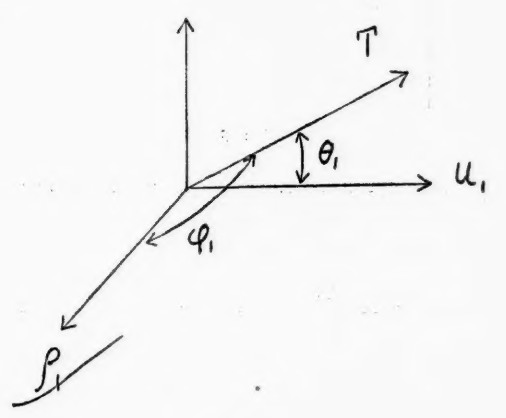

FIG. 2.

In order to get the value of j_1 it is necessary to find the pressure against the surface. This pressure is

$$\left|-m_1\frac{v^2+v_1^2}{\rho_1}+T\cos\phi_1\right|,$$

and we have at once the equation

$$m_1j_1=-\mu_1\left|-m_1\frac{v^2+v_1^2}{\rho_1}+T\cos\phi_1\right|+T\cos\theta_1.$$

The acceleration of m_2 is directed along P_1P_2 and is equal to $-\frac{T}{m_2}$. Now as the distance between the two particles is constant, the projections of their accelerations along P_1P_2 differ by

$$\omega^2 l = \frac{v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta}{l},$$

ω being the angular velocity of l and consequently equal to the geometric sum of v_1 and $-v_2$ divided by l.

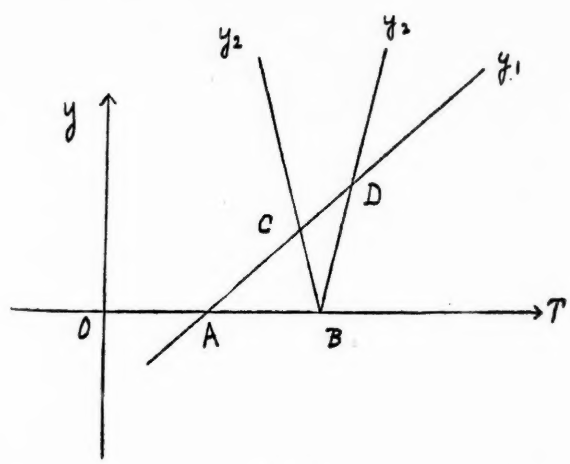

FIG. 3.

In case motion is possible we must therefore be able to determine T so as to satisfy the equation

$$\frac{T}{m_1}\cos^2\theta_1 - \mu_1\cos\theta_1 \left| \frac{T}{m_1}\cos\phi_1 - \frac{u_1^2}{\rho_1} \right| + \frac{u_1^2}{\rho_1}\cos\phi_1 + \frac{T}{m_1}(1-\cos^2\phi_1-\cos^2\theta_1)$$
$$-\frac{v_1^2+v_2^2-2v_1v_2\cos\theta}{l} + \frac{T}{m_2} = 0.$$

This equation may be rendered a trifle more compact by substituting k^2 and k_1^2 for $\frac{v_1^2+v_2^2-2v_1v_2\cos\theta}{l}$ and $\frac{u_1^2}{\rho_1}$. It can then be written

$$T\left(\frac{\sin^2\phi_1}{m_1}+\frac{1}{m_2}\right)+k_1^2\cos\phi_1-k^2=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|.$$

This condition is easily interpreted geometrically. In Fig. 3 the values of

$$y_1 = T\left(\frac{\sin^2\phi_1}{m_1}+\frac{1}{m_2}\right)+k_1^2\cos\phi_1-k^2$$

are represented by points on a straight line, while the values of

$$y_2=\mu_1 \cos \theta_1 \left| \frac{T}{m_1} \cos \phi_1 - k_1^2 \right|$$

are represented by points on a broken line which will lie above or below the T axis according as θ_1 is acute or obtuse. The values of T at the points C and D satisfy the conditions of our problem. If the data were so chosen that A coincided with B and the slopes of y_1 and one segment of the broken line were equal, we should have an infinite number of solutions. If θ_1 is acute and A lies to the left of B, there will be but one solution if the slope of the line y_1 is greater than the slope of the broken line; while if A lies to the right of B and the slope of y_1 is less than the slope of the broken line, our equation has no solution. The problem may therefore have 0, 1, 2, or an infinite number of solutions depending on how the data are chosen.

Case (b): *m_1 is Constrained to a Curve and m_2 is Free.*

The acceleration of m_1 lies in the osculating plane of the twisted curve to which m_1 is constrained. The component along the radius of curvature has the same value as in case (a), but the value of j_1 is different, as the pressure against the curve is now the geometric sum of $-m_1 \dfrac{u_1^2}{\rho_1}$ and the projection of T on the normal plane to the curve at P_1. This pressure (Fig. 2) is equal to

$$\sqrt{T^2 \sin^2 \theta_1 - 2Tm_1 \frac{u_1^2}{\rho_1} \cos \phi_1 + m_1^2 \frac{u_1^4}{\rho_1^2}}.$$

Therefore

$$j_1 = \frac{T}{m_1} \cos \theta_1 - \mu_1 \sqrt{\frac{T^2}{m_1^2} \sin^2 \theta_1 - 2\frac{T}{m_1} \frac{u_1^2}{\rho_1} \cos \phi_1 + \frac{u_1^4}{\rho_1^2}}.$$

If we now write the condition that the projection of the acceleration of m_1 along the rod is equal to $-\dfrac{T}{m_2} + \omega^2 l$ and use the same notation as in (a), we have

$$T\left(\frac{\cos^2 \theta_1}{m_1} + \frac{1}{m_2}\right) + k_1^2 \cos \phi_1 - k^2 = \mu_1 \cos \theta_1 \sqrt{\frac{T^2}{m_1^2} \sin^2 \theta_1 - 2\frac{T}{m_1} k_1^2 \cos \phi_1 + k_1^4}$$

as the condition which T must satisfy.

The possible number of solutions is the same as in case (a); *i. e.*, it may be 0, 1, 2, or an infinite number. Instead of having the points of intersection

of a straight line and a broken line as in Fig. 3, we now have a straight line and an arc of an hyperbola. In order to have an infinite number of solutions it is necessary that the hyperbola degenerate. In that case $\cos\phi_1 = \pm\sin\theta_1$, which means that the radius of curvature lies in the plane of the connecting rod and the direction of the velocity of m_1. This condition alone is of course not sufficient to give an infinite number of solutions. In addition two conditions corresponding to those of case (a) must be satisfied.

There is no difficulty in satisfying these conditions. For instance take $\cos\phi_1 = \sin\theta_1$ and suppose θ_1 acute. The other two conditions are

$$\frac{m_1k_1^2}{\sin\theta_1} = \frac{k^2 - k_1^2\sin\theta_1}{\frac{\cos^2\theta_1}{m_1} + \frac{1}{m_2}},$$

$$\frac{\cos^2\theta_1}{m_1} + \frac{1}{m_2} = \frac{\mu_1\cos^2\theta_1\sin\theta_1}{m_1}.$$

The first might be regarded as an equation to determine k^2 and the second as an equation which determines μ_1.

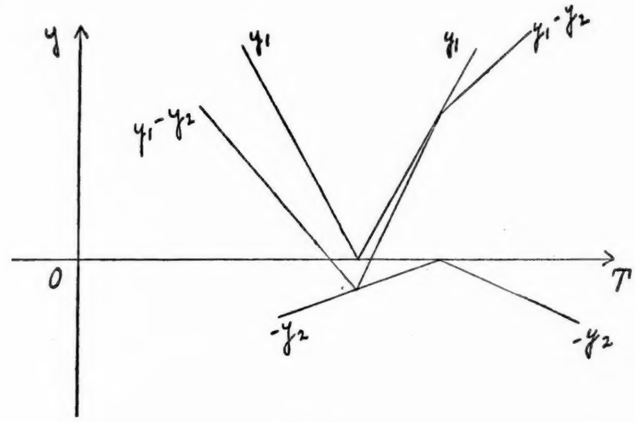

FIG. 4.

Case (c): *m_1 and m_2 are Both Constrained to Surfaces.*

According to (a) the projection of the acceleration of m_1 along P_1P_2 is

$$\frac{T}{m_1}\sin^2\phi_1 + k_1^2\cos\phi_1 - \mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1 - k_1^2\right|.$$

As the projection of the acceleration of m_2 along this line has the same form provided we replace T by $-T$, θ_1 by θ_2, ϕ_1 by ϕ_2, and k_1^2 by $\frac{u_2^2}{\rho_2} = k_2^2$, the equation

which determines T may be written

$$T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\sin^2\phi_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$
$$=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|-\mu_2\cos\theta_2\left|\frac{T}{m_2}\cos\phi_2+k_2^2\right|,$$

or $y=y_1-y_2$. It is no restriction to suppose the angles θ_1 and θ_2 acute. The values of

$$y_1-y_2=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|-\mu_2\cos\theta_2\left|\frac{T}{m_2}\cos\phi_2+k_2^2\right|$$

will then lie on a broken line composed of three rectilinear segments as in Fig. 4.

The desired values of T are found as the values of T at the points of intersection of the broken line with the line

$$y=T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\sin^2\phi_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2.$$

This gives rise to 0, 1, 2, 3, or an infinite number of solutions, depending on how the values of the constants are chosen. Some of the constants, as μ_1, μ_2, m_1, m_2, k_1^2, k_2^2, k^2, are from their nature positive, but even with this restriction cases with the different numbers of solutions are readily constructed.

As an illustration let us build a case with an infinite number of solutions We might take the vertices of the broken lines at the same point and have the line

$$y=T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\sin^2\phi_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$

coincide with that part of the broken line y_1-y_2 which lies to the right of the common vertex of y_1 and y_2 (Fig. 5). This can readily be accomplished by taking

$$m_1=m_2=1,\qquad \cos\phi_2=-\cos\phi_1,$$

$$\mu_1=\frac{\mu_2\cos\theta_2\cos\phi_1+2\sin^2\phi_1}{\cos\theta_1\cos\phi_1},$$

$$k^2=2k_1^2\frac{\cos^2\phi_1+\sin^2\phi_1}{\cos\phi_1}=\frac{2k_1^2}{\cos\phi_1};$$

μ_1 and k^2 being positive, we add the further restrictions that θ_1, θ_2, ϕ_1 are acute.

Case (d): *m_1 is Constrained to a Surface and m_2 to a Curve.*

From (a) and (b) the projection of the acceleration of m_1 along the rod is

$$T\frac{\sin^2\phi_1}{m_1}+k_1^2\cos\phi_1-\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|$$

when m_1 is constrained to a surface and

$$T\frac{\cos^2\theta_1}{m_1}+k_1^2\cos\phi_1-\mu_1\cos\theta_1\sqrt{\frac{T^2}{m_1^2}\sin^2\theta_1-2\frac{T}{m_1}k_1^2\cos\phi_1+k_1^4}$$

when constrained to a curve. If m_2 is constrained to a curve, the value of the projection of its acceleration along the rod is obtained from the corresponding

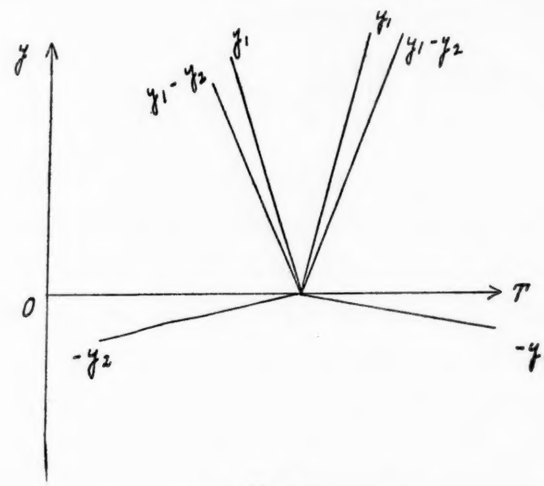

FIG. 5.

expression for m_1 by replacing T by $-T$ and making the subscripts 2 in place of 1. From this it follows at once that when m_1 is constrained to a surface and m_2 to a curve, the stress in the rod must satisfy the equation

$$T\left(\frac{\sin^2\phi_1}{m_1}+\frac{\cos^2\theta_2}{m_2}\right)+k_1^2\cos\phi_1-k_2^2\cos\phi_2-k^2$$

$$=\mu_1\cos\theta_1\left|\frac{T}{m_1}\cos\phi_1-k_1^2\right|-\mu_2\cos\theta_2\sqrt{\frac{T^2}{m_2^2}\sin^2\theta_2+2\frac{T}{m_2}k_2^2\cos\phi_2+k_2^4},$$

or more briefly, $y=y_1-y_2$.

Suppose the angles θ_1 and θ_2 are obtuse; the values of y_1-y_2 corresponding to the different values of T will then lie on two arcs of hyperbolas as illus-

trated in Fig. 6. The values of T which satisfy our problem are found as the values of T at the points where the curve y_1-y_2 is cut by the line

$$y=T\left(\frac{\sin^2 \phi_1}{m_1}+\frac{\cos^2 \theta_2}{m_2}\right)+k_1^2 \cos \phi_1-k_2^2 \cos \phi_2-k^2.$$

It is easily shown that there are 0, 1, 2, 3, 4 or an infinite number of solutions depending on how the data of the problem are chosen.

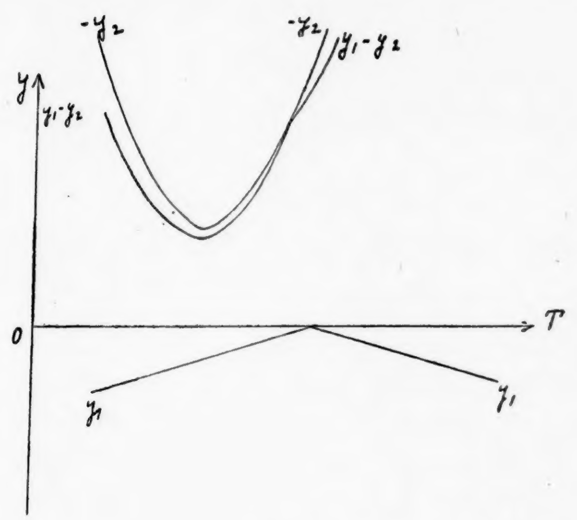

FIG. 6.

Case (e): *m_1 and m_2 are Both Constrained to Curves.*

By making use of the results already derived we have at once the equation for T; *viz.*,

$$T\left(\frac{\cos^2 \theta_1}{m_1}+\frac{\cos^2 \theta_2}{m_2}\right)+k_1^2 \cos \phi_1-k_2^2 \cos \phi_2-k^2$$
$$=\mu_1 \cos \theta_1\sqrt{\frac{T^2}{m_1^2}\sin^2 \theta_1-2\frac{T}{m_1}k_1^2 \cos \phi_1+k_1^4}$$
$$-\mu_2 \cos \theta_2\sqrt{\frac{T^2}{m_2^2}\sin^2 \theta_2+2\frac{T}{m_2}k_2^2 \cos \phi_2+k_2^4},$$

or $y=y_1-y_2$.

The values of y_1-y_2 corresponding to different values of T will lie on an arc of a quartic curve, and we have 0, 1, 2, 3, 4 or an infinite number of solutions for our problem, just as in case (d).

An Elastic Connecting Rod.

Although the case where the connecting rod is elastic does not come within the scope of this paper, as we are dealing with a rigid system, it may be worth mentioning that it is at once apparent that when the rod is supposed elastic, the stress can have but one value, which is determined by the strain, and consequently the accelerations at P_1 and P_2 have but a single value.

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Covariant Curves of the Plane Rational Quintic.

By J. I. Tracey.

Introduction.

Let a plane rational curve ρ^n of order n be given parametrically by the equations

$$x_i=(\alpha_i t)^n \qquad (i=0, 1, 2), \tag{1}$$

then any line of the plane intersects ρ^n in n points whose equation may be represented symbolically by the binary n-ic.

$$(a\xi)(\alpha t)^n=0. \tag{2}$$

We shall designate this form by $C_{1,n}$, since it is of the n-th order in the parameter t and involves the coefficients of the line ξ to the first degree.

If I_k is an invariant of $C_{1,n}$, it contains the coefficients of the line to the k-th degree and, for a varying ξ, represents a curve of class k. This curve is the locus of line sections of ρ^n for which I_k vanishes.

Any covariant $C_{l,m}$ of $C_{1,n}$ is of degree l in the ξ's and of order m in the t's. Now for a given ξ we obtain m points $\tau_1, \tau_2, \ldots, \tau_m$ on ρ^n. These are the roots of the equation $C_{l,m}=0$. However, for a given point of ρ^n, say $t=\tau$, $C_{l,m}$ becomes a curve of class l which is the locus of line sections of ρ^n containing τ for a root of its $C_{l,m}$.

If for a given line $C_{l,m}$ vanishes identically, then the coefficients of this m-ic give a linear system of at most $m+1$ curves of class l, each containing all line sections of ρ^n for which $C_{l,m}$ vanishes identically. By this means we can frequently express in concise form all the independent curves of a given class which contain special line sections of ρ^n, and account for all degenerate curves whose degeneracy is due to the multiplicity of the system.

It is important to keep in mind that when a covariant $C_{l,m}$ vanishes identically, all invariants and covariants of $C_{l,m}$ likewise vanish. Interpreted geometrically, if a linear system of curves given by $C_{l,m}$ contains a set of lines L, then any curve or system of curves derived from invariants and other covariants of $C_{l,m}$, respectively, will contain lines L and contain them multiply.

Thus, if I_{ql} is an invariant of $C_{l,m}$, each line L is a q-fold line of the curve represented by this invariant. A covariant of $C_{l,m}$ containing its coefficients to the r-th degree will determine a linear system of curves of class rl, and L will be r-fold lines of this system. Finally, any operation of $C_{l,m}$ on a second form, or conversely, will give a system of curves on L.

§ 1. *Covariant Curves on the Double Lines and Stationary Lines of* ρ^n.

The curve ρ^n has in general $3(n-2)$ stationary lines and $2(n-2)(n-3)$ double lines. Now there is a covariant $C_{2(n-2),2(n-2)}$ which vanishes when $C_{1,n}$ has a triple root, and also vanishes if $C_{1,n}$ contains two double roots. This covariant gives the $2n-3$ independent curves of class $2(n-2)$ on the $(n-2)(2n-3)$ stationary lines and double lines of ρ^n. In terms of the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $C_{1,n}$ it is

$$C_{2(n-2),2(n-2)} \equiv \sum^{n} (\alpha_1-\alpha_2)^2 (\alpha_1-\alpha_3)^2 \ldots (\alpha_1-\alpha_{n-1})^2 (\alpha_2-\alpha_3)^2 \ldots (\alpha_{n-2}-\alpha_{n-1})^2 (t-\alpha_n)^{2(n-2)}. \quad (3)$$

For the rational cubic this covariant is the $C_{2,2}$,* the Hessian of the $C_{1,3}$, and its coefficients give three independent conics on the three stationary lines.

In the case of the ρ^4 the corresponding covariant is †

$$C_{4,4} \equiv 3I_3C_{1,4} - 2I_2C_{2,4}, \quad (4)$$

and its coefficients equated to zero give five independent quartic curves on the six stationary lines and four double lines of the base curve.

Similarly for the ρ^5 the covariant is ‡

$$C_{6,6} \equiv 4C_{2,2}^3 + 12C_{2,2}C_{4,4} + C_{2,6}I_4 - 216C_{3,3}^2. \quad (5)$$

From the coefficients of this covariant we obtain a linear system of seven independent sextics, by means of which the six-fold infinity of sextics on the nine stationary lines and twelve double lines of ρ^5 may be expressed.

§ 2. *Covariant Curves on the Double Lines of* ρ^n.

The condition for $C_{1,n}$ to have two double roots is the identical vanishing of a $C_{2n-5,3(n-2)}$. The coefficients of this covarient give $3n-5$ curves of class $2n-5$ on the $2(n-2)(n-3)$ double lines of ρ^n. Now since there are only $3(n-2)$ linearly independent curves of class $2n-5$ on $2(n-2)(n-3)$ fixed lines, we

* The fundamental forms used throughout are, unless otherwise stated, the same as given by Salmon, "Higher Algebra," fourth edition, or by Elliott, "Algebra of Quantics." The notation is the same as Elliott's.

† Salmon, "Higher Algebra," p. 199.

‡ This covariant corresponds in form to the one given by Cayley, *Collected Mathematical Papers*, Vol. IV, p. 276.

shall expect the existence of a linear relation among these $3n-5$ coefficients. Expressed in terms of the roots of $C_{1,n}$ this covariant is

$$C_{2n-5,3(n-2)} \equiv \sum^{\frac{n(n-1)(n-2)}{6}} (\alpha_1-\alpha_4)\ldots\ldots(\alpha_1-\alpha_n)(\alpha_2-\alpha_4)\ldots\ldots \\ (\alpha_2-\alpha_n)(\alpha_3-\alpha_4)\ldots\ldots(\alpha_3-\alpha_n)(\alpha_4-\alpha_5)^2\ldots\ldots \\ (\alpha_{n-1}-\alpha_n)^2(t-\alpha_1)^{n-2}(t-\alpha_2)^{n-2}(t-\alpha_3)^{n-2}. \quad (6)$$

For the ρ^4 this covariant is the $C_{3,6}$ giving seven cubic curves on the four double lines. It has been proven* that this form is apolar to the sextic equation whose roots give the six points of inflection of the ρ^4, thus accounting for the linear relation which has been mentioned.

The condition that $C_{1,5}$ have two double roots is the vanishing of the covariant†

$$C_{5,9} \equiv 50C_{2,6}C_{3,3}-C_{1,5}(C_{4,4}+3C_{2,2}^2). \quad (7)$$

These coefficients give ten curves of class 5 on the twelve double lines of the ρ^5, and the apolarity relation existing between this covariant and the equation which gives the nine points of inflection has been proven by direct calculation.

The covariant whose vanishing is the condition that $C_{1,6}$ have two double roots is as follows:‡

$$C_{7,12} \equiv 36C_{3,12}I_2^2-35C_{4,6}C_{1,6}I_2+25[19C_{2,4}C_{5,8}-2C_{2,4}C_{3,8}I_2 \\ +21C_{3,2}C_{4,10}-18C_{3,6}C_{4,6}+C_{3,8}C_{4,4}], \quad (8)$$

and in a similar manner as above these coefficients give septimic curves on the twenty-four double lines of the rational sextic. That the apolarity relation exists between the $C_{7,12}$ and the equation giving the twelve points of inflection of ρ^6 is due to the following general theorem. We shall now prove that:

The covariant $C_{2n-5,3(n-2)}$, whose vanishing is the condition for two double roots, is apolar to the $3(n-2)$-ic which gives the points of inflection of ρ^n.

Each curve of class $2n-5$ determined by the coefficients of the covariant has $2(3n-7)$ common lines with the ρ^n in addition to the double lines. Now the base curve ρ^n may be represented in point form by a function of $\{A, \xi, t^n\}$,§ and in line form by $\{A^2, x, t^{2(n-1)}\}$. The $C_{2n-5,3(n-2)}$ is similarly represented

* Thomsen, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, p. 230.

† Cayley, *Collected Mathematical Papers*, Vol. II, p. 469. Cayley's $C_{4,4}$ is Elliott's $C_{2,2}^2-C_{4,4}$.

‡ The fundamental forms of $C_{1,6}$ used in the calculation of this covariant were taken from Cayley, *Collected Mathematical Papers*, Vol. XI, p. 377.

§ This is another way of writing $(a\xi)(at)^n$, the parametric equation of ρ^n, A representing the coefficients of the binary n-ics cut out by the reference triangle. The exponent of any quantity indicates the degree to which that quantity enters in the given expression, the brackets, $\{\ \}$, meaning a function of the enclosed quantities.

by $\{A^{2n-5}, \xi^{2n-5}, t^{3(n-2)}\}$, and on writing its apolarity relation with an arbitrary form $(\alpha t)^{3(n-2)}$ the resulting expression is $\{A^{2n-5}, \xi^{2n-5}, \alpha\}$. This represents a curve of class $2n-5$ on the double lines of ρ^n. The contacts of common lines to these two curves are given by the incidence condition of ξ and x in the forms $\{A^{2n-5}, \xi^{2n-5}, \alpha\}$ and $\{A^2, x, t^{2(n-1)}\}$, respectively, the result being the equation

$$\{A^{3(2n-5)}, \alpha, t^{2(n-1)(2n-5)}\}=0. \tag{9}$$

Equation (9) contains, as we have already seen, the contacts of the double lines which involve A and t to the degrees $6(n-3)$ and $4(n-2)(n-3)$, respectively. On removing this factor we are left with the contacts of the extra common lines which are given by the equation

$$\{A^3, \alpha, t^{2(3n-7)}\}=0. \tag{10}$$

Now it must be possible to obtain equation (10) by means of the transvectant process, so we shall examine the covariant sets of points along ρ^n.

There is no such covariant which involves the A's to a degree less than 3, and the highest order of such in the variable is $3(n-2)$. This of course is the form giving the points of inflection of ρ^n, which in this notation is $\{A^3, t^{3(n-2)}\}$. Therefore equation (10) is the first transvectant or the Jacobian of the arbitrary form $(\alpha t)^{3(n-2)}$ and the linear covariant $\{A^3, t^{3(n-2)}\}$.

In general, then, for every set of points given by $(\alpha t)^{3(n-2)}$ there is a curve of class $2n-5$ on the double lines formed by its apolarity relation with $C_{2n-5,3(n-2)}$. However, if $(\alpha t)^{3(n-2)}\equiv\{A^3, t^{3(n-2)}\}$ their Jacobian vanishes and there is no such curve corresponding to this set of points. This proves the theorem.

§ 3. *Covariant Curves on the Stationary Lines.*

The covariant which vanishes when $C_{1,n}$ has a triple root is a $C_{2(n-3),4(n-4)}$. From this form there are $4n-15$ curves of class $2(n-3)$ on the $3(n-2)$ stationary lines. There are, however, $2(n-2)(n-4)$ curves of class $2(n-3)$ on $3(n-2)$ lines, and for values of $n \gtrless 6$, we shall show how all the remaining curves are given by invariants which vanish when $C_{1,n}$ has a triple root.

In terms of the roots of $C_{1,n}$ this covariant is

$$C_{2(n-3),4(n-4)}\equiv \sum^{\frac{n(n-1)}{2}} (\alpha_1-\alpha_2)^2\ldots(\alpha_1-\alpha_{n-2})^2(\alpha_2-\alpha_3)^2\ldots(\alpha_{n-3}-\alpha_{n-2})^2 (\alpha_{n-1}-\alpha_n)^2(t-\alpha_{n-1})^{2(n-4)}(t-\alpha_n)^{2(n-4)}. \tag{11}$$

When $n=4$ this becomes the invariant I_2 of the $C_{1,4}$, and it is well known that this conic touches the six stationary lines of ρ^4.

The condition that $C_{1,5}$ have a triple root is the vanishing of *

$$C_{4,4} \equiv 3C^1_{4,4} - C^2_{2,2}. \tag{12}$$

The invariant I_4 also vanishes for a triple root; hence the $C_{4,4}$ and the I_4 give the linear system of six independent quartic curves on the nine stationary lines of ρ^5.

The covariant in question becomes a $C_{6,8}$ for the $C_{1,6}$, and this gives nine sextic curves on the twelve stationary lines. However, there are two invariants, an I_4 and an I_6, which vanish when $C_{1,6}$ has a triple root.† The quartic curve given by the I_4, together with any conic in the plane, is a degenerate sextic, and hence a five-fold infinity of the required system is given by this invariant. We have thus located sixteen independent sextic curves from which the entire linear system on the twelve stationary lines of the ρ^6 is determined.

§ 4. *Special Line Sections of ρ^5.*

To examine the curves given by the covariants and invariants of $C_{1,5}$ and study their relation to the base curve, we shall enumerate certain line sections of ρ^5, together with the reduced form which the corresponding binary quintics can be made to assume.

α. There are six so-called cyclic lines whose intersections with ρ^5 are given by a binary quintic which is apolar to a unique quadratic. Such line sections are reducible to the form x^5+y^5.

β. Fifteen lines such that four of the intersections are two harmonic pairs, while the remaining intersection is a root of the Jacobian of the two pairs. This form is reducible to $x(x^4+y^4)$.

γ. There are fourteen lines of ρ^5 such that the point of contact and three remaining intersections are a self-apolar quartic. The corresponding binary form may be written $x^2(x^3+y^3)$.

δ. Ten lines whose intersections with ρ^5 constitute a binary cubic and its Hessian. These can take the form $xy(x^3+y^3)$.

ϵ. There are nine stationary lines giving forms reducible to $x^3(x^2+y^2)$.

ξ. Twenty-four lines of ρ^5 such that the contact of each forms with the three remaining intersections a binary quartic which breaks up into harmonic pairs. These binary quintics can take the form $x^2y(x^2+y^2)$.

η. Twelve double lines which can be reduced to the form $x^2y^2(x+y)$.

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* Cayley, *Selected Mathematical Papers*, Vol. II, p. 469. The $C^1_{4,4}$ is Elliott's $C_{4,4}$.

† Salmon, "Higher Algebra," p. 263.

It is observed that the $C_{1,5}$ in each of these reduced forms consists of only two terms.

§ 5. *The First Osculants of ρ^5.*

Let the rational quintic be given by

$$x_i = a_i t^5 + 5b_i t^4 + 10c_i t^3 + 10d_i t^2 + 5e_i t + f_i \equiv (\alpha_i t)^5, \tag{13}$$

where $i=0, 1, 2$. Then the osculant quartic of a point τ on the base curve is obtained by polarizing equations (13) with respect to τ. It is

$$x_i \equiv (\alpha_i \tau)(\alpha_i t)^4. \tag{14}$$

Now on forming the conic g_2 of this quartic osculant we see at once that it is the $C_{2,2}$ of a line section when $t=\tau$. Therefore:

The locus of lines cutting out self-apolar quartics from the osculant curve (14) *is the conic $C_{2,2}$ formed for the given point τ.*

Similarly if we form the g_3 of a line section of (14), it is the canonizant of $C_{1,5}$ when $t=\tau$. Hence:

The locus of lines cutting harmonic pairs from the osculant quartic is the cubic curve $C_{3,3}$ formed for the point τ.

And from these we have immediately:

The sextic $C^3_{2,2} - 27C^2_{3,3}$, for any point τ of ρ^5, gives the line equation of the quartic osculant at that point.

The osculant cubic at the point τ is

$$x_i = (\alpha_i \tau)^2 (\alpha_i t)^3, \tag{15}$$

and on taking the discriminant of a line section of this curve we find that:

$C_{4,8} \equiv C_{2,2} C_{2,6} - C_{1,5} C_{3,3}$, *for a point τ, gives the line equation of the corresponding cubic osculant.*

Polarizing $(\alpha_i t)^5$ three times with respect to τ we obtain the osculant conic, namely:

$$x_i = (\alpha_i \tau)^3 (\alpha_i t)^2, \tag{16}$$

and it is readily seen that:

The Hessian $C_{2,6}$ formed for τ is the line equation of the osculant conic.

§ 6. *Other Covariant Curves.*

Taking up the curves given by the fundamental covariants of $C_{1,5}$, we shall consider first the system of conics given by the $C_{2,2}$. Every point τ of the base curve determines a conic, and for every line ξ of the plane there are two definite points t_1 and t_2. Hence there are four lines associated with the same two points, namely, the common lines of the conics determined by t_1 and

t_2, respectively. If we ask that the tangent at t_1 be a line of its conic, it is readily seen that t_1 must be a point of inflection. For let the parameter of t_1 be zero and the tangent be the line $x_0=0$, then the conic for $t=0$ is

$$(b\xi)(f\xi)-4(c\xi)(e\xi)+3(d\xi)^2=0, \tag{17}$$

and if $x_0=0$ is a line of (17), then d_0 must be zero, whence $x_0=0$ is a stationary line, and the roots of $C_{2,2}=0$ coincide, showing, as we have already seen, that I_4, its discriminant, contains the stationary lines. From a point τ there will be two lines to its conic, and the locus of these lines as τ varies is a curve of class 12 given by the eliminant of $C_{1,5}$ and $C^2_{2,2}$.*

If ρ^5 has an undulation, all the conics of the system touch the undulation tangent and this becomes a double line of the I_4. Hence for three undulations the linear system of conics given by the $C_{2,2}$ is the net of conics on the three undulation tangents and I_4 becomes rational.

The six cyclic lines α are the sections of ρ^5 for which the canonizant vanishes, and it has already been shown that the $C_{3,3}$ gives the linear system of cubic curves on these lines.† Now each line α has a unique apolar quadratic, and if t_1 and t_2 be the roots of such a quadratic, the $C_{3,3}$ formed for t_1 and t_2, respectively, will each have the given line α for a double line.

The $C_{3,5}$ is the Jacobian of $C_{2,2}$ and $C_{1,5}$, and for a given τ is a cubic curve. The lines from τ to this curve are the two lines to the $C_{2,2}$ and the tangent at τ. In the same way $C_{3,9}$ is another cubic curve, being the Jacobian of $C_{1,5}$ and $C_{2,6}$. The tangent at τ is a double line of this cubic with one of the contacts there. The osculant conic $C_{2,6}$ likewise has contact with ρ^5 at that point.

The pencil of lines through any point τ of ρ^5 cuts out a pencil of binary quartics, and two members of this pencil are self-apolar quartics. Now the $C_{4,4}$ which gives the quartics on the stationary lines, when formed for a given point, has contact with ρ^5 at that point and contains the two lines through it cutting out self-apolar quartics. Similarly the $C_{4,6}$ determines a quartic curve containing the lines to the corresponding $C_{3,3}$ and having contact with ρ^5 at that point. These quartics contain the lines α.

The quintic curve given by the $C_{5,1}$ and formed for a point τ will contain the lines to the $C_{3,3}$ and also the two self-apolar lines through τ; while the quintic given by $C_{5,3}$ contains the lines to the $C_{3,3}$ and the $C_{2,2}$, since it is the Jacobian of these two forms. Now the $C_{5,1}$ and the $C_{5,3}$ both vanish identically when $C_{1,5}$ contains a triple root or when $C_{1,5}$ is a cyclic quintic. Therefore:

* Salmon, "Higher Algebra," 4th ed., p. 260.

† Conner, Johns Hopkins University Circular, February, 1911, p. 67.

The coefficients of $C_{5,1}$ and $C_{5,3}$ give the linear system of six independent quintic curves on the fifteen lines α and ε.

It is obvious that the sextic curve $C_{6,4}$ contains the five lines to the $C_{5,1}$ and the tangent at τ, since it is formed by operating with $C_{5,1}$ on $C_{1,5}$. In general when any covariant is thus formed from two other known covariants, the lines from any point τ to the particular covariant curve it determines are likewise known.

Let us now take one of the cyclic lines for a reference line, say

$$x_0 = a_0 t^5 + f_0, \tag{18}$$

and form the $C_{3,3}$ and $C_{5,1}$ for any point τ; the terms in ξ_0^3 and ξ_0^5, respectively, in these equations do not appear, since both contain the reference line. Now by taking the linear polar of line (18) with reference to the $C_{3,3}$ we obtain the equation of the point where the curve touches the given line.* Since there is no term in ξ_0^3, to obtain the linear polar we need only consider the term in ξ_0^2, which is

$$a_0 f_0 \tau \{ (c_1\tau + d_1)\xi_1 + (c_2\tau + d_2)\xi_2 \} \xi_0^2.$$

Therefore the point of contact of $C_{3,3}$ with (18) is given by

$$(c_1\tau + d_1)\xi_1 + (c_2\tau + d_2)\xi_2 = 0. \tag{19}$$

In the same way the linear polar of (18) with reference to $C_{5,1}$ is obtained. The highest term in ξ_0 being $a_0^2 f_0^2 \{ (c_1\tau + d_1)\xi_1 + (c_2\tau + d_2)\xi_2 \} \xi_0^4$, we see that the required polar is given by equation (19). Hence it follows that:

For a given τ the $C_{3,3}$ and the $C_{5,1}$ have the same contact on lines α.

These lines then represent twelve common lines of the two curves, the remaining three being the lines from τ to its $C_{3,3}$, as we have seen that these are lines of the $C_{5,1}$. By the same method it is readily seen that for a given point:

The $C_{5,3}$, $C_{5,7}$, $C_{7,1}$ and $C_{7,5}$ have the same contact on lines α; also the $C_{7,1}$ and $C_{8,2}$ have the same contact on lines ε.

§ 7. *The Invariants.*

Since all the invariants vanish identically when $C_{1,5}$ has a triple root, it follows that all the invariant curves contain the lines ε. The quartic curve I_4 contains also lines β and γ as I_4 vanishes for the canonical form of the binary quintic cut out by any of these lines. Now the line equation of ρ^5 is given by the discriminant of $C_{1,5}$, namely, $I_4^2 - 128 I_8$. Therefore lines γ and ε represent all the lines common to I_4 and ρ^5.

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* Dually, the linear polar of any point on a given curve is the tangent to the curve at that point.

The lines ε are double lines of the curve I_8. This is easily seen by taking two stationary lines and the line joining the corresponding points of inflection for a triangle of reference. Calling the stationary lines $x_0=0$ and $x_2=0$ and the parameters of the points of inflection 0 and ∞, respectively, any line section will be given by

$$(a\xi)(\alpha t)^5 \equiv a_0\xi_0 t^5+5(b_0\xi_0+b_1\xi_1)t^4+10(c_0\xi_0+c_1\xi_1)t^3 +10(d_1\xi_1+d_2\xi_2)t^2+5(e_1\xi_1+e_2\xi_2)t+f_2\xi_2. \tag{20}$$

On forming the I_8 of this we observe that the two highest powers of ξ_0 and ξ_2 do not appear, showing that the stationary lines are double lines. In a similar manner lines γ are seen to be lines of I_8, for if

$$x_0=a_0t^5+10d_0t^2 \tag{21}$$

be a reference line, from the explicit form of I_8* the highest term in ξ_0 is $-27a_0^2d_0^5(f_1\xi_1+f_2\xi_2)\xi_0^7$, and taking the linear polar of (21) with reference to the curve the equation of the point of contact is

$$f_1\xi_1+f_2\xi_2=0. \tag{22}$$

Now taking the line equation of ρ^5 † we see that its contact on (21) is likewise given by (22). Therefore:

The curves I_8 and ρ^5 have the same contact on lines γ.

The invariant I_{12}‡ is the discriminant of the $C_{3,3}$ and represents a curve of class 12, α being four-fold lines and ε double lines. If we examine the relation of I_{12} to the lines ε, by means of (20), we find that I_{12} has contact with ρ^5 at its points of inflection. The nature of these contacts may be determined by considering the conjugate curve, which will be designated by $\bar{\rho}^5$. This curve has its line sections apolar to the line sections of ρ^5. In general every line section of ρ^5 will determine a $C_{1,5}$ which has a unique apolar cubic, its canonizant, and this cubic will be represented by three points on a line of $\bar{\rho}^5$. Now since I_{12} is the locus of line sections of ρ^5 for which the canonizant has a double root, it follows that to every line of I_{12} there corresponds a line of $\bar{\rho}^5$. On the other hand, since the point of contact of a line of $\bar{\rho}^5$ taken with any one of the three extra intersections will represent the canonizant which gives a line of I_{12}, we see that to every line of $\bar{\rho}^5$ there correspond three lines of I_{12}. Hence:

The curve I_{12} is in a one-to-three correspondence with the line equation of the conjugate curve.

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* Salmon, "Higher Algebra," p. 229.

† Salmon, "Higher Algebra," p. 230.

‡ The explicit form is given in Cayley, Vol. II, p. 294. All the forms of $C_{1,5}$ are given in full in this volume.

Now when $C_{1,5}$ has a triple root, its canonizant has the same triple root, and to a stationary line of ρ^5 must correspond a stationary line of I_{12}. This curve then has points of inflection which coincide with the points of inflection of ρ^5, and lines ε each represent six common lines of the two curves.

I_{12} also contains the lines β, and it is readily seen that I_4 and I_{12} have the same contact on these lines. Hence the lines β and ε are all the lines common to these two curves.

The skew invariant I_{18} gives a curve of class 18 containing all the sets of lines mentioned previously except γ. From the equation of this curve one observes that lines α are five-fold lines and δ and ε are triple lines. However, the binary equation giving the lines from a point of inflection of ρ^5 to I_{18} shows that from this point the stationary line represents five lines to the I_{18}. This indicates that the curve has a point of inflection and an ordinary contact at every point of inflection of the ρ^5, whence all the common lines of I_{12} and I_{18} are given by lines α, β and ε.

The lines α, δ and ε are the base lines of the pencil of quintic curves given by the $C_{5,1}$, and, for any member of the pencil, these base lines are equivalent to 87 common lines with the I_{18}. There are then three other common lines and it is obvious that they are given by the lines from the point τ, which determines the $C_{5,1}$, to the $C_{3,3}$. Or, since I_{18} is the eliminant of $C_{5,1}$ and $C_{3,3}$, we can define this curve to be the locus of lines from a varying point τ to its $C_{3,3}$.

§ 8. *The Pencil* $I_4^2+\lambda I_8$.

It has been stated that I_4 contains lines γ and ε, and since I_8 has ε for double lines and has contact with ρ^5 on γ it is evident that $I_4^2+\lambda I_8$ forms a pencil of octavics with ε as double lines and having simple contact on γ. These two sets of lines are equivalent to 64 common lines and are the base lines of the pencil. The base curve $I_4^2-128I_8$ is a member of this pencil, and of other members the I_4 and I_8 have been mentioned. It may be added that the lines α are double lines of I_8, and this curve may be defined as the locus of line sections of ρ^5 for which the $C_{2,2}$ and $C_{6,2}$ are harmonic pairs.

Another member of this pencil is of especial interest. We found that from a point τ there are two lines whose intersections with ρ^5 are self-apolar quartics, and the locus of these two lines for a varying τ is the curve $I_4^2-3I_8$.* Lines δ contain a binary cubic and its Hessian, and it is obvious that when τ is either of the Hessian points the remaining four are self-apolar, hence the lines δ are double lines of $I_4^2-3I_8$. These with lines ε make nineteen double

* Salmon, "Higher Algebra," p. 259.

lines of this curve; it is therefore of genus 2 and has a pencil of adjoint curves of class 5.* Since these lines are part of the base lines of the pencil $C_{5,1}$, it follows that:

The pencil of adjoint quintics of the invariant curve $I_4^2—3I_8$ is given by the covariant $C_{5,1}$.

For a given τ the two extra lines, which are a set of the involution, are the two lines through τ containing self-apolar quartics, since if four roots of $C_{1,5}$ are self-apolar the remaining root is the covariant $C_{5,1}$. Hence the pairs of lines in the involution intersect on the base curve and are the lines by whose locus the $I_4^2—3I_8$ was defined.

The following table is given to show which covariants contain any of the special line sections and their multiplicity.

Lines	$C_{3,3}$	I_4	$C_{4,6}$	$C_{5,1}$	$C_{5,3}$	$C_{5,7}$	$C_{6,2}$	$C_{6,4}$	$C_{7,1}$	$C_{7,5}$	I_8	ρ^5	$C_{8,2}$	$C_{9,3}$	$C_{11,1}$	I_{12}	$C_{13,1}$	I_{18}
α	1	..	1	1	1	1	2	1	1	1	2	..	2	3	3	4	2	5
β	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..	1
γ	..	1	..	..	..	..	..	..	..	..	1	1	..	..	..	..	..	..
δ	..	..	..	1	..	..	..	..	1	1	..	..	1	..	1	..	2	3
ε	..	1	..	1	1	..	1	1	1	1	2	2	1	1	2	2	2	3
ζ	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..	..	1
η	..	..	..	..	..	..	..	..	..	..	..	2	..	..	..	..	..	1

§ 9. *The Discriminant of the $C_{4,4}$.*

If $C_{1,5}$ has a double root, this is also a double root of its $C_{4,4}$. Interpreted geometrically this means that for a given point τ of ρ^5 the $C_{4,4}$ will have contact there, and the discriminant of $C_{1,5}$ must be a factor of the discriminant of the $C_{4,4}$. Again, if four points of a binary quintic are self-apolar, the remaining point is a double root of the $C_{4,4}$. This is readily seen by making the one point $t=0$; then in the $C_{1,5}$ we have $f=0$, and the last two terms of the $C_{4,4}$ become

$$-(2det+e^2)(6ae-15bd+10c^2). \tag{23}$$

Now the second factor is the condition that the four remaining points of $C_{1,5}$ be self-apolar, and hence $t=0$ is a double root of the $C_{4,4}$. We have seen that

* Dually a curve of order n and genus 2 has a pencil of adjoint curves of order $(n—3)$ on the $\frac{1}{2}(n—1)(n—2)—2$ double points. These have two extra intersections each, and set up an involution $I_{1,1}^2$ on the base curve.

$I_4^2-3I_8$ gives the locus of lines whose intersections with ρ^5 contain self-apolar quartics, and this invariant must also be a factor of the discriminant of the $C_{4,4}$.

The discriminant is of degree 24 in the original coefficients and is found from the two invariants of the $C_{4,4}$. These are:

$$\left.\begin{aligned} 12g_2 &\equiv I_4^2+72I_8, \\ 216g_3 &\equiv I_4(I_4^2-378I_8). \end{aligned}\right\} \tag{24}$$

These invariants have well-known meanings, and the discriminant of which two factors have been found is

$$g_2^3-27g_3^2\equiv I_8(I_4^2-3I_8)(I_4^2-128I_8). \tag{25}$$

If we consider the $C_{1,5}$ as given by a line of γ, the $C_{4,4}$ will have a triple root at the point of contact, and it is obvious that the quartic curve formed for that point will have a point of inflection there, the line γ being a stationary line.

§ 10. *The Hermitian of the $C_{4,4}$.*

Some further relations in connection with the $C_{4,4}$ are found by means of the Hermitian of the system of curves. In general, if we ask for the locus of lines which intersect a linear system of $(n+1)$ curves of order n in sets of n points in an involution, we find the locus to be a curve of class

$$\frac{n(n+1)}{2},$$

which I will call the Hermitian of the system of n-ics.

If the curves all have a point in common, that point becomes a part of the locus, and the class of the remaining part is reduced by one. The same is true for every point which the system may have in common.

Now consider the dual statement, and take for illustration a net of conics. The Hermitian is a curve of order 3 and, in this case, is the locus of the degenerate members of the net. From any point of this locus the tangents to the conics are in an involution.* If the conics are given symbolically by $(a\xi)^2$, $(b\xi)^2$ and $(c\xi)^2$, respectively, the required locus is

$$|abx|\,|bcx|\,|cax|=0. \tag{26}$$

In general, for an n-fold infinity of n-ics given by

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* Salmon, "Conic Sections," p. 360.

$$(a_0\xi)^n,\ (a_1\xi)^n,\ \ldots,\ (a_n\xi)^n$$

the Hermitian is

$$\prod^{\frac{n(n+1)}{2}} |a_0a_1x|\,|a_0a_2x|\ldots.|a_{n-1}a_nx|=0, \tag{27}$$

and this involves the coefficients of each of the n-ics linearly. If the three conics have a line in common, they will be seen in an involution from any point of the line; and if the conics are a net on three lines, the Hermitian becomes the product of the three lines.

Similarly, for a triple infinity of cubic class curves the locus of points from which these appear in an involution is a curve of order 6. If the cubics touch six lines, as, for example, the cubics on lines α given by the $C_{3,3}$, then the Hermitian is the product of those lines. If the cubics have five lines in common, the given curve consists of these five lines and another line, the locus of a point which, with the conic on the five lines, is a cubic of the system.

In a system of ∞^4 quartics the required locus is a curve of order 10. Let the quartics under consideration be given by the $C_{4,4}$ of a general line section of ρ^5. This, being a system of curves on lines ε, must have these nine lines as part of the Hermitian. The remaining part must be a line such that any point of it, together with the unique cubic on the nine lines, is a degenerate quartic of the system. The required line can be identified as follows:

The $C_{4,4}$, being of degree 4 in the coefficients of ρ^5, and in the ξ's, and of order 4 in the t's, may be represented by $\{A^4, \xi^4, t^4\}$. The Hermitian is of order 10 and is given by

$$\prod^{10} |abx|\,|acx|\ldots.|dex|\equiv\{A^{20}, x^{10}\}, \tag{28}$$

since it involves the coefficients of the five quartics linearly, and each of these is of degree 4 in the A's. Hence equation (28) must contain the lines ε as a fac tor, and the form of the remaining factor is easily found. The ρ^5 in point form is $\{A, \xi, t^5\}$ and in line form is $\{A^2, x, t^8\}$, while the equation giving the points of inflection is represented by $\{A^3, t^9\}$. Now the eliminant of $\{A^2, x, t^8\}$ and $\{A^3, t^9\}$, which is $\{A^{42}, x^9\}$, gives the product of the stationary lines and an additional factor, the cusp-invariant; for if x is a point on a stationary line, t will be a common root of these two forms, while if ρ^5 has a cusp, the cusp-parameter will factor out for any point x of the plane. The cusp-invariant, then, which is of degree 24 in the A's, must be a factor of the eliminant, the remaining factor being the stationary lines, or:

$$\{A^{42}, x^9\}\equiv\{A^{24}\}\cdot\{A^{18}, x^9\}. \tag{29}$$

The same result may be obtained by taking the discriminant of $\{A^2, x, t^8\}$,

which gives the equation of ρ^5 together with the product of the stationary lines. The discriminant is $\{A^{28}, x^{14}\}$, and since the equation of ρ^5 is $\{A^{10}, x^5\}$, the remaining factor is the same as found above.

Since the stationary lines are a part of the Hermitian of the system of quartics given by the $C_{4,4}$, we have

$$\{A^{20}, x^{10}\} \equiv \{A^{18}, x^9\} \cdot \{A^2, x\}. \tag{30}$$

The remaining part of the locus must be a covariant line, and there is only one covariant line of degree 2 in the coefficients; namely, the line whose intersection with ρ^5 is a binary quintic apolar to all line sections. Its equation is

$$|abx|\,|\alpha\beta| \equiv |afx| - 5\,|bex| + 10\,|cdx| = 0. \tag{31}$$

Therefore from any point of this line the system of quartics given by the $C_{4,4}$ appears in an involution.

Then there will be a set of four lines apolar to all sets of four lines in the involution, and any point of the line (31) with the cubic on the nine stationary lines is a quartic of the system given by the $C_{4,4}$.

§ 11. *A Quartic Curve for Every Point in the Plane.*

Following is a method by which a curve of the system given by the $C_{4,4}$ is associated with every point of the plane. Take any two lines ξ and η and form the $C_{4,4}$ for the pencil $\xi + \lambda\eta$; this gives a quartic in λ, each coefficient of which involves t to the fourth degree. Suppose ξ and η cut out the two binary quintics $(\alpha t)^5$ and $(\beta t)^5$, respectively. These quintics have a combinant quartic which we can write $|abx|\,|\alpha\beta|^3(\alpha t)^2\,(\beta t)^2$, x being the point of intersection of the two lines. It is seen at once that:

If x is a point of line (31), *then each coefficient of the quartic in λ is apolar to the given combinant.*

This is most easily verified by taking for a reference triangle two stationary lines and line (31). Then if x is a point of (31) the $C_{4,4}$ of $(\alpha t)^5 + \lambda(\beta t)^5$, considered as a quartic in λ, has each coefficient apolar to $|abx|\,|\alpha\beta|^3(\alpha t)^2(\beta t)^2$. Now if we operate with this combinant formed for a point x of (31) on the $C_{4,4}$ of the general line section $(a\xi)\,(\alpha t)^5$, the result is a quartic in the ξ's from which the point x factors, leaving the cubic on the stationary lines.

The combinant above is a function of $\{A^2, x, t^4\}$, while the $C_{4,4}$ of a line section has the form $\{A^4, \xi^4, t^4\}$. Now the apolarity relation of these is $\{A^6, \xi^4, x\}$, but we have seen that this form vanishes when the line ξ and point x are incident, provided x is a point of (31). In other words $\{A^6, \xi^4, x\} = 0$ when both $(\xi x) = 0$ and $\{A^2, x\} = 0$. Hence there exists the relation

$$\{A^6, \xi^4, x\} = \{A^6, \xi^3\}(\xi x) + \{A^4, \xi^4\}\{A^2, x\}, \tag{32}$$

and this indicates that the cubic on the stationary lines is of degree 6 in the coefficients; also $\{A^4, \xi^4\}$ must be the I^4, since it is the only invariant of degree 4 in the coefficients.

The I_4 and the cubic $\{A^6, \xi^3\}$ each contain the stationary lines and hence have three other common lines giving rise to an invariant triangle of ρ^5. The degree of this in the coefficients is calculated by taking the eliminant of $\{A^4, \xi^4\}$, $\{A^6, \xi^3\}$, and (ξx). The result on eliminating ξ is $\{A^{36}, x^{12}\}$, and on removing the factor which is the product of the stationary lines there remains the invariant triangle which is $\{A^{18}, x^3\}$.

From the relation (32) found above there is a double infinity of quartic curves picked out of the system given by the $C_{4,4}$. Take any point x of the plane and form the combinant quartic of the pencil of line sections on x. Then operating with this combinant on the $C_{4,4}$, the result is a quartic curve on the stationary lines. The pencil of this quartic and the I_4 contains the cubic on the stationary lines and the point x as a degenerate member. There is then a quartic curve of the system given by the $C_{4,4}$, associated uniquely with every point of the plane, and the pencil of quartics formed by this curve and the I_4 will have for base lines the stationary lines, the invariant triangle mentioned above, and the four lines from the given point to the I_4.

§ 12. *The Cubic Curve on the Stationary Lines of* ρ^5.

There are three sets of covariant points on the ρ^5 of degree 3 in the coefficients and of orders 9, 5, and 3 in the variable.* These we shall write in the forms $\{A^3, t^9\}$, $\{A^3, t^5\}$, and $\{A^3, t^3\}$, respectively. Now the $C_{1,5}$ has three fundamental forms of the third degree in the coefficients, namely, $C_{3,9}$, $C_{3,5}$, and $C_{3,3}$, which for a line section of ρ^5 are in the forms $\{A^3, \xi^3, t^9\}$, $\{A^3, \xi^3, t^5\}$, and $\{A^3, \xi^3, t^3\}$, respectively. On writing the apolarity relation of each pair of quantics having the same order in the variable t, the result in each case gives a cubic curve of degree 6 in the A's. Since the cubic on the stationary lines is in the form $\{A^6, \xi^3\}$, we shall determine whether it belongs to the system of these cubic curves.

The roots of the covariant form $\{A^3, t^9\}$ equated to zero give the parameters of the nine points of inflection, the symbolical expression for this covariant being

$$|abc|\,|\alpha\beta|\,|\beta\gamma|\,|\gamma\alpha|\,(\alpha t)^3\,(\beta t)^3\,(\gamma t)^3; \tag{33}$$

*See Coble, "Symmetric Binary Forms and Involutions," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, p. 359.

and when the ρ^5 is given by equation (13), the developed form is

$$\begin{aligned}|abc|t^9&+3|abd|t^8+3\{2|acd|+|abe|\}t^7\\ &+\{8|ace|+10|bcd|+|abf|\}t^6\\ &+3\{2|ade|+5|bce|+|acf|\}t^5\\ &+3\{2|bcf|+5|bde|+|adf|\}t^4\\ &+\{8|bdf|+10|cde|+|aef|\}t^3\\ &+3\{2|cdf|+|bef|\}t^2+3|cef|t+|def|=0.\end{aligned} \tag{34}$$

The covariant of order 5, $\{A^3, t^5\}$, gives the parameters of the five points cut out by the covariant line $|abx||\alpha\beta|^5$; the symbolical expression is

$$|abc||\alpha\beta|^3|\beta\gamma||\gamma\alpha|(\alpha t)(\beta t)(\gamma t)^3, \tag{35}$$

and the expanded form

$$\begin{aligned}\{|abe|&-2|acd|\}t^5+\{|abf|-10|bcd|\}t^4+2\{|acf|-5|bce|\}t^3\\ &+2\{|adf|-5|bde|\}t^2+\{|aef|-10|cde|\}t+\{|bef|-2|cdf|\}=0.\end{aligned} \tag{36}$$

The remaining covariant form is

$$|abc||\alpha\beta|^3|\beta\gamma|^2|\gamma\alpha|(\alpha t)(\gamma t)^2, \tag{37}$$

of which the expanded form is found to be

$$\begin{aligned}\{2|abf|&-5|ace|+20|bcd|\}t^3+3\{|acf|-5|ade|+5|bce|\}t^2\\ &-3\{|adf|-5|bcf|+5|bde|\}t-\{2|aef|-5|bdf|+20|cde|\}=0.\end{aligned} \tag{38}$$

Now take the reference triangle so that any line section $C_{1,5}$ is given by equation (20) and form the $C_{3,9}$, $C_{3,5}$, and $C_{3,3}$, respectively, of this equation. Let the apolarity relation of (34) and $C_{3,9}$ be denoted by C_1, that of (36) and $C_{3,5}$ by C_2, and of (38) and $C_{3,3}$ by C_3. These are the three cubic curves each involving the original coefficients to the sixth degree; and since $x_0=0$ is a stationary line, it is only necessary to find a covariant curve which does not contain the term in ξ_0^3, as such a curve will contain all the stationary lines. We observe then if there is a member of the system

$$\lambda C_1+\mu C_2+\nu C_3=0 \tag{39}$$

which is free from the term in ξ_0^3. Obtaining C_1, C_2, and C_3, it is easily seen that the required curve is a linear combination of these three curves, the numerical coefficients being readily determined, and hence:

$$C\equiv 84C_1-35C_2-80C_3=0 \tag{40}$$

is the cubic line curve on the stationary lines of the rational quintic.

The Group of Isomorphisms of an Abelian Group and Some of its Abelian Subgroups.

By G. A. Miller.

§ 1. *Introduction.*

Let G represent any abelian group, while I represents its group of isomorphisms. It is known that a necessary and sufficient condition that I be abelian is that G is cyclic. Moreover, the invariant operators of I are composed of those which transform every operator of G into the same power of itself, and hence the order of the central of I is $\phi(m)$, m being the largest order of an operator contained in G.* In the present paper we aim to determine a few new properties of I, especially as regards its Sylow subgroups. This paper has close contact with an article by the same author entitled "Isomorphisms of a Group whose Order is a Power of a Prime," *Transactions of the American Mathematical Society*, Vol. XII (1911); and a paper by Burnside entitled "On Some Properties of Groups whose Orders are Powers of Primes," *Proceedings of the London Mathematical Society*, Vol. XI (1912).

Let A_0 represent any abelian subgroup of I. All the operators of G which are invariant under one of the operators of A_0 constitute an invariant subgroup under each one of the operators of A_0. If t_1 and t_2 are any two operators of A_0 while s is any operator of G, there result equations of the form:

$$t_1^{-1}st_1=s_1s, \quad t_2^{-1}st_2=s_2s, \quad t_1^{-1}s_2t_1=s_1^1s_2,$$

where s_1, s_2 and s_1^1 are also operators of G. Since $t_1t_2=t_2t_1$, there result the following equations:

$$t_2^{-1}t_1^{-1}st_1t_2=t_2^{-1}s_1t_2s_2s=t_1^{-1}t_2^{-1}st_2t_1=s_1^1s_1s_2s.$$

As $t_2^{-1}s_1t_2=s_1^1s_1$, we have the theorem: *If any two commutative operators t_1 and t_2 of the group of isomorphisms of an abelian group G transform a given*

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* *Transactions of the American Mathematical Society*, Vol. II (1901), p. 260; cf. Ranum, *ibid.*, Vol. VIII (1907), p. 84.

operator of G into itself multiplied by s_1 and s_2 respectively, then the commutator of t_1 and s_2 is equal to the commutator of t_2 and s_1. On the other hand, it is easy to see that t_1 and t_2 must be commutative whenever these commutators are equal, so that the given condition is a sufficient as well as a necessary condition that t_1 and t_2 be commutative, provided s may represent any operator of G.

As a special case of the theorem of the preceding paragraph, it may be observed that every subgroup of I which is composed of operators transforming all the operators of G into themselves multiplied by operators which are invariant under all the operators of this subgroup is necessarily abelian, but an abelian subgroup of I is not always composed of such operators. The commutators of G whose elements are composed of a particular operator t of I and of all the operators of G, taken successively, constitute a subgroup T of G which may be associated with t.* In this way every operator of I may be associated with a particular subgroup of G. The identity of I is the only operator in I which corresponds to the identity of G, but the subgroups of G which correspond to other operators of I are not necessarily distinct when these operators are distinct. On the other hand, two operators of I are clearly distinct whenever their associated, or corresponding, subgroups are distinct.

The subgroup of G which is associated with t^{α} is clearly contained in T for every value of α. The subgroups of G which correspond to the operators of any cyclic subgroup of I must therefore all be contained in each of the subgroups which correspond to the generators of this cyclic subgroup of I. In particular, I involves at least two operators which correspond to the same subgroup of G whenever I involves an operator whose order exceeds 2. If G is the cyclic group of order 12, it is evident that any two distinct operators of I correspond to two distinct subgroups of G; but if G is an abelian group which is not contained in this cyclic group, then the I of G cannot have the property that every pair of its distinct operators corresponds to a pair of distinct subgroups.

When T is composed of operators which are invariant under t, the order of t is the same as the largest order of an operator of T, and the subgroup of G which corresponds to t^{α} is composed of the α-th power of the operators of T. Since the group of isomorphisms of any abelian group is the direct product of the group of isomorphisms of its Sylow subgroups, we may confine ourselves to a study of the case when the order of G is a power of a prime number.

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* *Bulletin of the American Mathematical Society*, Vol. VI (1900), p. 337.

§ 2. *Order of G is p^m, p being any Prime Number.*

We shall first determine the order of a Sylow subgroup of order $p^{m'}$ in the group of isomorphisms of any abelian group of order p^m, p being any prime number. Suppose that the independent generators of G are of orders p^{α_1}, p^{β_1}, ..., p^{λ_1} ($\alpha_1 > \beta_1 > \ldots\ldots > \lambda_1$), and that the number of the independent generators of these orders is $\alpha, \beta, \ldots\ldots, \lambda$ respectively. Hence

$$m = \alpha\alpha_1 + \beta\beta_1 + \ldots\ldots + \lambda\lambda_1.$$

It will be convenient to use the following abbreviations:

$$\begin{aligned} m = m_\alpha &= \alpha\alpha_1 + \beta\beta_1 + \ldots\ldots + \lambda\lambda_1, \\ m_\beta &= (\alpha+\beta)\beta_1 + \ldots\ldots + \lambda\lambda_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ m_\lambda &= (\alpha+\beta+\ldots\ldots+\lambda)\lambda_1. \end{aligned}$$

The orders of the groups generated by all the operators of G whose orders divide p^{α_1}, p^{β_1}, ..., p^{λ_1} are evidently p^{m_α}, p^{m_β}, ..., p^{m_λ} respectively.

To determine the value of m' we observe that G has a series of invariant subgroups of orders $p, p^2, \ldots\ldots, p^{m-1}$ under the given Sylow subgroup of order $p^{m'}$ in its group of isomorphisms. If we represent this series of invariant subgroups as follows:

$$H_1, H_2, \ldots\ldots, H_{m-1},$$

it is clear that H_1 is any one of the subgroups of order p generated by an independent generator of highest order. In fact, $H_1, H_2, \ldots\ldots, H_\alpha$ are generated respectively by $1, 2, \ldots\ldots, \alpha$ such subgroups. The subgroup $H_{\alpha+1}$ is generated by H_α and the subgroup of order p generated by an arbitary independent generator of order p^{β_1}, while $H_{\alpha+\beta}$ is the subgroup generated by the operators of order p in the subgroup of G generated by its independent generators of orders p^{α_1} and p^{β_1}. In general, $H_1, H_2, \ldots\ldots, H_{m-1}$ is a series of subgroups such that each is included in all those which follow it, but a characteristic subgroup of G is not always in this series. A subgroup in the given series which involves operators of order p^k must succeed every subgroup in this series which does not involve any operators of this order.

By means of the given notation it is easy to obtain the following formula:

$$\begin{aligned} m' &= m_\alpha - 1 + m_\alpha - 2 + \ldots\ldots + m_\alpha - \alpha + m_\beta - 1 + m_\beta - 2 + \ldots\ldots + m_\beta - \beta \\ &\qquad + m_\gamma - 1 + \ldots\ldots + m_\gamma - \gamma + \ldots\ldots + m_\lambda - 1 + \ldots\ldots + m_\lambda - \lambda \\ &= \alpha m_\alpha - \frac{\alpha(\alpha+1)}{2} + \beta m_\beta - \frac{\beta(\beta+1)}{2} + \ldots\ldots + \lambda m_\lambda - \frac{\lambda(\lambda+1)}{2} \\ &= \alpha^2\alpha_1 + (2\alpha\beta+\beta^2)\beta_1 + \ldots\ldots + (2\alpha\lambda + 2\beta\lambda + \ldots\ldots + \lambda^2)\lambda_1 \\ &\qquad - \left(\frac{\alpha(\alpha+1)}{2} + \frac{\beta(\beta+1)}{2} + \ldots\ldots + \frac{\lambda(\lambda+1)}{2}\right). \end{aligned}$$

This result may be expressed as follows: *If an abelian group of order p^m is generated by α independent generators of order p^{α_1}, β of order p^{β_1}, ..., λ of order p^{λ_1} ($\alpha_1 > \beta_1 > \ldots. > \lambda_1$), the order of a Sylow subgroup of its group of isomorphisms is $p^{m'}$, where*

$$m' = \alpha^2\alpha_1 + (2\alpha\beta + \beta^2)\beta_1 + \ldots. + (2\alpha\lambda + 2\beta\lambda + \ldots. + \lambda^2)\lambda_1 \\ - \left(\frac{\alpha(\alpha+1)}{2} + \frac{\beta(\beta+1)}{2} + \ldots. + \frac{\lambda(\lambda+1)}{2}\right).$$

Let $P_{m'}$ represent this Sylow subgroup of order $p^{m'}$. It is clear that $P_{m'}$ can always be represented as a transitive substitution group of degree p^{m-1}, since one of the largest independent generators s of G is transformed into itself multiplied by each of the operators of a subgroup of order p^{m-1} under $P_{m'}$, and G is generated by this subgroup and s. The regular subgroup R of $P_{m'}$, when $P_{m'}$ is represented as such a substitution group, which is formed by all the substitutions of $P_{m'}$ which are commutative with each of the independent generators of G except s, is of especial interest.

Let r_1 and r_2 be any two substitutions of R. Since all the operators of G may be written in the form ts^a, where t is an operator in the group generated by all the independent generators of G with the exception of s, it results that

$$r_1^{-1}sr_1 = t_1s^{\alpha-1}s, \quad r_2^{-1}sr_2 = t_2s^{\beta-1}s,$$

where t_1 and t_2 are commutative with both r_1 and r_2, and both $\alpha-1$ and $\beta-1$ are divisible by p. From the equations

$$r_1^{-1}s^{1-\beta}r_1s^{\beta-1} = t_1^{1-\beta}s^{\alpha-\alpha\beta+\beta-1}, \quad r_2^{-1}s^{1-\alpha}r_2s^{\alpha-1} = t_2^{1-\alpha}s^{\alpha-\alpha\beta+\beta-1}$$

it results that the abelian subgroup of R generated by those substitutions for which $\alpha=\beta=1$ is a maximal abelian subgroup of R whenever G has more than one largest invariant. That is, it is not contained in a larger abelian subgroup of R whenever G contains more than one independent generator whose order is equal to the order of s.

If G contains only one independent generator of highest order and if the quotient obtained by dividing the order of s by the order of an independent generator of next to the highest order is p^γ, then r_1 and r_2 are commutative whenever α and β are such that both $\alpha-1$ and $\beta-1$ are divisible by the order of a generator of next to the highest order. The order of a maximal abelian subgroup of R in this case is therefore p^γ times the order of the subgroup formed by all the substitutions of R for which $\alpha=\beta=1$, provided G is non-cyclic. This completes a proof of the following theorem: *A necessary and sufficient condition that the subgroup R of order p^{m-1} be abelian is that G involves not more than one independent generator whose order exceeds p.*

As the subgroup R is invariant under $P_{m'}$, it results that $P_{m'}$ is contained in the holomorph of R. When R is abelian its invariants are the same as the invariants of G, with the exception that the largest invariant of G must be divided by p to obtain the corresponding invariant of R. In this case R is clearly a maximal abelian subgroup of $P_{m'}$, since $P_{m'}$ is always contained in the holomorph of R. A necessary and sufficient condition that $P_{m'}$ be a Sylow subgroup of the holomorph of R, when R is abelian, is that all the invariants of R are equal to p. As an illustrative example we may cite the fact that the group of degree 8 and order 1344 is the holomorph of R when G is the abelian group of order 16 and of type $(1, 1, 1, 1)$. In this case $P_{m'}$ is a Sylow subgroup of order 64 contained in the given group of order 1344.

Another important invariant subgroup of $P_{m'}$ is composed of all the operators of I which transform each operator of G into itself multiplied by an operator of its subgroup of order p which is invariant under $P_{m'}$. In the given representation of $P_{m'}$ this subgroup must clearly have p^{m-2} transitive constituents of degree p, and its order is p^{δ}, δ being the number of invariants of G if at least one of these invariants exceeds p. If all these invariants are equal to p, then δ is one less than the number of invariants of G; that is, the order of the given invariant subgroup is p^{m-1} in this case. This result is a direct consequence of the important theorem that every abelian group has exactly as many subgroups of index p as it has subgroups of order p. A necessary and sufficient condition that the given invariant subgroup be a maximal abelian subgroup under $P_{m'}$ is that G involves no invariant that is divisible by p^3 and no more than one that is divisible by p^2.

As a very special case of what precedes we have the theorem: *A necessary and sufficient condition that a Sylow subgroup of order $p^{m'}$ of the group of isomorphisms of an abelian group of order p^m, $m>2$, be abelian, is that this group of order p^m is cyclic.* When no two invariants of G are equal to each other, the given series of invariant subgroups $H_1, H_2, \ldots, H_{m-1}$ is completely determined by G. On the other hand, this series is not completely determined by G whenever G has two equal invariants. As each such series corresponds to a Sylow subgroup in the group of isomorphisms of G, we have the following theorem: *A necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m must contain only one Sylow subgroup of order $p^{m'}$ is that this group of order p^m does not contain two equal invariants.**

The subgroup of G which corresponds to a particular operator of $P_{m'}$ has always an order which divides p^{m-1}. When G is cyclic, this order is evidently

* American Journal of Mathematics, Vol. XXVII (1905), p. 15.

p^{m-1} for some operator of $P_{m'}$. Suppose that G contains two different invariants and that the order of the larger exceeds p^2. It is clear that such a G contains a characteristic subgroup which involves operators of order p^2 without involving all the operators of order p contained in G. Hence there is no operator in a Sylow subgroup of order $p^{m'}$ of the group of isomorphisms of such a G, which corresponds to a subgroup of order p^{m-1} in G. In fact, such a correspondence implies that every two characteristic subgroups of G must have the property that one of them is contained in the other.

*Self-Projective Rational Curves of the Fourth and Fifth Orders.**

By R. M. Winger.

§ 1. *Introduction.*

In the study of a rational quintic, first pointed out by Cayley, through the consideration of special cases I was led to a curve which is invariant under a group of six collineations.† The question naturally arose as to other curves unaltered by the group, and this was later extended to the problem of the discovery of rational curves invariant under other finite collineation groups. Such curves have been called *self-projective.* Many of these have found their way into the literature, but usually quite apart from or incidental to their connection with the groups. And while the general quartics and quintics have been discussed by Ciani ‡ and Snyder § respectively, the rational case has not been systematically treated. Neither is this phase of the question to be ignored as a special case of the work cited above, for the order of imposing the conditions materially affects the results.

As the variety of self-projective curves is rather large, some criterion of classification had to be adopted. The purpose of this paper is to present in canonical form *all of the projectively distinct types of the most general rational curves of the fourth and fifth orders invariant under the different finite collineation groups.*

Of fundamental importance in the study for rational curves is the

Theorem. *To every ternary collineation of the points of the curve there is a corresponding binary collineation of the parameter, and conversely.*

Both of these must leave the curve unaltered if it is to be self-projective. The groups to be considered, then, are those generated by the regular body groups: ||

cyclic	g_n,
dihedral	g_{2n},
tetrahedral	g_{12},
octahedral	g_{24},
icosahedral	g_{60}.

* Read before the American Mathematical Society, January 2, 1913.

† Discussed, § 14.

‡ *Istituto Lombardo Rendiconti,* Series II, Vol. XXXIII, p. 1170.

§ American Journal of Mathematics, Vol. XXX.

|| Klein, "Ikosaeder."

It is no condition on a conic to admit any of these groups.

Again, *every member of the syzygetic pencil of rational cubics*

$$x_0^3 + x_1^3 + \lambda x_0 x_1 x_2 = 0$$

is invariant under a dihedral g_6.

The cubic with a cusp,

$$x_0 = t^3, \quad x_1 = t^2, \quad x_2 = 1,$$

is unchanged by the substitution

$$t = \alpha t' \text{ or } \begin{cases} x_0' = \alpha^3 x_0, \\ x_1' = \alpha^2 x_1, \\ x_2' = x_2. \end{cases}$$

That is, *the cuspidal cubic is invariant under a one-parameter group.*

I. THE RATIONAL QUARTIC.

§ 2. *The Stahl Binary Sextic.*

The first case of considerable interest, then, is the rational quartic, ρ^4, which will not admit a group without restriction.

The labor is greatly facilitated by reason of the existence of a *fundamental sextic* (binary) of which line sections are second polars and upon which the invariant theory of the curve depends.*

Bolza † has tabulated the canonical forms of the non-singular binary sextics with linear transformations into themselves. He exhibits the following types with the corresponding groups:

(1)	$t^6 + \alpha t^4 + \beta t^2 + 1$	cyclic	g_2,
(2)	$t(t^5 + 1)$	cyclic	g_5,
(3)	$t(t^4 + \alpha t^2 + 1)$	dihedral	g_4,
(4)	$t^6 + \alpha t^3 + 1$	dihedral	g_6,
(5)	$t^6 + 1$	dihedral	g_{12},
(6)	$t(t^4 + 1)$	octahedral	g_{24}.

We supplement this with the singular form ‡

(7)	$t^2(t^4 + 1)$	cyclic	g_4.

To obtain the parametric equations of the different classes of those self-projective quartics which have Stahl sextics, we need only select three linearly

* Stahl, *Journal für Math.*, Vol. CIV, p. 302. This is the unique sextic to which the fundamental involution is apolar, referred to hereafter as the *Stahl sextic*.

† AMERICAN JOURNAL OF MATHEMATICS, Vol. X, p. 70.

‡ Other singular forms are not given, for they furnish no new types. See § 8.

independent second polars of these sextics, written homogeneously. Usually the most convenient are

$$\frac{\partial^2}{\partial t_1^2}, \quad \frac{\partial^2}{\partial t_1 \, \partial t_2}, \quad \frac{\partial^2}{\partial t_2^2}.$$

Designate henceforth by S_n the sextic that admits a group of order n and by ρ_n^4 the quartic curve derived from it. Note that we get a proper curve for every case except (5).

We proceed now to the consideration of the others in greater detail. In particular, we shall characterize them by means of the fundamental invariants.* We use Professor Morley's set, indicating Salmon's equivalents and the geometric meaning of their vanishing, thus:

Morley:	*Salmon:*	*Condition for:*
I_2	$-B$	triple point,
I'_2	$A + 12B$	flex conic to touch,
I_4	R	undulation,
I_6	D	cusp,
I_9	M	self-reflexion.

§ 3. *The Cyclic Groups.*

Professor Morley has shown that *the necessary and sufficient condition for a* ρ^4 *to be reflected into itself, i. e., to admit a cyclic* g_2, *is*

$$I_9 = 0.†$$

On this supposition simply certain properties of the curve are readily inferred. Thus the center is the intersection of two double lines. The axis cuts out one double point, the intersection of the other double lines and the contacts of tangents from the center, whose parameters are the double points of the involution. *The six points of inflexion are harmonically perspective from the center*, accounting for the two relations that must exist on them.

Consider next the cyclic g_3

$$t' = \omega t, \qquad \omega^3 = 1.$$

A sextic to admit this group can contain only cube terms. It is therefore of the form

$$t^6 + \alpha t^3 + \beta.$$

* F. Morley, "Projective Geometry Notes" (1910), Chapter 6. J. E. Rowe, *Trans. Amer. Math. Soc.*, Vol. XII, p. 304. Salmon, "Higher Algebra," 4th ed., §§ 220 ff., where the system of two quartics is to be interpreted as the fundamental involution of the curve.

† *Loc. cit.* Instead of *involutory collineation* we use *reflexion*, the abstract projective notion of physical reflexion.

We can choose the unit point so that $\beta = 1$, reducing the sextic to type (4), and we have the theorem: *A ρ^4 that is invariant under a cyclic g_3 admits at the same time a dihedral g_6.* *

If the sextic is unaltered by the cyclic g_4

$$t' = i\,t, \qquad i^4 = 1,$$

it takes the form

$$t\,(t^4 + \alpha).$$

The reference scheme may be so selected that $\alpha = 1$, which gives at once the octahedral sextic S_{24}. The ρ^4 derived therefrom has the same property. Therefore:

If a rational quartic is invariant under a cyclic g_4, it admits also the octrahedral g_{24}; in other words, has three inflexional nodes.

Berzolari† has deduced the properties of this curve from the consideration of the g_{24} of permutations that leaves the double lines unaltered.

Three second polars of the S_5 determine the quartic

$$x_0 = t^4, \qquad x_1 = t, \qquad x_2 = 1,$$
$$x_1^4 - x_0 x_2^3 = 0.$$

This curve is unaffected by the one-parameter group of § 1.‡ Hence:

If a ρ^4 admits a cyclic g_5, it admits an infinite group.

There is an undulation at $t = 0$ and at ∞ the dual singularity, a special triple point whose three parameters have come together, counting for a node and two cusps. All of the point and line singularities are concentrated at these two points.

All of the invariants vanish. §

§ 4. *The Metrical Aspect of Dihedral Collineation Groups.*

We shall now explain a very convenient metrical representation of dihedral collineation groups of any order, *in which all of the conjugate points of a set are in general real*, and examine the quartics invariant under them. We shall speak in metrical terms at our convenience, leaving the projective interpretation to be supplied if desired.

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* The theorems of this section are predicated on the supposition that the Stahl sextic exists and has no repeated roots. Otherwise some of them are not valid.

† "Sulla Lemniscata Projettiva," *Ist. Lomb. Rend.*, Series II, Vol. XXXVII, pp. 277 and 304.

‡ Cf. Ciani, *loc. cit.*, p. 1175.

§ Cf. the case of the cuspidal cubic.

Using the absolute coordinates

$$\begin{cases} x = X + iY, \\ \bar{x} = X - iY, \quad (X \text{ and } Y \text{ rectangular coordinates}), \end{cases}$$

and an elliptic naming in which the parameter t runs around the unit circle, a point on a rational curve is given by the single equation

$$(1) \qquad x = f(t),$$

where f is a rational algebraic function.* This carries with it the conjugate

$$(2) \qquad \bar{x} = \bar{f}(t),$$

derived by writing for each complex number, including t, its conjugate. If x and $\bar{x}$ are interchanged when $t' = \frac{1}{t}$ the curve is symmetrical about the base line, admits a g_2. If in addition

$$x' = \varepsilon^a x \text{ when } t = \varepsilon t', \qquad 1 \leqq a \leqq n-1,$$

where ε is a primitive n-th root of unity, the curve is invariant under a rotation about the origin of period n, admits a cyclic g_n. The two generate a dihedral g_{2n} containing n rotations, including the identity, and n reflexions.†

Now a g_2 means a line of symmetry, and the cyclic group carries this line into n distinct positions when n is odd. Hence, in that case:

A dihedral g_{2n} means symmetry about n equispaced lines through the origin.

These are axes of the n reflexions, the centers being at infinity.

When n is even there are two sets of $\frac{n}{2}$ lines of symmetry, not conjugate to each other, which make up n equispaced axes of reflexion. Furthermore, the curve has the origin for center.

In either case the centers lie on a line, the line at infinity, and the axes meet at a point, the origin, point and line being fixed elements of the group. When n is even there is a center on each axis.

The elements of the cyclic g_n binary and ternary are

$$(4) \qquad t = \varepsilon^a t', \quad \begin{cases} x' = \varepsilon^a x, \\ \bar{x}' = \varepsilon^{-a} \bar{x}, \end{cases} \qquad a = 1, \ldots, n.$$

* See a memoir by Professor Morley, *Trans. Amer. Math. Soc.*, Vol. I, p. 97.

† Cf. Hilton, "Finite Groups," p. 53, Ex. 21 (iii).

The products of these with the g_2's

$$(5) \qquad t = \frac{1}{t'}, \quad \begin{cases} x' = \bar{x}, \\ \bar{x}' = x, \end{cases}$$

are the n involutions and reflexions respectively.

The application of the ternary g_n to a point and its image affords an easy geometrical construction for the general set of $2n$ conjugate points. Such sets break up into two concentric regular polygons of n vertices. If, however, a point is at a center or lies on an axis of reflexion, it assumes only n distinct positions.

Among these special sets of n points on ρ^m are the two sets whose parameters are the fixed points of the involutions and which may be taken as the basis of determination of the conjugate sets of the binary group. In the canonical form employed here, the general set of $2n$ conjugate points of the binary group is given by

$$(6) \qquad (t^n + 1)^2 - \lambda\,(t^n - 1)^2,$$

and therefore factors into two sets of n.*

Among the curves left invariant by the general dihedral group is the system of concentric circles (double contact conics)

$$(7) \qquad x\bar{x} = \lambda^2.$$

These will intersect ρ^m in $2m$ invariant points which break up into conjugate sets, their parameters being conjugate sets of the binary group.

This furnishes a method not only for a geometrical construction of conjugate sets of the ternary group, but also for finding the analytic representation of such sets of the binary group.

§ 5. *The Dihedral* ρ_4^4.

From the typical sextics we get immediately the equations of the dihedral quartic curves

$$(1) \qquad x = \frac{\dfrac{\partial^2 S}{\partial t_2^2}}{\dfrac{\partial^2 S}{\partial t_1 \partial t_2}}, \qquad \bar{x} = \frac{\dfrac{\partial^2 S}{\partial t_1^2}}{\dfrac{\partial^2 S}{\partial t_1 \partial t_2}}.$$

We may write the quartic with a dihedral g_4,

$$(2) \qquad x = \frac{t^3}{t^4 + a t^2 + 1}, \qquad \bar{x} = \frac{t}{t^4 + a t^2 + 1}.$$

* Klein, *loc. cit.*, p. 49.

Since $n = 2$, the curve is symmetrical about each of two perpendicular lines and their point of intersection. A general set of four conjugate points therefore will form a rectangle with center at the origin.

The three elements of this group, omitting the identity, are involutory, since the rotation is of period 2 and therefore a reflexion from the origin in the line at infinity.

Now the parametric equations show that there is a bi-flecnode, isolated however, at the origin. Hence, *if a* ρ^4 *admits a dihedral* g_4, *it has a bi-flecnode.* Conversely, *if a* ρ^4 *has a bi-flecnode, it is invariant under a dihedral* g_4, for its equation can be written

$$y^2 z^2 + z^2 x^2 + x^2 y^2 + 2 f x^2 y z = 0, \tag{3}$$

which is unaltered by changing the sign of x, interchanging y and z, or changing the signs of y and z interchanged; *i. e.*, by a dihedral g_4.

Here, then, is an example of a rational quartic that is reflected into itself from a point on it. This transformation is singled out from the other two which are symmetrical in their effects.

Denote the three reflexions by R_1, R_2, R_3, their centers by a_1, a_2, a_3, and the axes by $\alpha_1, \alpha_2, \alpha_3$; a_3 being the bi-flecnode and a_2 on the base line.

Two of the double lines meet at a_1 and the other two at a_2. Again, the other four flexes lie on three pairs of lines, one pair on each center, and the flex tangents meet in three pairs of points, one pair on each axis. Or we may say:

The diagonal 3-point of the four inflexional points and the diagonal 3-line of the four inflexional lines constitute a single triangle, the centers and axes of reflexion.

This is a fixed triangle of the group and characterizes it geometrically.

The fundamental involution is

$$(a t^4 - 6 t^2) + \lambda (6 t^2 - a),$$

and the invariants are

$$\begin{aligned} I_2 &= -a^2, & I_2' &= -a^4, & I_4 &= a^4 (a^2 - 36)^2, \\ I_6 &= -16 a^6, & I_9 &= 0, & I_6' &= 64 a^6 (a^2 - 4), \end{aligned}$$

whence the invariant conditions on the curve are

$$16 I_2^3 - I_6 = 0, \qquad I_4 - (I_2' - 36 I_2)^2 = 0. \tag{4}$$

The second of these is the condition for four collinear flexes, which we saw is twice satisfied.

Since an absolute constant remains, the curve may be further specialized. A cusp can not occur, for that would imply too many intersections with the axis α_3. *The two double points may come together, however, forming a tac-node at either a_1 or a_2 with α_3 as the nodal tangent, the condition being* $a^2 = 4$, the I_6' above. We have then a second center of reflexion lying on the curve; *viz.*, at the tac-node.

Equations (2) become

$$(5) \qquad x = \frac{t^3}{(t^2 \pm 1)^2}, \qquad \bar{x} = \frac{t}{(t^2 \pm 1)^2},$$

which give, on eliminating t,

$$(6) \qquad (x \pm \bar{x})^4 - x\bar{x} = 0.$$

If $a^2 = 36$, $I_4 = 0$ and the curve acquires two undulations, given by $t^2 \mp 1$ and lying on α_1 or α_2, corresponding to $\pm a$. Thus the two undulations, the inflexional node and the intersection of the two double lines are collinear.

The single condition $a = 0$ suffices for two additional bi-flecnodes, and the curve becomes a lemniscate.

§ 6. *The Dihedral* ρ_6^4.

Taking for the S_6

$$t^6 + 5at^3 + 1,$$

we obtain the equations of ρ_6^4 in the canonical form

$$(1) \qquad x = \frac{at^3 + 1}{t^2}, \qquad \bar{x} = \frac{t^4 + at}{t^2}.$$

Here n is odd and equal to 3, so that the curve has symmetry about three equispaced lines through the origin. On each of these axes of reflexion lies a double point, while all three centers are on a double line.

*The group is characterized geometrically as the g_6 that permutes the sides of the triangle of double points.**

All of the single conics, of which there is a multitude, attached to the ρ^4, belong to the system of concentric circles

$$x\bar{x} = \lambda^2.$$

* Brusotti, *Ist. Lomb. Rend.* (II), Vol. XXXVII, p. 888, discusses a class of rational curves that admit this group, namely those with three *hyper-osculation* points, undulations for the ρ^4.

To the different values of a correspond projectively distinct ρ_6^4's. The line at infinity is, however, a common double line having contacts with each at the circular points.

The fundamental involution is

$$(4t^3 + a) + \lambda (a t^4 + 4t).$$

The values of the invariants and the geometrical significance of their vanishing are

$I_2 = a^2 - 1,$	triple point,
$I_2' = -(a^2 + 2)^2,$	3 bi-flecnodes,
$I_4 = a^2 (a^2 - 16)^2,$	3 undulations,
$I_6 = -a^2 (a^2 - 4)^3,$	3 cusps.

Thus, on account of the triangular symmetry, if a condition is satisfied once, it is, generally speaking, satisfied three times.

The invariant conditions on ρ_6^4 are

$$I_9 = 0,$$
$$[I_4 + (4I_2 - I_2')(60 I_2 + I_2')]^2 - 768 I_6' (12 I_2 + I_2') = 0,$$

where I_6' is the tac-node condition.

The equation in absolute coordinates of the curve with a triple point $(a = 1)$ is

$$x^3 + \bar{x}^3 - x^2 \bar{x}^2 = 0.$$

The sixteen intersections with the deltoid $(a = 2)$, in terms of the parameter of the latter, are given by

$$t^2 (t^6 + t^3 + 1)^2.$$

Hence, *the two curves have eight contacts at which the eighteen common lines are accounted for, the common double tangent counting for six.*

We note that, *if a ρ^4 have three cusps or three undulations, it must admit a dihedral g_6, since it is then determined by a binary cubic.* *

§ 7. *The Projective Lemniscate with a Real Dihedral g_8.*

The octahedral group contains as a sub-group a dihedral g_8. We include a brief mention of the ρ_{24}^4 with reference to the g_8, since in this scheme of representation the sets of eight conjugate points are in general real.

* Cf. S_6 with three double roots, below.

The metrical equations, found at once from S_{24}, are

$$x = \frac{t}{t^4+1}, \qquad \bar{x} = \frac{t^3}{t^4+1},$$

whence

$$(x^2+\bar{x}^2)^2 - x\bar{x} = 0.$$

The curve is symmetrical about each of two pairs of perpendicular lines which make up a set of four equispaced lines through the origin, which is an isolated node.

The fundamental involution is

$$a\,(t^4+1) + \lambda b\, t^2,$$

a quartic and its Hessian.

The invariant relations are

$$I_2' = 0, \qquad I_9 = 0,$$
$$I_4 - 1296\, I_2^2 = 0,$$
$$I_2 I_4 - 81\, I_6 = 0.$$

The curve cuts the base circle at the twelfth roots of unity, exclusive of the fourth roots. Bearing in mind the symmetry it is readily drawn.*

§ 8. *Singular Stahl Sextics.*

The table of Bolza gives only non-singular sextics. We shall now examine those with multiple roots. A triple root can not occur, for the second polars all have a common factor, the ∞^2 line sections reducing to a pencil, and the curve is degenerate. There may be, however, one, two or three double roots.

Starting with Bolza's types, we may impose the further conditions for one or more repeated factors, obtaining thus singular S_i's. But if we first insist on repeated roots and then ask that the sextics admit the different groups, will we get the same tabulation? In other words, are the processes commutative?

Beginning with the extreme case, suppose the sextic has three double roots. It is then the square of a cubic and may be taken as

$$(t^3+1)^2.$$

Hence it is a special case of S_6, $\alpha = 2$. We saw that the three axes of reflexion contain each a double point of ρ_6^4. In this case the other intersections of the axes are given by

$$t^3+1 \quad \text{and} \quad t^3-1.$$

* This is the *right circular cross curve* of Loria, "Spezielle Ebene Kurven," Vol. I, p. 226.

Starting with $(t^3 - 1)^2$, we get the same curve, since the two sextics are projectively equivalent.

A sextic with two double roots is made up of one quadratic and the square of another, and may be taken as

$$t\,(t^2 + \alpha t + 1)^2.$$

Since there is always an involution leaving two quadratics unaltered, *a sextic with two double roots admits a* g_2. This is a special case of S_2.* If we ask further that the sextic admit the involution $t' = -t$, the double roots must be interchanged and it may be reduced to the form

$$t\,(t^2 \pm 1)^2,$$

and is a special case of type (3), $a = \pm 2$. In this case the double points of the three involutions of the g_4 are the two quadratic factors of the sextic and their Jacobian, three mutually apolar pairs. *The corresponding* ρ^4 *is an instance of one whose Stahl sextic is cut out by a line*, the axis α_1 or α_2.

The sextic with one double root may be written

$$t^2\,(t^4 + a t^3 + b t^2 + c t + 1).$$

Now the double root 0 must be a fixed point in any transformation that leaves the sextic unaltered, while all the transformations that leave one point unaltered constitute a cyclic group that leaves a second point unaltered. Two cases are to be distinguished.

CASE I. *The second fixed point not a root of S.* Then the involution $t' = -t$ will interchange the four simple roots in pairs and S becomes

$$t^2\,(t^4 + a t^2 + 1),$$

which is a special case of type (1), as is readily verified. S can not admit a cyclic g_3, for one of the simple roots would be a fixed point contrary to hypothesis. The four points, however, might be permuted cyclicly by the transformation

$$t' = i t, \qquad i^4 = 1,$$

when S assumes the form

$$t^2\,(t^4 + 1).$$

The ρ^4 corresponding to this may be written

$$x_0 = t, \quad x_1 = t^2, \quad x_2 = t^4 + 1,$$
$$x_0^4 + x_1^4 - x_0^2 x_1 x_2 = 0.$$

It has a triple point with a cusp at $0, \infty$.

---

* Writing S_2, $a t_1^6 + b t_1^5 t_2 + t_1^4 t_2^2 + c t_1^3 t_2^3 + t_1^2 t_2^4 + b t_1 t_2^5 + a t_2^6$, we obtain, on placing $a = b = 0$, $t_1^2 t_2^2 (t_1^2 + c t_1 t_2 + t_2^2)$, a form to which a sextic with two double roots may be reduced.

CASE II. *The second fixed point a root of S.* The sextic can now admit the cyclic g_3

$$t' = \omega t, \qquad \omega^3 = 1,$$

which permutes cyclicly the remaining roots producing

$$S_3, \quad t^2(t^3+1).$$

The corresponding ρ^4 may be written

$$x_0 = 4t^3 + 1, \quad x_1 = t^4 + 4t, \quad x_2 = t^2,$$

which is a curve with three undulations, and is therefore a special case of the ρ_6^4 already noticed, § 6.*

There are then sextics with one repeated root invariant under cyclic groups of orders 2, 3 and 4, only the last of which, however, determines a new type of curve.

§ 9. *Cases of Failure.*

There remains the consideration of the ρ^4 with a singularity such that the tangent cuts out a quartic with four equal roots, for then the fundamental involution has a common factor and the Stahl sextic is no longer defined.

If ρ^4 have an undulation, say at the point 0, it can admit only a cyclic group of which the undulation is a fixed point. The group will have another fixed point, say ∞. Referred to the tangents at these two points and the line joining them, the equations of the curve are

$$\begin{cases} x_0 = t^4, \\ x_1 = t^2 + bt + 1, \\ x_2 = t(t^2 + ct + a). \end{cases}$$

This will admit the involution

$$t' = -t, \quad \text{if} \quad b = c = 0,$$

and we have as the equations of a ρ_2^4 with an undulation

$$x_0 = t^4, \quad x_1 = t^2 + 1, \quad x_2 = t(t^2 + a),$$

which is a special case of ρ_2^4 of § 3.

If the other fixed point is an inflexion, ρ^4 may admit the cyclic g_3, $t' = \omega t$. The equations are

$$x_0 = t^4, \quad x_1 = t, \quad x_2 = t^3 + 1,$$

or

$$x_1(x_0 + x_1)^3 - x_0 x_2^3 = 0.$$

* It should be remarked that while the sextic admits only a g_3 there is no contradiction in our theory, for the Stahl sextic in this case is not unique (see below).

The flex tangent x_1 meets again at the undulation and x_2 cuts out a triple point. Three flexes and three double lines remain.

Again, the quartic with a tac-node-cusp at 0 may admit the cyclic g_3, giving rise to a new type of ρ_3^4,

$$x_0 = t^4, \quad x_1 = t^2, \quad x_2 = t^3 + 1,$$

or

$$x_1 (x_0^3 - x_1^3 + 2\, x_0 x_1 x_2) - x_0^2 x_2^2 = 0.$$

The multiple point consumes all of the singularities of the curve except three flexes.

A ρ^4 with two undulations, say at 0 and ∞, may be written

$$x_0 = t^4, \quad x_1 = 1, \quad x_2 = t\,(t^2 + 2\,a\,t + 1),$$

which admits the involution

$$t' = 1/t.$$

Hence, *a* ρ^4 *with two undulations is invariant under a* g_2. In this case the undulations are interchanged. They may be fixed points of an involution when the curve takes the form

$$x_0 = t^4, \quad x_1 = 1, \quad x_2 = t\,(t^2 + 1),$$

which is invariant under a dihedral g_4 and is the same as that noticed, § 5.

The ρ^4 with three undulations is Brusotti's curve and admits a dihedral g_6, § 6, foot-note. It may be derived uniquely from the special S_6

$$t^6 + 20\,t^3 + 1.$$

But if we attempt to recover the sextic, we obtain the doubly infinite system

$$(t^6 + 20\,t^3 + 1) + \lambda\,t^2\,(2\,t^3 + 5) + \mu\,t\,(5\,t^3 + 2),$$

any member of which determines the same curve.

We have thus arrived at a complete solution of the first part of our fundamental problem, having found eight types of self-projective rational quartics. Moreover, the nature of the investigation is such that we are in a position to say that *these eight types include as special cases all others.*

For convenience of reference they have been collected in a table at the end.

II. The Rational Quintic.

§ 10. *General Considerations.*

We begin our study of the ρ^5 with a generalization of a theorem enunciated for the ρ^4, § 3, viz.:

If a rational curve ρ^m *of order* m

$$x_i = (\alpha_i\, t)^m, \qquad i = 0, 1, 2,$$

is invariant under a cyclic group of order n, $n > m$, it is invariant under a one-parameter group.

Proof: A ternary collineation of period n can always be written in the canonical form

$$x'_i = \varepsilon_i x_i,$$

where the ε's are n-th roots of unity. The triangle of reference is then a fixed triangle of the group.* Now each side of this triangle must cut ρ^m in m points whose parameters form a conjugate set under the corresponding binary cyclic g_n. If $n > m$, the only conjugate sets of m points are

$$t_1^a t_2^b, \qquad a + b = m.$$

The only three sets having no common factors are

$$t_1^m, \quad t_2^m, \quad t_1^{m-r} t_2^r, \qquad 1 \leqq r \leqq m - 1.$$

The equations of the curve then become

$$x_0 = t_1^m, \quad x_1 = t_1^{m-r} t_2^r, \quad x_2 = t_2^m,$$

which admits an infinite group.

If $n = m$, the only conjugate sets of n points are

$$t_1^a t_2^b \quad \text{and} \quad \alpha t_1^n + \beta t_2^n, \qquad a + b = n.$$

There can not be three independent sets of the latter kind. If there are two independent sets, they can be replaced linearly by t_1^n or t_2^n, and we have the one-parameter type. If there is one such set and a set t_1^n or t_2^n, we can obtain by a transformation the preceding case. The only new types then are

$$\alpha t_1^n + \beta t_2^n, \quad t_1^r t_2^{n-r}, \quad t_1^s t_2^{n-s}, \qquad r \neq s, \ 1 \leqq \left\{ \begin{matrix} r \\ s \end{matrix} \right\} \leqq n - 1.$$

If there is no set of the form $\alpha t_1^n + \beta t_2^n$, we have again the one-parameter case.

That is, *the equations of a ρ^n invariant under a cyclic g_n may be written (non-homogeneously) in canonical form*

$$x_0 = t^r, \quad x_1 = t^s, \quad x_2 = t^n + 1,$$

with the above restrictions imposed on r and s.

---

* Every such collineation has at least one proper fixed triangle, an homology having ∞^1.

These equations are the canonical form of ρ^n invariant under a dihedral g_{2n} if $r + s = n$.

Incidentally, for varying r in the one-parameter case, we obtain the classes of curves invariant under an infinite group. These are the self-dual binomial curves, so-called parabolas of higher order,

$$x_1^n - x_0^{n-r}\, x_2^r = 0, \qquad 0 < r < n.$$

The singularities are all concentrated at the two fixed points 0 and ∞. There are two such types of ρ^5, $r = 1, 2$.* If $r = 3, 4$, we obtain curves metrically distinct but projectively equivalent.†

Our theorem excludes for the rational quintic those cyclic g_n's and dihedral g_{2n}'s, $n > 5$. Furthermore, a group that transforms a ρ^5 into itself must leave the flex ninic unaltered. From a study of the characteristic invariant forms of the binary group — *i. e.*, those special sets of parameters which assume fewer than n values — we readily conclude that

A ρ^5 can not admit a dihedral g_{2n}, n even, nor the tetrahedral, the octahedral or the icosahedral group, even though its flex equation have multiple roots.

There are left, then, only cyclic groups of orders 2, 3, 4 and 5 and dihedral groups of orders 6 and 10. It turns out that there are rational quintics belonging to all of these.

§ 11. *The Cyclic Groups.*

Using the fixed triangle of the group as triangle of reference, we obtain without difficulty the equations of the cyclic quintics in canonical form.

Thus the ρ_2^5 is

$$\begin{cases} x_0 = t^5 + a\,t^3 + b\,t, \\ x_1 = t^4 + c\,t^2, \\ x_2 = t^2 + 1. \end{cases}$$

The only fixed points under the involution are 0 and ∞. One of these is an inflexion — the center of the reflexion. The other lies on the axis, being the contact of a tangent from the center. The parameters of the other contacts of tangents from the center and those of the extra intersections of the axis must be interchanged by the involution. That is, *two double lines meet at the center and two double points lie on the axis.*

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* Cf. Snyder's G_{20} and G_{30}, *loc. cit.*, paragraphs 5 and 6. It may be remarked that his curves

$$a\,x\,y^4 + b\,y^5 + c\,z^5 = 0 \quad \text{and} \quad a\,x\,y^4 + b\,z^5 = 0$$

are projectively equivalent.

† See Wieleitner, "Algebraische Kurven," p. 135, Beisp. 5.

The equations of ρ_3^5 are

$$\begin{cases} x_0 = t^5 + a\,t^2, \\ x_1 = b\,t^3 + 1, \\ x_2 = t^4 + t. \end{cases}$$

The points of the curve are arranged in triads, but as the singularities occur in multiples of 3 this can happen without any striking geometrical specialization. The line x_2 is the apolar line and cuts out the fixed points of the generating binary transformation.

We write the ρ_4^5

$$\begin{cases} x_0 = a\,t^5 + t, \\ x_1 = t^3, \\ x_2 = t^4 + 1. \end{cases}$$

Now the binary cyclic g_4 generated by

$$t' = i\,t, \qquad i^4 = 1,$$

contains as a subgroup the involution $t' = -t$. Hence the ρ_4^5 admits a reflexion into itself. The line x_1 is seen to be a touching flex tangent, the contacts being the fixed points of the binary group. The line x_2 is the axis of reflexion and cuts out as above two double points. In addition to the special flex tangent two double lines meet at 0. Thus, *the center of reflexion is a syzygetic point.**

For particular values of r and s of the previous section we obtain two types of ρ^5 invariant under the g_5 generated by

$$t' = \varepsilon\,t, \qquad \varepsilon^5 = 1,$$

$$(1) \qquad x_0 = t, \quad x_1 = t^2, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 + x_1^5 - x_0^3 x_1 x_2 = 0;\dagger$$

$$(2) \qquad x_0 = t^3, \quad x_1 = t, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 - x_1^5 + x_0 x_1^2 x_2^2 - 2\,x_0^3 x_1 x_2 = 0.$$

The first has a fourfold point at $0, \infty$, with three coincident parameters, consuming four flexes and seven double lines. The parameters are the fixed points of the binary group.

---

* By a *syzygetic point* we mean a point whose lines cut out the syzygetic pencil of binary quartics $U + \lambda H$, where H is the Hessian of U. See *Johns Hopkins University Circular*, February, 1911, p. 101. Snyder's curve, paragraph 5, is a special case of this and may be written parametrically

$$x_0 = t^5 + t, \quad x_1 = t^4, \quad x_2 = 1,$$

when 0 names the undulation, ∞ the hyper-osculation point and $t^4 + 1$ the fourfold point.

† Cf. Snyder, *loc. cit.*, paragraph 5.

The second has a cusp of higher order at a point of inflexion which uses up all of the double points, four flexes and seven double lines, the parameters being fixed under the binary group.

Sets of five conjugate points on either curve are cut out by the pencil of double contact conics

$$x_0^2 + \lambda x_1 x_2 = 0,$$

five of the intersections being taken up at the multiple point.

§ 12. *The Dihedral ρ_{10}^5's.*

We find likewise two types of quintics invariant under the dihedral g_{10} obtained by combining with the g_5 above the involution

$$t' = 1/t.$$

They are

(1) $$x_0 = t^3, \quad x_1 = t^2, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 + x_1^5 - x_0^2 x_1^2 x_2 = 0;^*$$

(2) $$x_0 = t, \quad x_1 = t^4, \quad x_2 = t^5 + 1,$$

or

$$x_0^5 + x_1^5 + x_0 x_1 x_2 (3 x_0 x_1 - x_2^2) = 0.$$

The first has a fourfold point with two cusps; the parameters being fixed under the cyclic g_5. The line x_2 cuts out the five flexes which are the centers of the five reflexions. From each inflexion run one of the five double lines and one simple tangent. The contacts of the latter, together with the corresponding flexes, make up the five pairs of fixed points of the involutions.

In the metrical representation of § 4 the curve has fivefold symmetry. The equations are †

$$x = \frac{t^3}{t^5+1}, \qquad \bar{x} = \frac{t^2}{t^5+1}.$$

The multiple point is isolated at the origin but the five double lines are real, their equations being

$$5\,(\varepsilon^i x + \varepsilon^{-i}\bar{x}) + 4 = 0, \qquad i = 0, \ldots, 4;\ \varepsilon^5 = 1.$$

There are five inflexional asymptotes whose equations are

$$5\,(\varepsilon^i x + \varepsilon^{-i}\bar{x}) - 1 = 0.$$

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* This is Snyder's curve (2), paragraph 5. † Cf. Hulburt, "Calculus," Ex. 14, p. 125.

We saw that sets of conjugate points were cut out by the system of concentric circles

$$x\bar{x} = \lambda^2.$$

Among these may be mentioned the circle on

(a)	5 flex lines,	$10\lambda = 1$,
(b)	5 lines from flexes,	$2\lambda = 1$,
(c)	5 double lines,	$5\lambda = 2$,
(d)	10 contacts of double lines.	

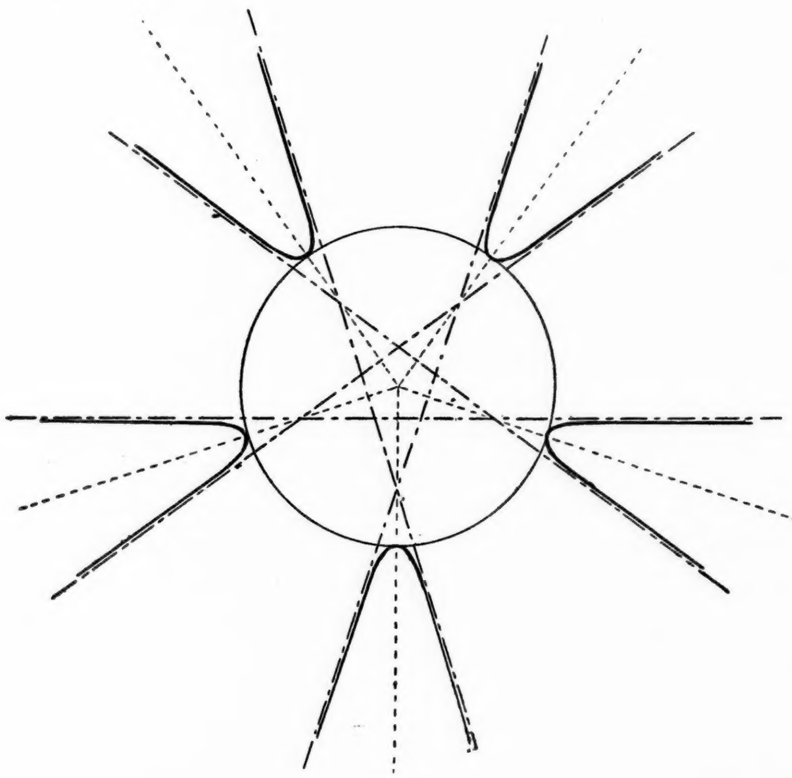

FIG. 1.

(b) has five contacts with ρ^5, lying on the axis and given by $t^5 - 1$.

The curve cuts the base circle at the fifteenth roots of -1, exclusive of the fifth roots. A figure (Fig. 1) has been drawn, using the equation $x' = 5x$.

Quintic (2) has a double point of which each branch is an undulation,* the nodal parameters being the fixed points of the binary cyclic g_5. The line x_2

* This may be called a *bi-und-node*.

cuts out the remaining five flexes which are the centers of the five reflexions. From each flex run one ordinary tangent and two of the ten double lines.

Metrically the equation of the curve is

$$x = \frac{t}{t^5 + 1},$$

The bi-und-node, at which the five axes of reflexion meet, is isolated at the origin. The other five double points are real, one lying on each axis and together forming a special conjugate set of the ternary group and their parameters a general set of the binary group. That is,

The six double points lie at the vertices and center of a regular pentagon. They are therefore perspective in ten ways, two ways corresponding to each axis of symmetry.

There are five real inflexional asymptotes:

$$5\,(\varepsilon^i x + \varepsilon^{-i}\bar{x}) + 3 = 0.$$

Among the system of invariant circles we note that on

(a)	5 flex lines,	$10\lambda = 3$,
(b)	5 lines from flexes,	$2\lambda = 1$,
(c)	5 double points,	$\lambda = 1$,

besides two each on

(d) 5 double lines,
(e) 10 contacts of double lines.

(b) is the perspective conic, or we may say, *the perspective conic is inscribed in the five tangents from the flexes, the contacts lying on the axes of symmetry, given by* $t^5 - 1$. These contacts and the flex parameters, therefore, are the fixed points of the involutions.

A figure (Fig. 2) is given.

§ 13. *The Dihedral* ρ_6^5.

The equations of ρ^5 invariant under the dihedral g_6 generated by

$$t' = \omega t, \qquad t' = 1/t, \qquad\qquad \omega^3 = 1.$$

are found at once from those of ρ_3^5 by writing $a = b$:

$$x_0 = t^5 + a t^2, \qquad x_1 = a t^3 + 1, \qquad x_2 = t^4 + t.$$

The centers of reflexion are points of inflexion and are given by $t^3 + 1$, That is, *the quadratic giving the fixed points of the binary cyclic* g_3 *is the*

Hessian of the cubic which names the centers of reflexion, and both are cut out by the apolar line. Denote the cubic by $C_{1,3}$, the cubicovariant by $C_{3,3}$ and the Hessian by H.

From each center run one tangent and two double lines, the contacts $C_{3,3}$ of the former lying on the axes. The residual intersections of each axis, therefore, are made up of two double points. Hence, *the double points are perspective.*

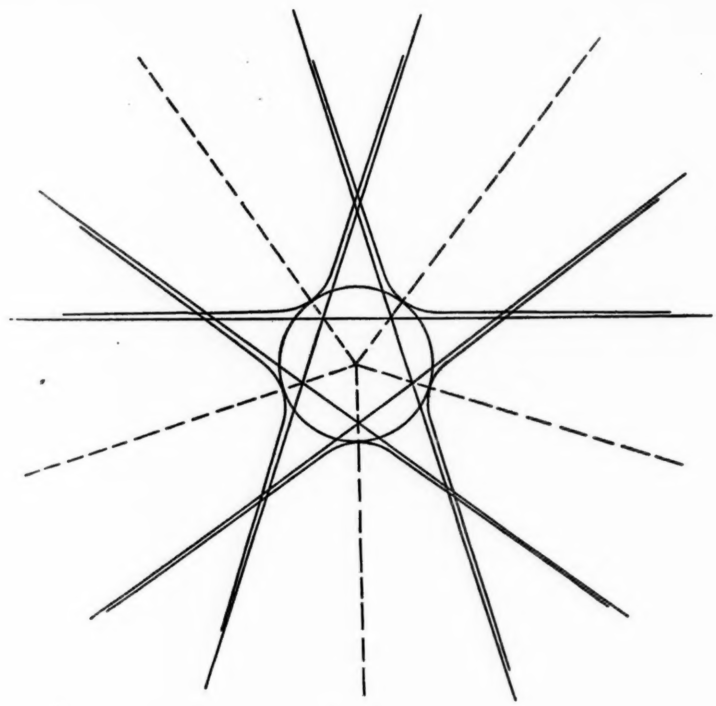

FIG. 2.

The tangents at the centers are

$$3\,(\omega^i x_0 + \omega^{-i} x_1) + (a+5)\,x_2 = 0, \qquad i = 0, 1, 2.$$

The invariant conics all touch the curve at the Hessian points. Among them is the perspective conic

$$\xi_0 = 1, \quad \xi_1 = t^2, \quad \xi_2 = -(a+1)\,t,$$

or

$$(a+1)^2 \xi_0 \xi_1 - \xi_2^2 = 0;$$

$$x_0 = (a+1)\,t^2, \quad x_1 = (a+1), \quad x_2 = 2\,t,$$

or

$$4\,x_0 x_1 - (a+1)^2 x_2^2 = 0,$$

which touches again at the cubicovariant points. Hence, *the perspective conic*

is inscribed in the three tangents from the centers of reflexion and touches the curve at the same points.

The group may be characterized geometrically as the one that permutes the sides of this triangle.

The flex form, exclusive of t^3+1, is

$$a t^6 + (a^2 - 9a + 10) t^3 + a,$$

and names a general set of six conjugate points. Owing to the symmetry, the vanishing of the discriminant

$$(a-1)(a-2)(a-5)(a-10)$$

means three equal roots and is therefore the condition that they reduce to a special set of three.

Many interesting special cases of ρ_6^5 arise for numerical values of a. In particular are those corresponding to the roots of the discriminant above. We note the following:

(1)	$a=1$,	degenerate,		
(2)	$a=2$,	undulations	at	$C_{3,3}$,
(3)	$a=5$,	cusps	"	$C_{3,3}$,
(4)	$a=10$,	hyper-osculation points	"	$C_{1,3}$,
(5)	$a=-2$,	syzygetic points	"	$C_{1,3}$,
(6)	$a=-8$,	syzygetic points	"	H.

(3) is self-dual. The cuspidal tangents are the axes and therefore meet at a point.

(4) is the quintic case of Brusotti's curves. From each hyper-osculation point run one tangent and one of the three double lines, contacts of the former being t^3-1. Many of the facts brought out by him in connection with the group apply to all quintics invariant under it.

§ 14. *Three Syzygetic Points.*

I shall conclude the discussion of the quintic with an account of the special ρ_6^5 which suggested this investigation.

Cayley* proved that the locus of the foci of parabolas on three points is a rational quintic passing through the circular points. Now at each of these points three double lines meet,† *i. e.*, they are syzygetic points.

* "Collected Papers," Vol. VII, p. 568.

† Loria, "Ebene Kurven," Vol. I, p. 239 a.

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In absolute coordinates the equation can be written *

$$(1) \qquad x = t + \sum_1^3 \frac{A_i}{t - \alpha_i},$$

where α_i are the parameters of the three points on the unit circle and

$$A_i = (\alpha_i - \alpha_j)(\alpha_i - \alpha_k), \qquad i \neq j \neq k.$$

In this form the coefficients of (1) can be expressed in symmetric functions of the α's.

The question naturally arises as to the nature of the locus when a particular arrangement of the three base points is selected. For example, if the triangle is equilateral, say -1, $-\omega$, $-\omega^2$, so that $s_1 = s_2 = 0$, $s_3 = -1$, equation (1) becomes

$$(2) \qquad x = \frac{t^4 - 8t}{t^3 + 1},$$

which is the metrical representation of the sixth type of the previous section. Hence, *the locus of the foci of parabolas on the vertices of an equilateral triangle is a special* ρ_6^5, as is evident also from other considerations. All twelve double lines cut ρ^5 at points of x_2, two each at the inflexions $t^3 + 1$ and three each at the Hessian points.

Again, if the vertices of the triangle are $-\omega, -\omega^2, 1$, whence $s_1 = s_2 = 2s_3 = 2$, the equations of the curve written homogeneously are

$$(3) \qquad \begin{cases} x_0 = t^5 - 2t^4 + 4t^2 - 2t, \\ x_1 = 2t^4 - 4t^3 + 2t - 1, \\ x_2 = t^4 - 2t^3 + 2t^2 - t. \end{cases}$$

It is readily verified that *there is a third syzygetic point*, $t = 1$, *lying on a line with the other two.*

For the sake of symmetry we apply to the parameter the transformation

$$T: \quad t' = \frac{\omega t + 1}{t + \omega}, \qquad T^{-1}: \quad t = \frac{1 - \omega t'}{t' - \omega},$$

so that the syzygetic points are named by $t^3 - 1$. Call this cubic $\bar{C}_{1,3}$, the cubicovariant $\bar{C}_{3,3}$ and the Hessian H. Likewise transfer to a new triangle of reference by the substitution

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* *Johns Hopkins University Circular*, February, 1911, p. 105. The proof is given in the manuscript of my dissertation, Johns Hopkins University, 1912. The parameters of the absolute points on circle and quintic are 0 and ∞, while the α's on the quintic are cut out by the line joining them.

$$X: \begin{cases} x_0' = x_0 - \omega x_1 + (\omega - 1)\, x_2, \\ x_1' = -\omega x_0 + x_1 + (\omega - 1)\, x_2, \\ x_2' = (\omega - 1)\, x_2, \end{cases} \qquad X^{-1}: \begin{cases} x_0 = x_0' + \omega x_1' + \omega^2 x_2', \\ x_1 = \omega x_0' + x_1' + \omega^2 x_2', \\ x_2 = \omega^2 x_2'. \end{cases}$$

Under the transformations T and X equations (3) reduce, on dropping primes,* to

$$(4) \qquad x_0 = t^5 + 2t^2, \quad x_1 = -2t^3 - 1, \quad x_2 = t^4 - t,$$

or, eliminating t,

$$(5) \qquad x_0 x_1 (x_0^3 + 3x_0 x_1 x_2 + x_1^3) + x_2^2 (2x_0^3 + x_0 x_1 x_2 + 2x_1^3) - x_2^5 = 0.$$

We saw, § 13, that the syzygetic points are inflexions. It is readily found that *the tangents at the syzygetic points touch again at* $\overline{C}_{3,3}$. These three pairs of parameters, then, are the fixed parameters of the involutions. Now each such touching flex tangent is made up of one stationary line and two double lines. *The tangents from the syzygetic points therefore account for the twelve double lines of the curve.*

The equations of the perspective conic are

$$(6) \qquad \begin{cases} \xi_0 = 1, \quad \xi_1 = t^2, \quad \xi_2 = t, \\ x_2^2 - 4x_0 x_1 = 0. \end{cases}$$

Substituting the values of the x's from (4), the contacts are found to be

$$t\,(t^3 + 1).$$

Hence, *the perspective conic is inscribed in and has contacts in common with the five tangents at the points* $H\,\overline{C}_{1,3}$, *cut out by the apolar line, the contacts being given by* $H\,\overline{C}_{3,3}$. Since each syzygetic tangent counts for four common lines of the two curves, these five defining tangents constitute the total of sixteen.

We proceed in the next instance to a consideration of the double points. The coordinates of a node of (1) were found to be†

$$\begin{cases} x_0 = s_3 \alpha_3 (\alpha_2 + \alpha_3) - \alpha_2 \alpha_3^3 (\alpha_2 - \alpha_3) - s_3 \alpha_1 \alpha_2, \\ x_1 = \alpha_3^2 (\alpha_2 - \alpha_3) - \alpha_2^2 (\alpha_3 - \alpha_1) + s_3, \\ x_2 = s_3 \alpha_3, \end{cases}$$

* On writing $t = -t$ and changing the signs of x_0 and x_1, this becomes the special case (5) of ρ_6^5 above.

† *Johns Hopkins University Circular, loc. cit.*

whence the others can be derived by permutations of the α's. For this case they become, substituting the particular values of the α's,

$$\begin{array}{ccc} 1-\omega^2, & 1-\omega^2, & \omega^2-\omega, \\ 1-\omega^2, & \omega^2-\omega, & 1-\omega^2, \\ 1, & 1, & 1; \end{array}$$

$$\begin{array}{ccc} 1-\omega, & \omega-\omega^2, & 1-\omega, \\ 1-\omega, & 1-\omega, & \omega-\omega^2, \\ 1, & 1, & 1. \end{array}$$

Under the transformation X these reduce to

$$\begin{array}{ccc} (1, 1, \omega), & (1, \omega, 1), & (\omega, 1, 1); \\ (1, 1, \omega^2), & (1, \omega^2, 1), & (\omega^2, 1, 1). \end{array}$$

Denote the first row by d_1, d_2, d_3 and the second by d'_1, d'_2, d'_3.

These two triangles of double points, together with

$$\begin{array}{ccc} (1, 0, 0), & (0, 1, 0), & (0, 0, 1); \\ (1, 1, 1), & (1, \omega, \omega^2), & (1, \omega^2, \omega), \end{array}$$

have the canonical form of four triangles, any two of which are in sixfold perspection, the centers of perspection being the other two. Hence, *the six double points form two fully perspective triangles. In other words, they define a Hessian configuration in the plane, namely, that associated with a syzygetic pencil of cubic curves.** This just accounts for the four relations that must exist among them.

The little table exhibits in rows and columns the six centers of perspection, together with the lines passing through them.

$d_1 d'_1$	$d_2 d'_3$	$d'_2 d_3$	$0, 0, 1$
$d_2 d'_2$	$d_3 d'_1$	$d'_3 d_1$	$0, 1, 0$
$d_3 d'_3$	$d_1 d'_2$	$d'_1 d_2$	$1, 0, 0$
$1, 1, 1$	$1, \omega, \omega^2$	$1, \omega^2, \omega$	

The residual intersections of the fifteen lines joining the double points are given by $H^3 \bar{C}^2_{1,3} \bar{C}_{3,3}$. Hence, two of the centers are on the curve, *viz.*, at the Hessian points.

* Three of the base points of the pencil of cubics are the syzygetic points of the quintic.

Since the curve admits three reflexions, the other six inflexions must lie in harmonic pairs on three lines from each of the syzygetic points. That is, *they are in triple harmonic perspection, which just accounts for the four conditions on them.*

Thus, from each syzygetic point run, besides the touching flex tangent, two double lines and two lines carrying each two double points, and three lines carrying each two flexes.

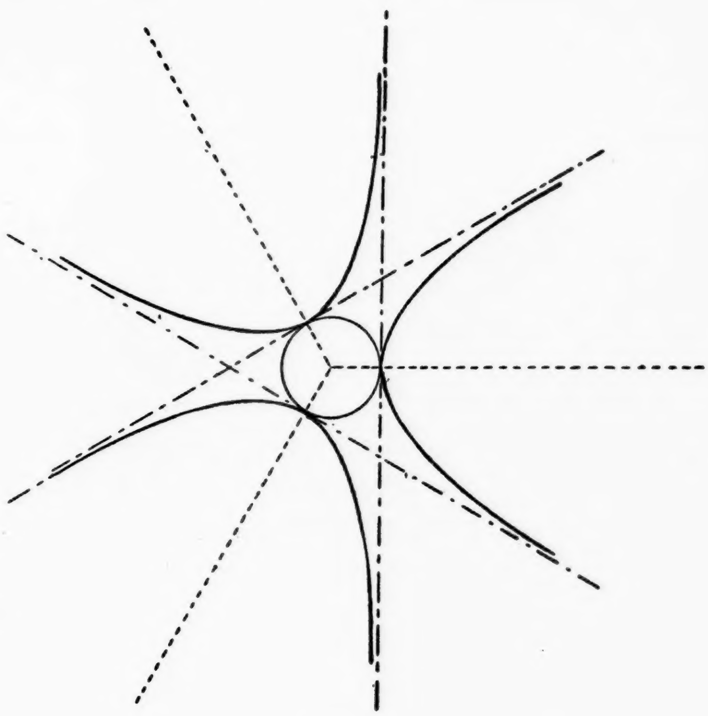

FIG. 3.

The metrical equation of the curve, reflected in the origin, is

$$x = \frac{t^4 - 2t}{t^3 + 1}.$$

The syzygetic line is now the line at infinity, and there are three real inflexional asymptotes (the tangents at the syzygetic points which touch again) and two ordinary ones (the circular rays, imaginary of course). The perspective conic becomes a circle of radius 1/2, touching ρ^5 where the real asymptotes touch. In other words, the triangle of asymptotes is inscribed in the unit circle and circumscribed about the perspective.

The intersections with the base circle are at the ninth roots of -1, exclusive of the cube roots. The curve is now readily drawn and a figure is shown above (Fig. 3).

§ 15. *Summary.*

To summarize, we have found, according to our criterion, ten types of self-projective rational quintics, belonging to cyclic groups of orders 2, 3, 4, 5 (two types) and dihedral groups of orders 6 and 10 (two types), besides two with infinite groups.

The parametric equations in canonical form of these as well as of the quartics are shown in the following table:

Self-projective Rational Quartics.

		x_0	x_1	x_2	
cyclic	g_2	t^4+at^2,	t^2+1,	t^3+bt.	(1)
cyclic	g_3	t^4,	t,	t^3+1	(2)
cyclic	g_3	t^4,	t^2,	t^3+1.	(3)
cyclic	g_4	t,	t^2,	t^4+1.	(4)
dihedral	g_4	t^3,	t,	t^4+at^2+1.	(5)
dihedral	g_6	at^3+1,	t^4+at,	t^2.	(6)
octahedral	g_{24}	t,	t^3,	t^4+1.	(7)
	g_∞	t^4,	t,	1.	(8)

Self-projective Rational Quintics.

cyclic	g_2	t^5+at^3+bt,	t^4+ct^2,	t^2+1.	(1)
cyclic	g_3	t^5+at^2,	bt^3+1,	t^4+t.	(2)
cyclic	g_4	at^5+t,	t^3,	t^4+1.	(3)
cyclic	g_5	t,	t^2,	t^5+1.	(4)
cyclic	g_5	t^3,	t,	t^5+1.	(5)
dihedral	g_6	t^5+at^2,	at^3+1,	t^4+t.	(6)
dihedral	g_{10}	t^3,	t^2,	t^5+1.	(7)
dihedral	g_{10}	t,	t^4,	t^5+1.	(8)
	g_∞	t^5,	t^4,	1.	(9)
	g_∞	t^5,	t^3,	1.	(10)

Iterated Limits in General Analysis.

By Ralph E. Root.

Introduction.

In a former note* we have briefly indicated a method for the investigation of iterated limits of functions on an abstract range. It is the purpose of the present paper to give a more comprehensive account of the method there proposed. The paper has its origin in the thought that in most of the definitions of limit that are employed in current mathematics a notion analogous to that of "neighborhood" or "vicinity" of an element is fundamental. In the domain of general analysis† various ways of determining a neighborhood of an element have been employed, notably the notion of *voisinage* used by M. Fréchet,‡ and the relations K_1 and K_2 used by E. H. Moore, either as undefined or as defined in terms of a "development" of the class of elements constituting the fundamental domain.§

A definite class of elements being assumed, the notion of "neighborhood" of an element is essentially that of a subclass having a special relation to the element. In taking this relation as undefined and at the basis of our system of postulates we occupy a position intermediate, as regards generality, between the extreme position of those who take the notion of "limit" itself as undefined,|| and that of those who define "limit" by means of other relations which give rise to notions analogous to that of "neighborhood." The character and form of the postulates adopted are determined largely by two fundamental require-

* *Bull. Am. Math. Soc.*, Vol. XVII (July, 1911), p. 538.

† The term "general analysis" is here used in a technical sense to indicate mathematical analysis pertaining to a class of elements whose character is not specified.

‡ "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII.

§ E. H. Moore, "Introduction to a Form of General Analysis," pp. 125 and 138.

|| For example, Fréchet in the first chapter of the paper referred to above, and F. Riesz in his paper before the International Congress of Mathematicians at Rome, 1908 ("Stetigkeitsbegriff und abstracte Mengenlehre," *Atti*, Vol. II, pp. 18–24).

ments;* first, to provide for an adequate treatment of ideal limiting elements, and second, to insure the persistence of the specified conditions under composition of ranges.

In Chapter I, we consider a class $\mathfrak{P}$ of elements and an undefined relation R between subclasses of $\mathfrak{P}$, the system $(\mathfrak{P}; R)$ being subjected to a set of postulates that permit the definition of ideal elements in such fashion that the system, when once extended by the adjunction of ideal elements, is closed to further extension in this manner. It is shown also that from two or more systems a composite system may be derived, and that the composite system satisfies the postulates if and only if the postulates are satisfied by every component system.

A somewhat less restrictive body of postulates, considered in Chapter II, pertain to a system $(\mathfrak{P}; \mathfrak{U}; T)$, $\mathfrak{P}$ being a class of elements, $\mathfrak{U}$ a class of ideal elements, and T a relation between subclasses of $\mathfrak{P}$ and individual elements of $\mathfrak{P}$ or $\mathfrak{U}$. A subclass $\mathfrak{R}$ of $\mathfrak{P}$ having the relation T to an element p of $\mathfrak{P}$ or to an ideal element u of $\mathfrak{U}$ may be thought of as a generalized neighborhood of p or u. The postulates of Chapter I, with the definition of ideal elements for the system $(\mathfrak{P}; R)$, lead to a system satisfying the postulates of Chapter II. We obtain for our system a generalization of a portion of the theory of point-sets by establishing relations between our postulates and the more general conditions involved in the notion of "limit" as used by Fréchet, and those involved in the "Verdichtungstelle" of F. Riesz.

In the third chapter a system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is supposed to satisfy the postulates of Chapter II, and functions μ defined on the range $\mathfrak{P}$, a subclass of $\overline{\mathfrak{P}}$, are studied relative to limits and continuity. The treatment is not intended to be exhaustive, the theorems developed being such as are suggested by familiar theorems on multiple sequences and functions of real variables. Interesting features of the general theory associate themselves with the presence of ideal elements in the system, and with the study of a property of functions which has much the same force as uniform continuity, but which we have called *extensible* continuity.

The fourth chapter is given to applications of the general theory through direct specialization of the system and particular determination of other arbitrary features. Special systems $(\mathfrak{P}; \mathfrak{U}; T)$ are specified, by consideration of

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* E. R. Hedrick (*Transactions*, Vol. XII (1911), p. 289) obtains by his "inclosable" property of the fundamental domain essentially a generalization of the notion "neighborhood," but his assumptions are made from a different point of view and, involving a certain uniformity, are more restrictive than the postulates of the present paper.

which the theorems of Chapter III pertain to: The theory of multiple sequences; functions of real variables; functions on a range for which there is defined a relation of the type of either of the relations K_1 and K_2 used by Professor Moore; functions on a range subject to the *voisinage* used by Fréchet; and functions on a range whose elements are real-valued functions on an arbitrary range. In some cases the system $(\mathfrak{P}; \mathfrak{U}; T)$ is reached by the mediation of a system $(\mathfrak{P}; R)$, and in some cases directly. In the applications, under certain restrictions on the class $\mathfrak{P}$, the property extensible continuity is found to be equivalent to uniform continuity in each case where the latter is defined.

We find it advantageous to draw largely upon the notation and terminology used by Professor Moore in his work on General Analysis. Convenience and economy of notation are conserved by the adoption of letters for elements, classes, etc., whose connotation renders frequent explanatory remarks unnecessary. Classes of elements are denoted by $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}$, etc., while their elements are denoted by p, q, r, etc., respectively. Classes of classes are, in general, denoted by u, v, w, etc.; properties and relations by P, Q, R, etc., or simply by the numerals attached to their definitions. Superscripts denote, in general, defining properties or conditions, the character of the superscript as well as of the base symbol serving to determine the nature of the limitation. Thus, $\mathfrak{R}^{\mathfrak{P}}$ states that $\mathfrak{R}$ is a subclass of $\mathfrak{P}$, $p^{\mathfrak{R}}$ that p is an element of $\mathfrak{R}$, $\mathfrak{P}^{P}$ that $\mathfrak{P}$ has the property P, etc. The symbol $\supset$ is a sign of implication, to be used in the statement of a proposition. That which precedes the sign of implication is hypothesis or given data, and that which follows is conclusion or a true statement concerning the given data. Thus, if A and B are propositions, $A . \supset . B$ is read "A implies B" or "if A then B," and if x represents a number in a certain interval and F a definite function on the interval, the proposition "for every two numbers x_1 and x_2 of the interval $F(x_1) - F(x_2) < k$" may be written, $x_1 . x_2 . \supset . F(x_1) - F(x_2) < k$. The reversed symbol $\subset$ denotes "is implied by" and $\backsim$ is the symbol of logical equivalence, "implies and is implied by." In a complex statement the symbols $\supset$, $\subset$ and $\backsim$ carry punctuation marks, ., :, .:, etc., the primary implication of the proposition being indicated by the greater number of dots. The mark $\exists$ is read "there exists," and the mark $\ni$ may be read "such that" or "where" as the sense of the proposition demands.

The independent use of the symbolical statement of propositions is confined largely to the proofs of theorems, where it is most useful in conserving precision and brevity, and where the technical symbols may be least objectionable to the general reader.

CHAPTER I.

The System $(\mathfrak{P}; R)$: Extension and Composition of Systems.

§ 1. *Introductory: The System* $(\mathfrak{P}; R)$.

In this chapter we consider a system $(\mathfrak{P}; R)$ consisting of a class $\mathfrak{P}$ of elements p and a relation R on ordered pairs of subclasses of $\mathfrak{P}$. While the relation R is of the definite type indicated, it is not further specifically defined. We specify a system $(\mathfrak{P}; R)$ by specifying the class $\mathfrak{P}$ and the relation R, *i. e.*, a criterion which determines for every two subclasses $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathfrak{P}$ whether or not $\mathfrak{R}_1$ has the relation R to $\mathfrak{R}_2$.

For example, take for $\mathfrak{P}$ the class of all points of an ordinary Euclidean plane. Consider a circle as the class of all points within and on its circumference, then we may specify a relation R in terms of geometry as follows: Every circle whose radius is different from zero has the relation R to the point at its center considered as singular subclass, and every two concentric circles whose radii are different from zero have the relation R to each other. In no other case does the relation R hold.

In this example we have a definite system $(\mathfrak{P}; R)$. The pertinence of the relation R as specified to the study of limits of functions defined for a set of points in the plane is obvious. A study of the current theory of real-valued functions, in particular in connection with questions of continuity and iterated limits, leads to a determination of bodies of postulates on systems $(\mathfrak{P}; R)$ which serve to validate a theory of continuous functions and multiple and iterated limits associated with such systems $(\mathfrak{P}; R)$ in general.

Subclasses of $\mathfrak{P}$ are, in general, denoted by $\mathfrak{R}$, and the notation $\mathfrak{R}_1 R \mathfrak{R}_2$ indicates that $\mathfrak{R}_1$ has the relation R to $\mathfrak{R}_2$, while $\mathfrak{R}_1 {}^{-}R \mathfrak{R}_2$ indicates that $\mathfrak{R}_1$ does not have the relation R to $\mathfrak{R}_2$. In case it is desired to imply that a subclass consists of a single element, we may for simplicity, and for our purposes without confusion, use the notation for single elements. Thus $\mathfrak{R} R p$ indicates that the class $\mathfrak{R}$ has the relation R to the singular subclass whose element is p. The letter v denotes a class of subclasses $\mathfrak{R}$ of $\mathfrak{P}$, and, for a given element p, v_p is the class of all subclasses $\mathfrak{R}$ having the relation R to p, *i. e.*,

$$v_p = [\text{all } \mathfrak{R} \ni \mathfrak{R} R p].$$

Thus, in the example above, v_p is the family of concentric circles whose common center is at the point p, excluding the point circle of the family.

§ 2. *The Postulates and Certain Fundamental Definitions.*

Preliminary to the statement of postulates for a system $(\mathfrak{P}; R)$, we note that a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$ may have one or more of the following properties:

1. Every member $\mathfrak{R}$ of the class v contains at least one element p.

2. The relation R holds between every two classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ that are members of v.

3. There exists a sequence $\{\mathfrak{R}_n\}$ of members of the class v such that for every $\mathfrak{R}$ of v there is a number $n_{\mathfrak{R}}$ such that for $n > n_{\mathfrak{R}}$ the class $\mathfrak{R}_n$ is contained in $\mathfrak{R}$.

4. For every $\mathfrak{R}$ of v there exists an $\mathfrak{R}_1$ of v such that for every p in $\mathfrak{R}_1$ there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ having the relation R to the singular class p.

5. If v_1 is a class containing v and having properties 1, 2, 3 and 4, then $v = v_1$.

6. If v_1 is a class having properties 1, 2, 3, 4 and 5, and not containing v, then there exists a member $\mathfrak{R}_1$ of v_1 and a member $\mathfrak{R}_2$ of v such that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have no common elements.

7. For every element p of $\mathfrak{P}$ there is an $\mathfrak{R}$ of v which does not contain p.

These definitions may be more concisely stated in symbols as follows:

1. $\mathfrak{R}^v . \supset . \exists \, p^{\mathfrak{R}}$.
2. $\mathfrak{R}_1^v . \mathfrak{R}_2^v . \supset . \mathfrak{R}_1 R \mathfrak{R}_2$.
3. $\exists \{\mathfrak{R}_n\} \ni [(n . \supset . \mathfrak{R}_n^v) . (\mathfrak{R}^v : \supset : \exists \, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} . \supset . \mathfrak{R}_n^{\mathfrak{R}})]$.
4. $\mathfrak{R}^v : \supset : \exists \, \mathfrak{R}_1^v \ni p^{\mathfrak{R}_1} . \supset . \exists \, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R p$.
5. $v^{v_1} . v^{1.2.3.4} . \supset . v = v_1$.

6.* $v_1^{1.2.3.4.5} . v^{-v_1} . \supset . \exists \, (\mathfrak{R}_1^{v_1} . \mathfrak{R}_2^v) \ni {}^{-}\exists \, p \ni p^{\mathfrak{R}_1} . p^{\mathfrak{R}_2}$.

7. $p . \supset . \exists \, \mathfrak{R}^v \ni p^{-\mathfrak{R}}$.

These properties, 1–7, may be called propositional properties.† It is not here asserted that any of the defining propositions are true with respect to any class v, but it is clear that the question whether or not a given one of these propositions is true with respect to a given class v is a question of the presence or absence of a definite property for the class.

The desired postulates might now be stated in the following form:

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* The minus sign here signifies negation. Thus ${}^{-}\exists$ is read "there does not exist," and $p^{-\mathfrak{R}}$ indicates that p is not an element of the subclass $\mathfrak{R}$. v^{-v_1} indicates that v is not a subclass of v_1.

† See E. H. Moore, *loc. cit.*, p. 20.

(A) *For every element p the class v_p has properties* 1–6, *i. e.*,

$$p \,.\supset.\, v_p^{1.2.3.4.5.6}.$$

(B) *For every element p it is true that every $\mathfrak{R}$ of v_p contains p, while if p_1 is distinct from p there is an $\mathfrak{R}$ of v_p not containing p_1.* In symbols:

$$p :\supset: (\mathfrak{R} R p \,.\supset.\, p^{\mathfrak{R}}) \,.\, (p_1 \neq p \,.\supset.\, \exists\, \mathfrak{R} \ni \mathfrak{R} R p \,.\, p_1^{-\mathfrak{R}}).$$

But for convenience of reference, as well as to provide for discussion of the independence of the conditions on the system, we separate these assumptions into simpler components, which we state explicitly in the following seven postulates:

I. $\mathfrak{R} R p \,.\supset.\, p^{\mathfrak{R}}$.

II. $\mathfrak{R}_1 R p \,.\, \mathfrak{R}_2 R p \,.\supset.\, \mathfrak{R}_1 R \mathfrak{R}_2$.

III. $p \,.:\supset:.\, \exists \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n R p) \,.\, (\mathfrak{R} R p :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})]$.

IV. $\mathfrak{R} R p :\supset: \exists\, \mathfrak{R}_1 \ni [\mathfrak{R}_1 R p \,.\, (p_1^{\mathfrak{R}_1} \,.\supset.\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R p_1)]$.

V. $v^{1.2.3.4} \,.\, (\mathfrak{R} R p \,.\supset.\, \mathfrak{R}^{v}) :\supset: v = v_p$.

VI. $v^{1.2.3.4.5} \,.\, v_p^{-v} \,.\supset.\, \exists\, (\mathfrak{R}_1^{v} \,.\, \mathfrak{R}_2^{v_p}) \ni {}^{-}\exists\, p_1 \ni p_1^{\mathfrak{R}_1} \,.\, p_1^{\mathfrak{R}_2}$.

VII. $p_1 \neq p \,.\supset.\, \exists\, \mathfrak{R}\, \epsilon\, \mathfrak{R} R p \,.\, p_1^{-\mathfrak{R}}$.

Postulates I and VII are together equivalent to the statement (B). A corollary of postulate I is that for every p the class v_p has property 1, while postulates II–VI state that for every p the class v_p has the respective properties 2–6.

The following examples are pertinent to the question of consistency and independence of the postulates. Example 0 is an instance of a system satisfying the seven postulates, and the remaining examples each violate one postulate and satisfy all the others, the examples being numbered in the order of the postulates violated.

Ex. 0. The class $\mathfrak{P}$ is the class of all complex numbers. The notation $\mathfrak{R}_{dp}$, where d is a positive real number and p is an element of $\mathfrak{P}$, stands for the subclass of $\mathfrak{P}$ consisting of all elements p_1 of $\mathfrak{P}$ such that $|p_1 - p| \leqq d$, that is,

$$\mathfrak{R}_{dp} \equiv [\text{all } p_1 \ni |p_1 - p| \leqq d].$$

The relation R is specified as follows: For every p and every d the relation $\mathfrak{R}_{dp} R p$ holds, and for every p and every d_1 and d_2 the relation $\mathfrak{R}_{d_1 p} R \mathfrak{R}_{d_2 p}$ holds. The relation holds in no other case.

Ex. 1. The system $(\mathfrak{P}; R)$ is specified as in example 0, except that $\mathfrak{R}_{dp}$ does not contain the element p, hence

$$R_{dp} \equiv [\text{all } p_1 \neq p \ni |p_1 - p| \leqq d].$$

Ex. 2. The class $\mathfrak{P}$ is the class of all complex numbers, and the notation $\mathfrak{R}_{dp}$ has the same significance as in example 0. For every p and for every d the relation $\mathfrak{R}_{dp} R p$ holds, but in no other case does the relation R hold.

In this example postulates V and VI are satisfied vacuously, *i. e.*, their hypotheses are incapable of fulfilment, there being no class v which has properties 2 and 3.

Ex. 3. The class $\mathfrak{P}$ is the class of all points of a given Euclidean plane. The designation "line" is used to indicate a subclass $\mathfrak{R}$ constituting the class of all points of a line. Every "line" has the relation R to every one of its points, and every two intersecting "lines" have the relation R to each other. In no other case does the relation R hold.

Here "intersecting" is interpreted as "having a point in common," so that a "line" has the relation R to itself. Postulates V and VI are again satisfied vacuously.

Ex. 4. The system $(\mathfrak{P}; R)$ is as specified in example 0, except that in the particular case $p = 0$ the classes R_{dp} consist only of real elements p_1, *i. e.*,

$$\mathfrak{R}_{d0} \equiv [\text{all real } p_1 \ni |p_1| \leqq d].$$

Ex. 5. Again, the class $\mathfrak{P}$ is the class of all complex numbers, and the notation $\mathfrak{R}_{dp}$ has the same significance as in example 0. The relation R is as specified in example 0 except that for the particular element $p_0 = 0$ the relation $\mathfrak{R}_{dp_0} R p_0$ holds only in case d is less than or equal to unity.

Ex. 6. The class $\mathfrak{P}$ consists of two elements, p_1 and p_2.* The cases in which the relation R holds are listed as follows:

$$p_1 R p_1, \quad p_2 R p_2, \quad \mathfrak{P} R \mathfrak{P}.$$

Ex. 7. Again, $\mathfrak{P}$ is a class consisting of two elements, p_1 and p_2. Following is the list of cases in which the relation R holds:

$$\mathfrak{P} R p_1, \quad \mathfrak{P} R p_2, \quad \mathfrak{P} R \mathfrak{P}.$$

In this instance postulate VI is satisfied vacuously, since any class v that has properties 1–5 possesses the single member $\mathfrak{P}$, and is therefore coincident with both v_{p_1} and v_{p_2}.

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* It should be remembered that elements do not enter in the relation R. The notation for elements is substituted for class notation as a matter of convenience. The class v_{p_1} consists of one member, the class $\mathfrak{R}$ having the single element p_1, and has no member in common with that class v whose only member is $\mathfrak{P}$.

§ 3. *Extension of the System by the Adjunction of Ideal Elements*

Making use of properties 1–7 defined in § 2, properties that may be possessed by a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$, we proceed to the definition of ideal elements for the system $(\mathfrak{P}; R)$.

Def. 1. An *ideal element* of the system $(\mathfrak{P}; R)$ is a class v of subclasses $\mathfrak{R}$ of $\mathfrak{P}$ having properties 1–7.

The letter u invariably stands for an ideal element.

THEOREM I. *If v is a class having properties* 1–6, *then v is an ideal element u, or there is an element p such that $v = v_p$.*

Proof: If v has property 7, it is a u by definition; if it has not property 7, then there is an element p common to all classes $\mathfrak{R}$ of v. Since v_p has properties 1–6, we clearly have $v = v_p$.

Let $\mathfrak{U}$ denote the class of all ideal elements of the system $(\mathfrak{P}; R)$, and let $\mathfrak{Q}$ be a class consisting of the elements of $\mathfrak{P}$, together with all ideal elements, *i. e.*, $\mathfrak{Q} \equiv \mathfrak{P} + \mathfrak{U}$. We denote elements of $\mathfrak{Q}$, in general, by q, subclasses of $\mathfrak{Q}$ by $\mathfrak{S}$, and classes of subclasses by w. A certain technical form of correspondence between classes is of frequent occurrence, and it is therefore convenient to adopt a special symbol, $\|$, to be read *corresponds to*, which we define as follows:

Def. 2. $\mathfrak{S} \| \mathfrak{R}$ indicates that $\mathfrak{S}$ consists of the elements of $\mathfrak{R}$ together with every ideal element u such that there is a subclass $\mathfrak{R}_1$ of $\mathfrak{R}$ which belongs to the class* u. In symbols:

$$\mathfrak{S} \| \mathfrak{R} : \equiv : \mathfrak{S} = \mathfrak{R} + [\text{all } u \ni \exists\, \mathfrak{R}_1^{\mathfrak{R}} \ni \mathfrak{R}_1^{u}].$$

Def. 3. $w \| v$ indicates that w consists of all classes $\mathfrak{S}$ for which there exist classes $\mathfrak{R}$ in v such that $\mathfrak{S} \| \mathfrak{R}$. In symbols:

$$w \| v : \equiv : w = [\text{all } \mathfrak{S} \ni \exists\, \mathfrak{R}^{v} \ni \mathfrak{S} \| \mathfrak{R}].$$

It is obvious that for every $\mathfrak{R}$ there is a unique $\mathfrak{S}$ such that $\mathfrak{S} \| \mathfrak{R}$, and that for two distinct classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the corresponding classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are distinct. It follows that for every v there is a unique w, and that for two distinct classes v_1 and v_2 the corresponding w_1 and w_2 are distinct.

Let S be a relation of the same type as R defined as follows:

Def. 4. The relation $\mathfrak{S}_1 S \mathfrak{S}_2$ holds if and only if one of the following conditions is fulfilled:

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* No confusion need arise from the fact that the letter u denotes at the same time an element of $\mathfrak{Q}$ and a class of subclasses of $\mathfrak{Q}$, as well as of $\mathfrak{P}$. It was this double rôle that led to the adoption of small letters as notation for classes of subclasses in general.

(a) $\exists\,(\mathfrak{R}_1 \,.\, \mathfrak{R}_2) \ni \mathfrak{R}_1 R \mathfrak{R}_2 \,.\, \mathfrak{S}_1 \parallel \mathfrak{R}_1 \,.\, \mathfrak{S}_2 \parallel \mathfrak{R}_2 .$

(b) $\exists\,(\mathfrak{R} \,.\, u) \ni \mathfrak{R}^u \,.\, \mathfrak{S}_1 \parallel \mathfrak{R} \,.\, (q^{\mathfrak{S}_2} \,.\supset.\, q = u).$

In condition (b) the ideal element u considered as a class of subclasses contains $\mathfrak{R}$, and considered as an element of $\mathfrak{Q}$ constitutes the singular subclass $\mathfrak{S}_2$.

We now have a definite system $(\mathfrak{Q}; S)$, which we shall call the *extended system* derived from $(\mathfrak{P}; R)$. We investigate the character of this extended system with respect to the seven postulates. The properties 1–7 defined for a class v are defined also for a class w, if in the notation we replace v by w, p by q, $\mathfrak{R}$ by $\mathfrak{S}$ and R by S, and with similar changes of notation we have the seven postulates stated for the system $(\mathfrak{Q}; S)$.

THEOREM II. *The seven postulates are satisfied by the extended system* $(\mathfrak{Q}; S)$.

In proving this theorem it it convenient to establish first the following lemma:

LEMMA. *The necessary and sufficient condition that a class w shall have properties* 1–4 *or* 1–5 *or* 1–6 *is that there shall exist a class v having the corresponding properties such that $w \parallel v$.*

In considering the necessity of the condition we have available in each case the fact that w has property 2. This is sufficient to secure the existence of a v such that $w \parallel v$. It is sufficient, then, to assume a definite w corresponding to a definite v and prove the following propositions:

(a)	$w^{1.2.3.4} \,.\supset.\, v^{1.2.3.4}$,	(b)	$v^{1.2.3.4} \,.\supset.\, w^{1.2.3.4}$,
(c)	$w^{1.2.3.4.5} \,.\supset.\, v^{5}$,	(d)	$v^{1.2.3.4.5} \,.\supset.\, w^{5}$,
(e)	$w^{1.2.3.4.5.6} \,.\supset.\, v^{6}$,	(f)	$v^{1.2.3.4.5.6} \,.\supset.\, w^{6}$.

(a): Since w has property 1, every $\mathfrak{S}$ of w contains a q, that is, either a p or a u. Every $\mathfrak{R}$ of v has a corresponding $\mathfrak{S}$ in w, hence it contains either this element p or an $\mathfrak{R}_1$ of this class u. Therefore v has property 1. That v has property 2 is evident from definition 4, and from property 3 of w there is a sequence $\{\mathfrak{S}_n\}$ such that the sequence $\{\mathfrak{R}_n\}$, where $\mathfrak{S}_n \parallel \mathfrak{R}_n$, is effective in establishing property 3 for v. Since w has property 4, we have

(1) $$\mathfrak{S}^w :\supset: \exists\, \mathfrak{S}_1^w \ni q^{\mathfrak{S}_1} \,.\supset.\, \exists\, \mathfrak{S}_2^{\mathfrak{S}} \ni \mathfrak{S}_2 S q,$$

and we wish to prove

(2) $$\mathfrak{R}^v :\supset: \exists\, \mathfrak{R}_1^v \ni p^{\mathfrak{R}_1} \,.\supset.\, \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2 R p.$$

Given an $\mathfrak{R}$ of v, take $\mathfrak{S}$ such that $\mathfrak{S} \parallel \mathfrak{R}$, then $\mathfrak{S}^{w}$ and (1) applies. Take $\mathfrak{R}_1$ so that $\mathfrak{S}_1 \parallel \mathfrak{R}_1$, then $\mathfrak{R}_1^{v}$ and every p of $\mathfrak{R}_1$ is in $\mathfrak{S}_1$. For every p of $\mathfrak{R}_1$, then, there is a subclass $\mathfrak{S}_2$ of $\mathfrak{S}$ such that $\mathfrak{S}_2 S p$, and there is an $\mathfrak{R}_2$ such that $\mathfrak{S}_2 \parallel \mathfrak{R}_2$, then clearly $\mathfrak{R}_2^{\mathfrak{R}}$ and $\mathfrak{R}_2 R p$. Thus v has property 4.

(b): The proof is obvious for properties 1, 2 and 3. As to property 4, we have condition (2) above and wish to prove (1). Given an $\mathfrak{S}$ of w, there is a corresponding $\mathfrak{R}$ in v, so that (2) applies to provide an $\mathfrak{R}_1$ fulfilling the conclusion of (2). Take $\mathfrak{S}_1$ to correspond to $\mathfrak{R}_1$, then $\mathfrak{S}_1^{w}$ and every p in $\mathfrak{S}_1$ is in $\mathfrak{R}_1$ and an $\mathfrak{S}_2$ corresponding to an $\mathfrak{R}_2$ furnished by (2) meets the requirements of (1). Further, every u in $\mathfrak{S}_1$ possesses a member $\mathfrak{R}_2$ which is a subclass of $\mathfrak{R}_1$, and since $\mathfrak{R}_1$ is necessarily a subclass of $\mathfrak{R}$, we see that the class $\mathfrak{S}_2$ corresponding to $\mathfrak{R}_2$ is a subclass of $\mathfrak{S}$; and clearly $\mathfrak{S}_2 S u$, therefore w has property 4.

Propositions (c) and (d) are easily verified by the use of (a) and (b).

(e): From property 6 of w we have

$$(3) \qquad w_1^{1.2.3.4.5} \,.\, w^{-w_1} \,.\, \supset \,.\, \exists\, (\mathfrak{S}_1^{w_1} \,.\, \mathfrak{S}_2^{w}) \ni -\exists\, q \ni q^{\mathfrak{S}_1} \,.\, q^{\mathfrak{S}_2},$$

and we wish to prove

$$(4) \qquad v_1^{1.2.3.4.5} \,.\, v^{-v_1} \,.\, \supset \,.\, \exists\, (\mathfrak{R}_1^{v_1} \,.\, \mathfrak{R}_2^{v}) \ni -\exists\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}.$$

If v_1 has properties 1–5 and does not contain v, then there is a w_1 such that $w_1 \parallel v_1$ which does not contain w and which, by (b) and (d), has properties 1–5. Proposition (3) is now applicable, and the $\mathfrak{S}_1$ and $\mathfrak{S}_2$ thus available have corresponding classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ which obviously fulfil the conclusion of (4).

The proof of (f) is similar to that of (e).

The proof of the theorem is now easily completed. In analogy with previous notation, we denote by w_q the class of all classes $\mathfrak{S}$ such that $\mathfrak{S} S q$, and we observe that for every p we have $w_p \parallel v_p$, while for every u we have $w_u \parallel u$. Since every v_p and every u have properties 1–6, it follows from the lemma that the class w_q has properties 1–6. Postulate I being obviously fulfilled, it remains to consider postulate VII. We wish to show that

$$q_1 \neq q_2 \,.\, \supset \,.\, \exists\, \mathfrak{S} \ni (\mathfrak{S} S q_1 \,.\, q_2^{-\mathfrak{S}}).$$

If both q_1 and q_2 are elements of $\mathfrak{P}$, postulate VII on the system $(\mathfrak{P}; R)$ assures us of a class $\mathfrak{R}$ such that the corresponding class $\mathfrak{S}$ is effective. If either q_1 or q_2 is a u, then, since by property 7 of u and postulate I the class u is not contained in any class v_p, the corresponding class w_u is not contained in

any class w_p, and therefore property 6 of w_u is effective, the desired conclusion being an immediate consequence.

It is desirable now to show that the extended system $(\mathfrak{Q}; S)$ is closed to this process of extension; that is, if we repeat the process of extension the second extended system coincides with the first. We may state the required theorem in the form:

THEOREM III. *No ideal elements arise in the extended system* $(\mathfrak{Q}; S)$.

Proof: Suppose a class w to have properties 1–6, then there is a class v having properties 1–6 such that $w \parallel v$. By theorem I v is a u or a v_p, and in either case w is a w_q and therefore does not have property 7.

Following is an instance of a system which illustrates effectively the operation of the foregoing definition of extension:

Example. The class $\mathfrak{P}$ is the class of all rational numbers. If p_1 and p_2 are two distinct rational numbers, then, if p_1 is less than p_2, the class $\mathfrak{R}_{p_1 p_2}$ is the class of all rational numbers on the interval $p_1 p_2$. That is,

$$\mathfrak{R}_{p_1 p_2} \equiv [\text{all } p \ni p_1 \leqq p \leqq p_2].$$

The relation $\mathfrak{R}_{p_1 p_2} R \mathfrak{R}_{p_3 p_4}$ holds if and only if the intervals $p_1 p_2$ and $p_3 p_4$ have a common sub-interval, *i. e.*, if $p_1 < p_4$ and $p_3 < p_2$; and the relation $\mathfrak{R}_{p_1 p_2} R p$ holds if and only if $p_1 < p < p_2$. In no other case does the relation R hold.

It is not difficult to see that the system $(\mathfrak{P}; R)$ here specified satisfies the seven postulates. We proceed, therefore, to investigate the matter of ideal elements. Consider a class v having properties 1–6. By a little attention to the requirements of properties 3 and 4 we see that there exists a sequence $\{\mathfrak{R}_n\}$ of members of v such that for every n the class $\mathfrak{R}_n$ is of the form $\mathfrak{R}_{\underline{p}_n \bar{p}_n}$, where the sequence $\{\underline{p}_n\}$ is an increasing monotonic sequence of distinct elements and the sequence $\{\bar{p}_n\}$ is a decreasing monotonic sequence of distinct elements; and, further, such that every member $\mathfrak{R}$ of v contains a member $\mathfrak{R}_n$ of the sequence. In view of property 5, then, the two sequences $\{\underline{p}_n\}$ and $\{\bar{p}_n\}$ have a common limit. If this limit is a rational number p, then v coincides with v_p, while if the limit is an irrational number a, then v consists of all classes $\mathfrak{R}_{p_1 p_2}$ such that $p_1 < a < p_2$. In the latter case v has property 7 and is an ideal element of the system $(\mathfrak{P}; R)$. Since it is obvious that for every irrational number a the class v consisting of all classes $\mathfrak{R}_{p_1 p_2}$ such that $p_1 < a < p_2$ has properties 1–7, we see that the ideal elements u of the system $(\mathfrak{P}; R)$ are in reciprocal one-to-one correspondence with the irrational numbers in such fashion that, if u corresponds to a, then

$$u \equiv [\text{all } \mathfrak{R}_{p_1 p_2} \ni p_1 < a < p_2].$$

We may therefore consider our definition of ideal elements, in this instance, as a definition of irrational numbers.

The extended system $(\mathfrak{Q}; S)$ is seen to be as follows: $\mathfrak{Q}$ is the class of all real numbers; two intervals with rational limits that have a common interior element have the relation S to each other, and every interval with rational limits has the relation S to every one of its interior elements (considered as singular class), but in no other case does the relation S hold.

§ 4. *Composition of Systems.**

Two classes of elements, $\mathfrak{P}'$ and $\mathfrak{P}''$, determine a "product" or composite class, $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, consisting of all elements p of the form $p = (p', p'')$, where p' and p'' belong to the classes $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively. It should be borne in mind that these bi-partite elements, $p = (p', p'')$, which we shall denote simply by $p'p''$, are not in any sense products of the elements p' and p'', but rather that p is a notation for the pair $p'p''$. From $\mathfrak{R}'$ and $\mathfrak{R}''$, subclasses of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, we have $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, a subclass of $\mathfrak{P}$ consisting of all elements p of the form $p = p'p''$, where p' and p'' belong to $\mathfrak{R}'$ and $\mathfrak{R}''$ respectively. Similarly, if v' is a class of subclasses $\mathfrak{R}'$ of $\mathfrak{P}'$, and v'' is a class of subclasses $\mathfrak{R}''$ of $\mathfrak{P}''$, we have the composite class $v = v'v''$, the class of all $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, where $\mathfrak{R}'$ and $\mathfrak{R}''$ belong to the respective classes v' and v''.

If R' and R'' are relations of the type discussed in § 1, defined for $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, then from the two systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$ we derive what we shall call the composite of these two systems, the system $(\mathfrak{P}; R)$, where $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and R is a relation of the same type as R' and R'' defined as follows:

Def. 4. For a system $(\mathfrak{P}; R)$, composite of $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, the relation $\mathfrak{R}_1 R \mathfrak{R}_2$ holds if and only if there exist classes $\mathfrak{R}_1'$, $\mathfrak{R}_1''$, $\mathfrak{R}_2'$ and $\mathfrak{R}_2''$ such that $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$, and such that the relations $\mathfrak{R}_1' R' \mathfrak{R}_2'$ and $\mathfrak{R}_1'' R'' \mathfrak{R}_2''$ hold.

It may be observed that the effectiveness of the foregoing definition is independent of the conditions imposed upon the component systems. It is convenient throughout the present section to regard the systems involved as unconditioned, except as conditions are specified in the hypotheses of the several theorems.

THEOREM IV. *The seven postulates are satisfied by the system* $(\mathfrak{P}; R)$, *composite of* $(\mathfrak{P}'; R')$ *and* $(\mathfrak{P}''; R'')$, *if and only if they are satisfied by both component systems.*

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* Compare T. H. Hildebrandt, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 250.

We consider first the following lemma:

LEMMA I. *If $v = v'v''$, then the following propositions hold:*

(a) $v^{1.2.3.4} . \supset . v'^{1.2.3.4} . v''^{1.2.3.4}$, (b) $v'^{1.2.3.4} . v''^{1.2.3.4} . \supset . v^{1.2.3.4}$,

(c) $v^{1.2.3.4.5} . \supset . v'^{5} . v''^{5}$, (d) $v'^{1.2.4.4.5} . v''^{1.2.3.4.5} . \supset . v^{5}$,

(e) $v^{1.2.3.4.5.6} . \supset . v'^{6} . v''^{6}$, (f) $v'^{1.2.3.4.5.6} . v''^{1.2.3.4.5.6} . \supset . v^{6}$.

(a) and (b) are sufficiently obvious if only we remember the significance of the relation $v = v'v''$.

(c): Suppose a class v_1' to contain v' and to have properties 1–4. The composite class $v_1 = v_1'v''$ then contains v and, by (a) and (b), has properties 1–4; by property 5 of v then $v_1 = v$, and therefore $v_1' = v'$. In similar manner it may be shown that v'' has property 5.

(d): Suppose v_1 to contain v and to have properties 1–4, and consider the two classes v_1' and v_1'' defined as follows:

$$v_1' = [\text{all } \mathfrak{R}' \ni \exists\, (\mathfrak{R}'' . \mathfrak{R}^{v_1}) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''],$$
$$v_1'' = [\text{all } \mathfrak{R}'' \ni \exists\, (\mathfrak{R}' . \mathfrak{R}^{v_1}) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''].$$

It is not difficult to see that these classes have properties 1–4, and also that v_1' contains v' and v_1'' contains v''; then by property 5 of v' and v'', we have $v_1' = v'$ and $v_1'' = v''$, and consequently $v_1 = v$.

(e): Suppose v_1' has properties 1–5 and $v'^{-v_1'}$, then if $v_1 = v_1'v''$ we have $v_1^{1.2.3.4.5}$ and v^{-v_1}; hence by property 6 of v,

$$\exists\, (\mathfrak{R}_1^{v_1} . \mathfrak{R}_2^{v}) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} . p^{\mathfrak{R}_2}.$$

The classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are of the forms $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$, where $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ are members of v_1' and v' respectively, and $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ are members of v''. By properties 1 and 3 of v'', $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ have common elements, therefore $\mathfrak{R}_1'$ and $\mathfrak{R}_2'$ can have no elements in common, and consequently v' has property 6. In like manner it may be shown that v'' has property 6.

(f): Suppose $v_1^{1.2.3.4.5}$ and v^{-v_1}; take v_1' and v_1'' defined as in the proof of (d) above, then as before they have properties 1–4, and by (b) the composite class $\bar{v}_1 = v_1'v_1''$ has these four properties. Since $\bar{v}_1$ clearly contains v_1, and v_1 has property 5, we have $\bar{v}_1 = v_1$; hence by (c) v_1' and v_1'' have property 5. Since v is not contained in v_1, either v' is not contained in v_1', or v'' is not contained in v_1''; suppose the former, then since v' has property 6, we have

$$\exists\, (\mathfrak{R}_1'^{v_1'} . \mathfrak{R}_2'^{v'}) \ni {}^{-}\exists\, p' \ni p'^{\mathfrak{R}_1'} . p'^{\mathfrak{R}_2'}.$$

Take $\mathfrak{R}_1''$ and $\mathfrak{R}_2''$ so that $\mathfrak{R}_1''$ is a member of v_1'' and $\mathfrak{R}_2''$ is a member of v'', and

take $\mathfrak{R}_1 = \mathfrak{R}_1'\mathfrak{R}_1''$ and $\mathfrak{R}_2 = \mathfrak{R}_2'\mathfrak{R}_2''$; then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are members of v_1 and v_2 respectively, and clearly they can have no common elements. Thus we show that v has property 6.

We record here for future reference certain results reached incidentally in the foregoing proofs:

LEMMA II. (a) *If v has properties* 1–4 *and*

$$v' = [\text{all } \mathfrak{R}' \ni \exists (\mathfrak{R}'' \,.\, \mathfrak{R}^v) \ni \mathfrak{R} = \mathfrak{R}'\mathfrak{R}''],$$

then v' has properties 1–4.

(b) *If v has properties* 1–5, *then there exist classes v' and v'' such that $v = v'v''$.*

Taking up now the proof of the theorem, we assume that the composite system $(\mathfrak{P}; R)$ satisfies the seven postulates and show* that as a consequence both component systems satisfy them. For a given p' there exist elements p and p'' such that $p = p'p''$ and $v_p = v'_{p'}v''_{p''}$, and since every $\mathfrak{R}$ of v_p contains p, clearly every $\mathfrak{R}'$ of $v'_{p'}$ contains p', so that postulate I is satisfied by $(\mathfrak{P}'; R')$. It follows also from lemma I that postulates II–VI are satisfied by this system. If $p_1' \neq p_2'$, then taking $p_1 = p_1'p''$ and $p_2 = p_2'p''$, where p'' is any element of $\mathfrak{P}''$, we have $p_1 \neq p_2$; hence there is an $\mathfrak{R}$ not containing p_2 such that $\mathfrak{R}\, R\, p_1$. But such an $\mathfrak{R}$ is of the form $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, where $\mathfrak{R}'\, R'\, p_1'$ and $\mathfrak{R}''\, R''\, p''$; and since p_1 is in $\mathfrak{R}$, we see that p_1' is in $\mathfrak{R}'$ and p'' is in $\mathfrak{R}''$. Clearly then, p_2' is not in $\mathfrak{R}'$, so that the system $(\mathfrak{P}'; R')$ satisfies postulate VII. In like manner the system $(\mathfrak{P}''; R'')$ is shown to satisfy the seven postulates.

The remainder of the theorem, *i. e.*, that if the postulates are satisfied by both component systems then they are satisfied by the composite system, is sufficiently evident without detailed discussion.

From the system $(\mathfrak{P}; R)$, composite of the two systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$, we may derive an extended system $(\mathfrak{Q}; S)$ by the process defined in § 3; or we may take first the extended systems $(\mathfrak{Q}'; S')$ and $(\mathfrak{Q}''; S'')$ and form their composite system, which we denote by $\bar{\mathfrak{Q}}; \bar{S})$. It is desirable to compare the two systems $(\mathfrak{Q}; S)$ and $(\bar{\mathfrak{Q}}; \bar{S})$ thus derived. Let it be assumed that the systems $(\mathfrak{P}'; R')$ and $(\mathfrak{P}''; R'')$ satisfy the seven postulates; then the systems $(\mathfrak{Q}; S)$ and $(\bar{\mathfrak{Q}}; \bar{S})$ also satisfy them. The class $\bar{\mathfrak{Q}}$ consists of all elements of the four forms

$$(1)\ \bar{q} = p'p'', \qquad (2)\ \bar{q} = u'u'', \qquad (3)\ \bar{q} = p'u'', \qquad (4)\ \bar{q} = u'p''.$$

* It is necessary to exclude the trivial case in which the class $\mathfrak{P}$ has no elements.

Now all elements $p = p'p''$ belong to $\mathfrak{P}$ and are therefore in $\mathfrak{Q}$; also, since all classes u' and u'' have properties 1–7, every class $u = u'u''$ has properties 1–7 and is an ideal element of $(\mathfrak{P}; R)$ and consequently appears in $\mathfrak{Q}$. Further, it is clear that a class v of the form $v'_{p'}u''$ or $u'v''_{p''}$ has properties 1–7 and is therefore an element of $\mathfrak{Q}$.

Conversely, every element of $\mathfrak{Q}$ is either a p or a u; every p is of the form $p = p'p''$ and therefore is in $\mathfrak{Q}$; every u is a class having properties 1–7, hence, by lemma II above, is of the form $u = v'v''$; and by lemma I above v' and v'' have properties 1–6; therefore by theorem I v' is a $v'_{p'}$ or a u, and v'' is a $v''_{p''}$ or a u'', so that every u of $\mathfrak{Q}$ is of the form $u'u''$ or $v'_{p'}u''$ or $u'v''_{p''}$.

We arrive, then, at the following theorem:

Theorem V. *If from two systems,* $(\mathfrak{P}'; R')$ *and* $(\mathfrak{P}''; R'')$, *which satisfy the seven postulates we derive* $(\mathfrak{Q}; S)$ *by composition then extension, and* $(\overline{\mathfrak{Q}}; \bar{S})$ *by extension then composition, the two systems* $(\mathfrak{Q}; S)$ *and* $(\overline{\mathfrak{Q}}; \bar{S})$ *are related as follows:*

(a) *The elements* q *of* $\mathfrak{Q}$ *are in reciprocal one-to-one correspondence with the elements* $\bar{q}$ *of* $\overline{\mathfrak{Q}}$ *in such manner that, if* q *corresponds to* $\bar{q}$, *then* $q = p'p''$ *and* $\bar{q} = p'p''$, *or* q *is of the form* $u = u'u''$, *where* $\bar{q} = u'u''$, *or of the form* $u = v'_{p'}u''$, *where* $\bar{q} = p'u''$, *or of the form* $u = u'v''_{p''}$, *where* $\bar{q} = u'p''$.

(b) *If under the correspondence of* (a) q *corresponds to* $\bar{q}$, *then the classes* $\mathfrak{S}$ *such that* $\mathfrak{S}Sq$ *are in reciprocal one-to-one correspondence with the classes* $\overline{\mathfrak{S}}$ *such that* $\overline{\mathfrak{S}}\bar{S}\bar{q}$ *in such manner that, if* $\mathfrak{S}$ *corresponds to* $\overline{\mathfrak{S}}$, *then under the correspondence of* (a) *the elements of* $\mathfrak{S}$ *correspond to elements of* $\overline{\mathfrak{S}}$, *and those elements of* $\overline{\mathfrak{S}}$ *which do not correspond to elements of* $\mathfrak{S}$ *are of the form* $p'u''$ *or* $u'p''$.

For a given $\bar{q}$ a class $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}}\bar{S}\bar{q}$ must be of the form $\overline{\mathfrak{S}} = \mathfrak{S}'\mathfrak{S}''$, and if $\mathfrak{S}'$ contains an element p' for which there is no subclass $\mathfrak{R}'$ of $\mathfrak{S}'$ such that $\mathfrak{R}'R'p'$, it is obvious that for every u'' in $\mathfrak{S}''$ the element $p'u''$ is in $\overline{\mathfrak{S}}$, while the corresponding element $v'_{p'}u''$ is not in the class $\mathfrak{S}$ that corresponds to $\overline{\mathfrak{S}}$ under the correspondence of (b).*

* An important fact to be noted here is that it is with respect to the relations S and $\bar{S}$ that this discrepancy appears. Our purpose in defining a relation S and an extended system $(\mathfrak{Q}; S)$ is to show more completely than would otherwise be possible the operation of our definition of ideal elements. We make no further use of relations S, but treat ideal elements u as associated with systems $(\mathfrak{P}; R)$. A considerable simplification is introduced in Chapter II. It is sufficient for our purpose that if $\mathfrak{U}'$ is the class of ideal elements arising in $(\mathfrak{P}'; R')$ and $\mathfrak{U}''$ is the class of ideal elements arising in $(\mathfrak{P}''; R'')$, then the class $\mathfrak{U}$ of ideal elements arising in the composite system $(\mathfrak{P}; R)$ may be regarded as identical with the sum of the composite classes $\mathfrak{U}'\mathfrak{U}''$, $\mathfrak{U}'\mathfrak{B}''$ and $\mathfrak{P}'\mathfrak{U}''$. And this is seen to be permissible, in the sense that $\mathfrak{U}$ consists of all classes u of the form $u'u''$ or $u'v''_{p''}$ or $v'_{p'}u''$.

Consider a set of r systems, $(\mathfrak{P}^1; R^1)$, $(\mathfrak{P}^2; R^2)$, ..., $(\mathfrak{P}^r; R^r)$. From the first two we may construct the composite system $(\mathfrak{P}^{1,2}; R^{1,2})$, and from this system and $(\mathfrak{P}^3; R^3)$ we obtain the composite system $(\mathfrak{P}^{1,2,3}; R^{1,2,3})$. Continuing in this way we arrive at what may be called the iterated composite system $(\mathfrak{P}^{1,\dots,r}; R^{1,\dots,r})$. From the definition of the composite of two given systems it is at once evident that this iterated composite system is as follows: $\mathfrak{P}^{1,\dots,r}$ is the class of all elements of the form $p = p^1 p^2 \dots p^r$, where $p^1, p^2, \dots$, and p^r belong to the respective classes $\mathfrak{P}^1, \mathfrak{P}^2, \dots, \mathfrak{P}^r$. That is, $\mathfrak{P}^{1,\dots,r} = \mathfrak{P}^1 \mathfrak{P}^2 \dots \mathfrak{P}^r$. If $\mathfrak{R}_1^{1,\dots,r}$ and $\mathfrak{R}_2^{1,\dots,r}$ are subclasses of $\mathfrak{P}^{1,\dots,r}$, then the relation $\mathfrak{R}_1^{1,\dots,r} R^{1,\dots,r} \mathfrak{R}_2^{1,\dots,r}$ holds if and only if there exist classes $\mathfrak{R}_1^1, \mathfrak{R}_1^2, \dots, \mathfrak{R}_1^r$ and $\mathfrak{R}_2^1, \mathfrak{R}_2^2, \dots, \mathfrak{R}_2^r$ such that $\mathfrak{R}_1^{1,\dots,r} = \mathfrak{R}_1^1 \mathfrak{R}_1^2 \dots \mathfrak{R}_1^r$ and $\mathfrak{R}_2^{1,\dots,r} = \mathfrak{R}_2^1 \mathfrak{R}_2^2 \dots \mathfrak{R}_2^r$ and the relations $\mathfrak{R}_1^1 R^1 \mathfrak{R}_2^1, \mathfrak{R}_1^2 R^2 \mathfrak{R}_2^2, \dots, \mathfrak{R}_1^r R^r \mathfrak{R}_2^r$ are all fulfilled.

It is clear that any other iterated composite system derived from this same set of r systems taken in different order can differ from the one considered in notation only. It will be observed, also, that if we form the composites of groups of systems into which this set of r systems may be divided, and then take the composite of these composites, we arrive at a composite system differing only in notation from the iterated composite system first considered. Thus we see that, aside from possible differences of notation, there is a unique composite system of any finite number of systems of the type $(\mathfrak{P}; R)$.

An obvious generalization of theorem IV is:

THEOREM VI. *The composite system of a finite number of systems satisfies the seven postulates if and only if every component system satisfies them.*

As a partial generalization of theorem V we have:

THEOREM VII. *If $(\mathfrak{P}; R)$ is the composite of the systems $(\mathfrak{P}^1; R^1)$, $(\mathfrak{P}^2; R^2)$, ..., $(\mathfrak{P}^r; R^r)$, then all elements of the form $q = q^1 q^2 \dots q^r$, where at least one q^i is an ideal element of the corresponding system, and the remainder are elements of the corresponding classes $\mathfrak{P}^i$, may be regarded as ideal elements of the system $(\mathfrak{P}; R)$, in the sense that the class $u = v_{q^1}^1 v_{q^2}^2 \dots v_{q^r}^r$ (in which, if q^i is a u^i, $v_{u^i}^i$ is understood to be identical with the class u^i) has the properties* 1–7. *And every ideal element of the composite system is a composite class of the type indicated.*

CHAPTER II.

PROPERTIES OF A SYSTEM FOR WHICH IDEAL ELEMENTS ARE DEFINED.

§ 5. *Postulates for a System* $(\mathfrak{P}; \mathfrak{U}; T)$, *an Instance of which is Associated with Every System* $(\mathfrak{P}; R)$.

We have seen in Chapter I that the postulates I–VII on the system $(\mathfrak{P}; R)$ permit of an effective definition of ideal elements for the system, and that the postulates persist under the process of composition, this process being suitably defined. A body of postulates more simple in form and more general in application, yet adequate for a development of a theory of multiple and iterated limits, may be stated for a system for which a definition of ideal elements is assumed to exist independently. Special instances of systems may arise, where ideal elements may be defined in terms of special features of the system in such manner that the enlarged system shall possess all the properties that are essential for the application of our general theory of functions, but where it may be very difficult or impossible to treat the system as a special instance of a system $(\mathfrak{P}; R)$ satisfying our postulates.* For this reason, as well as for the sake of simplicity, we specify here the conditions on which we rely for the development of the following theory.

The system consists of a class $\mathfrak{P}$ of elements p, a class $\mathfrak{U}$ of ideal elements u, and a relation T between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ and elements of the class $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$. The notation for this system is $(\mathfrak{P}; \mathfrak{U}; T)$, but the symbol T is largely suppressed in practice. That a given subclass $\mathfrak{R}$ has the relation T to a given element q is indicated by $\mathfrak{R}^q$, while $\mathfrak{R}^{-q}$ indicates that the relation T does not hold between $\mathfrak{R}$ and q. A relation T is said to be defined for the classes $\mathfrak{P}$ and $\mathfrak{U}$ when a criterion exists determining for every $\mathfrak{R}$ and every q whether $\mathfrak{R}^q$ or $\mathfrak{R}^{-q}$.

Following are the postulates: †

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* Instances of this sort are found in the systems $(\mathfrak{P}; K_2)$ of § 17 and $(\mathfrak{P}; V)$ of § 18.

† We discriminate notationally between the present postulates and those of § 2 by the use of parentheses with the numerals. The postulates may be read as follows:

(I) If $\mathfrak{R}$ has the relation T to p, then p is an element of $\mathfrak{R}$.

(II) Every $\mathfrak{R}$ having the relation T to an ideal element u contains at least one element p.

(III) For every q there exists a sequence $\{\mathfrak{R}_n\}$ of classes having the relation T to q such that for every $\mathfrak{R}$ having the relation T to q there exists a number $n_{\mathfrak{R}}$ such that for $n > n_{\mathfrak{R}}$ $\mathfrak{R}_n$ is a subclass of $\mathfrak{R}$.

(VI) For every $\mathfrak{R}$ having the relation T to an element q there exists another class $\mathfrak{R}_1$ having the relation T to q such that for every p in $\mathfrak{R}_1$ there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ having the relation T to p.

(V) If q_1 is distinct from q_2 there exist classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ having the relation T to q_1 and q_2 respectively such that there is no element common to $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

(I) $\mathfrak{R}^p . \supset . p^{\mathfrak{R}}$.

(II) $\mathfrak{R}^u . \supset . \exists\, p^{\mathfrak{R}}$.

(III) $q .\!:\supset:\!. \exists \{\mathfrak{R}_n\} \ni [(n . \supset . \mathfrak{R}_n^q) . (\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} . \supset . \mathfrak{R}_n^{\mathfrak{R}})]$.

(IV) $R^q : \supset : \exists\, \mathfrak{R}_1^q \ni (p^{\mathfrak{R}_1} . \supset . \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2^p)$.

(V) $q_1 \neq q_2 . \supset . \exists\, (\mathfrak{R}_1^{q_1} . \mathfrak{R}_2^{q_2}) \ni {}^{-}\exists\, p \ni p^{\mathfrak{R}_1} . p^{\mathfrak{R}_2}$.

A system $(\mathfrak{P}; \mathfrak{U}; T)$ may be derived from a system $(\mathfrak{P}; R)$ as follows: Let $\mathfrak{P}$ and $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ be respectively the class $\mathfrak{P}$ of the system $(\mathfrak{P}; R)$ and the class of ideal elements arising in $(\mathfrak{P}; R)$; and let a subclass $\mathfrak{R}$ have the relation T to an element p if and only if $\mathfrak{R} R p$, and let $\mathfrak{R}$ have the relation T to an ideal element u if and only if $\mathfrak{R}$ is a member of the class u which constitutes the ideal element of the system $(\mathfrak{P}; R)$.

It may easily be seen that if the system $(\mathfrak{P}; R)$ satisfies the seven postulates of § 2 the resulting system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the five postulates stated above. By the mediation of the foregoing definition of T in terms of R, either example 0 of § 2, or the example of § 3, may serve to establish the consistency of the present postulates; and in this same way examples 1, 3, 4 and 7 of § 2 serve as instances of systems satisfying respectively all but (I), all but (III), all but (IV) and all but (V). To complete the proof of the independence of the five postulates we have only to show a system failing to satisfy (II) but satisfying the remaining four postulates. Such a system is the following: $\mathfrak{P}$ is the class of positive integers, and $\mathfrak{U}$ consists of a single ideal element u. The relation $\mathfrak{R}^p$ holds if and only if $\mathfrak{R}$ contains only the single element p, while $\mathfrak{R}^u$ holds if and only if $\mathfrak{R}$ is the null class.

From two systems, $(\mathfrak{P}'; \mathfrak{U}'; T')$ and $(\mathfrak{P}''; \mathfrak{U}''; T'')$, we derive a composite system $(\mathfrak{P}; \mathfrak{U}; T)$, where $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ and $\mathfrak{U}$ consists of all elements of the form $u'u''$ or $u'p''$ or $p'u''$, *i. e.*, $\mathfrak{U} = \mathfrak{U}'\mathfrak{U}'' + \mathfrak{U}'\mathfrak{P}'' + \mathfrak{P}'\mathfrak{U}''$, so that if $\mathfrak{Q}' = \mathfrak{P}' + \mathfrak{U}'$ and $\mathfrak{Q}'' = \mathfrak{P}'' + \mathfrak{U}''$ and $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$, then $\mathfrak{Q}$ is the product or composite class $\mathfrak{Q}'\mathfrak{Q}''$. The relation $\mathfrak{R}^q$ holds if and only if there exist classes $\mathfrak{R}'$ and $\mathfrak{R}''$, subclasses of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively, such that $\mathfrak{R} = \mathfrak{R}'\mathfrak{R}''$, and $\mathfrak{R}'^{q'}$ and $\mathfrak{R}''^{q''}$, where $q = q'q''$. Obviously, this definition of the composite system is consistent with the definition employed in § 4 and the above definition of T in terms of R.

Analogous to theorem IV of Chapter I is the following theorem, the proof of which should cause no difficulty:

Theorem I. *The composite system* $(\mathfrak{P}; \mathfrak{U}; T)$ *of two systems* $(\mathfrak{P}'; \mathfrak{U}'; T')$ *and* $(\mathfrak{P}''; \mathfrak{U}''; T'')$ *satisfies the postulates* (I)–(V) *if and only if both component systems satisfy them.*

As in § 4, the definition of the composite of two systems leads to a unique composite system for any finite number of systems of the type considered. We clearly have here a theorem analogous to theorem VI of Chapter I.

§ 6. *Limiting Elements: Postulates of F. Riesz.*

The notion of limiting element of a subclass is of primary importance for the purpose in hand. From the point of view of the present investigation, it is sufficient to define limiting elements only for subclasses* $\mathfrak{R}$ of $\mathfrak{P}$.

Def. 1. An element q is a *limiting element* of the subclass $\mathfrak{R}_1$ of $\mathfrak{P}$ if every $\mathfrak{R}$ such that $\mathfrak{R}^q$ contains an element of $\mathfrak{R}_1$ distinct from q, *i. e.*, if

$$\mathfrak{R}^q \cdot \supset \cdot \exists\, p \neq q \ni p^{\mathfrak{R}_1} \cdot p^{\mathfrak{R}}.$$

If q is a p, it is an *actual* limiting element, and if a u, it is an *ideal* limiting element; and if q is an element of $\mathfrak{R}_1$, it is a *proper* limiting element of $\mathfrak{R}_1$, and if not an element of $\mathfrak{R}_1$, it is an *improper* limiting element of $\mathfrak{R}_1$. Evidently, a proper limiting element is always actual, and an ideal limiting element is always improper, but a limiting element may be for the same $\mathfrak{R}_1$ both actual and improper.

In his paper† on "Stetigkeitsbegriff und abstracte Mengenlehre" before the International Congress of Mathematicians at Rome, 1908, F. Riesz proposed a set of postulates on which to build, for an abstract class, a generalization of the theory of point-sets. He first assumes an abstract system which we may denote by $(\mathfrak{P}; C)$, where $\mathfrak{P}$ is a class of elements p, and C is a relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ and individual elements p, in the sense that p is a limiting element or element of condensation (Verdichtungstelle) of the class $\mathfrak{R}$. It is of interest here to note that if the abstract class of Riesz be identified with our class $\mathfrak{P}$, and if the relation C be assumed to hold if and only if the element p is a limiting element of $\mathfrak{R}$ by definition 1, then the resulting system is found to satisfy the postulates of Riesz. We establish this fact by proving the following theorem, the four propositions of the theorem being somewhat more general than the four conditions necessary to secure the postulates.

THEOREM II. (a) *Every limiting element of $\mathfrak{R}$ is a limiting element of every class $\mathfrak{R}_1$ containing $\mathfrak{R}$.*

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* To define limiting elements for a general subclass of $\mathfrak{Q}$ it would be necessary to resort to a situation analogous to that of §3; but since the extended system $(\mathfrak{Q}; S)$ is of the same character as $(\mathfrak{P}; R)$, we should thus revert essentially to the situation found in the special case when the class $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ contains no elements. This special case furnishes a close analogue to the usual method of procedure in the matter of limits.

† *Atti*, etc., pp. 18–24.

(b) *If* $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$, *then every limiting element of* $\mathfrak{R}$ *is a limiting element either of* $\mathfrak{R}_1$ *or of* $\mathfrak{R}_2$.

(c) *Only infinite classes have limiting elements.*

(d) *Each limiting element of* $\mathfrak{R}$ *is uniquely determined by the totality of all subclasses of* $\mathfrak{R}$ *of which it is a limiting element.*

Proof: Proposition (a) is immediately evident from definition 1.

(b): Let q be a limiting element of $\overline{\mathfrak{R}} = \overline{\mathfrak{R}}_1 + \overline{\mathfrak{R}}_2$. We are to show that q is a limiting element of $\overline{\mathfrak{R}}_1$ or of $\overline{\mathfrak{R}}_2$. By postulate (III) we have

$$(1) \qquad \exists \{\mathfrak{R}_n\} \ni [(n \,.\supset.\, \mathfrak{R}_n^q) \,.\, (\mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, \mathfrak{R}_n^{\mathfrak{R}})];$$

then by definition 1 we see that

$$(2) \qquad n \,.\supset.\, \exists\, p_n \ni p_n \neq q \,.\, p_n^{\mathfrak{R}_n} \,.\, p_n^{\overline{\mathfrak{R}}}.$$

A sequence $\{p_n\}$ thus secured satisfies the condition

$$(3) \qquad \mathfrak{R}^q :\supset: \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, p_n^{\mathfrak{R}}.$$

Since either $\overline{\mathfrak{R}}_1$ or $\overline{\mathfrak{R}}_2$ must contain an infinite subsequence of $\{p_n\}$, we may suppose that $\overline{\mathfrak{R}}_1$ contains the sequence $\{p_{n_m}\}$, where, if $m_1 \neq m_2$, then $n_{m_1} \neq n_{m_2}$. For a given $\mathfrak{R}$ only a finite number of terms of the sequence $\{p_{n_m}\}$ can precede the term $p_{n_{\mathfrak{R}}}$ in the sequence $\{p_n\}$; hence by (3) we have

$$\mathfrak{R}^q \,.\supset.\, \exists\, m \ni p_{n_m}^{\mathfrak{R}},$$

and since by (2) every p_{n_m} is distinct from q, we see that q is a limiting element of $\overline{\mathfrak{R}}_1$.

(c): Let q be a limiting element of $\overline{\mathfrak{R}}$. If possible, let $\overline{\mathfrak{R}}$ consist of a finite set of elements, $p_1, p_2, \ldots, p_n$. By postulate (V) we see that for every element p_i distinct from q there exists a class $\mathfrak{R}_i$ not containing p_i such that $\mathfrak{R}_i^q$. By an application of postulate (III) we secure a class $\mathfrak{R}$ such that $\mathfrak{R}^q$, which is a common subclass of all the classes $\mathfrak{R}_i$. This class $\mathfrak{R}$ clearly contains no element of $\overline{\mathfrak{R}}$ distinct from q; hence we reach a contradiction.

(d): We are to show that if q_1 and q_2 are distinct limiting elements of $\overline{\mathfrak{R}}$, then there is a subclass of $\overline{\mathfrak{R}}$ that has one of these as limiting element but not the other. By postulate (V)

$$\exists\, (\mathfrak{R}_1^{q_1} \,.\, \mathfrak{R}_2^{q_2}) \ni \overline{\exists}\, p \ni p^{\mathfrak{R}_1} \,.\, p^{\mathfrak{R}_2}.$$

Denote by $\overline{\mathfrak{R}}_1$ the greatest common subclass of $\overline{\mathfrak{R}}$ and $\mathfrak{R}_1$; then it is clear that q_2 is not a limiting element of $\overline{\mathfrak{R}}_1$. For a given $\mathfrak{R}$ such that $\mathfrak{R}^{q_1}$ there exists by

postulate (III) a common subclass $\mathfrak{R}_3$ of $\mathfrak{R}$ and $\mathfrak{R}_1$ such that $\mathfrak{R}_3^{q_1}$. Since q_1 is a limiting element of $\overline{\mathfrak{R}}$, $\mathfrak{R}_3$ must contain an element p of $\overline{\mathfrak{R}}$, distinct from q_1, and this element p is obviously in $\overline{\mathfrak{R}}_1$. Thus every $\mathfrak{R}$ such that $\mathfrak{R}^{q_1}$ contains an element of $\overline{\mathfrak{R}}_1$ distinct from q_1; that is, q_1 is a limiting element of $\overline{\mathfrak{R}}_1$.

The four postulates of F. Riesz are equivalent to the four propositions of this theorem if we restrict limiting elements q to actual limiting elements p, and if in (b) $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have no common elements.

For the purpose of introducing ideal elements into the abstract class $\mathfrak{P}$, Riesz considers a system which we may denote by $(\mathfrak{P}; V)$, where V is a relation between subclasses $\mathfrak{R}$ of $\mathfrak{P}$ of the same type as our relation R. He postulates for the system $(\mathfrak{P}; V)$ four properties as follows:

(1) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the relation V, and $\mathfrak{R}_3$ contains $\mathfrak{R}_1$ and $\mathfrak{R}_4$ contains $\mathfrak{R}_2$, then $\mathfrak{R}_3$ and $\mathfrak{R}_4$ have the relation V.

(2) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the relation V, and $\mathfrak{R}_1$ is divided into two classes, then at least one of these has the relation V to $\mathfrak{R}_2$.

(3) Two singular subclasses can not have the relation V.

(4) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ both have the relation V to a given singular class p, then they have the relation V to each other.

A definition of C in terms of V is given by Riesz, by which the relation C holds for a class $\mathfrak{R}$ and an element p if and only if the class $\mathfrak{R}$ and the singular class whose element is p have the relation V. A system $(\mathfrak{P}; C)$ thus obtained from a system $(\mathfrak{P}; V)$ which satisfies the first three conditions above has the first three properties postulated for the system $(\mathfrak{P}; C)$.

From a system $(\mathfrak{P}; \mathfrak{U}; T)$ we obtain a system $(\mathfrak{P}; V)$ as follows: The class $\mathfrak{P}$ of the system $(\mathfrak{P}; V)$ is identical with the class $\mathfrak{P}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$. The relation V holds between two subclasses of $\mathfrak{P}$ if and only if the two have a common limiting element (actual or ideal) or one subclass contains a limiting element of the other.*

It is easily seen that if the system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the five postulates of § 5, then the resulting system $(\mathfrak{P}; V)$ fulfils the four conditions prescribed by Riesz. In fact, the propositions (a), (b) and (c) of theorem II contain sufficient conditions on the system $(\mathfrak{P}; \mathfrak{U}; T)$ to secure this result.

Riesz defines an ideal element as a class v of subclasses $\mathfrak{R}$ which satisfies the following conditions:

(a) If $\mathfrak{R}$ belongs to v and $\mathfrak{R}_1$ contains $\mathfrak{R}$, then $\mathfrak{R}_1$ belongs to v.

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* Compare Riesz, *loc. cit.*, p. 23.

(b) If $\mathfrak{R}$ belongs to v and consists of two subclasses, $\mathfrak{R}_1$ and $\mathfrak{R}_2$, then either $\mathfrak{R}_1$ or $\mathfrak{R}_2$ belongs to v.

(c) Every two classes $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of v have the relation V.

(d) The class v is not contained in any different class v_1 having properties (a), (b) and (c).

(e) No element p is contained in every $\mathfrak{R}$ of v or has the relation V to every $\mathfrak{R}$ of v.

A special case of theorem II, (d), is the proposition: Every ideal element u is uniquely determined by the totality of all subclasses of $\mathfrak{P}$ of which it is a limiting element. It is not difficult to see that such a totality of classes for a given u constitutes a class v fulfilling the five conditions just given. In fact, condition (a) follows from proposition (a) of theorem II, condition (b) from (b) of theorem II, condition (c) from the definition of V in terms of T, and conditions (d) and (e) from (d) of theorem II. Thus every element of the class $\mathfrak{U}$ of the system $(\mathfrak{P}; \mathfrak{U}; T)$ corresponds uniquely to an ideal element of the system $(\mathfrak{P}; V)$.

It may be observed that by the mediation of the definition of a system $(\mathfrak{P}; \mathfrak{U}; T)$ in terms of a system $(\mathfrak{P}; R)$ there is associated with every system $(\mathfrak{P}; R)$ a definite system $(\mathfrak{P}; V)$, and that if the former satisfies the seven postulates of § 2, the latter must fulfil the conditions stated by Riesz. It is clear, also, that every ideal element arising in the system $(\mathfrak{P}; R)$ by our definition corresponds uniquely to an ideal element arising in the associated system $(\mathfrak{P}; V)$ by the definition given by Riesz.

§ 7. *The Fréchet Limit: Properties of Classes.*

In his thesis, "Sur quelques points du calcul fonctionnel," Paris, 1906, M. Fréchet* makes use of an undefined relation between sequences of elements and individual elements. By imposing certain conditions on this relation he is able to develop a theory, analogous to the theory of point-sets and of continuous functions, in which an element that has the undefined relation to a sequence plays the rôle of limit of the sequence. It is of advantage here to show that the notion of limit of a sequence of elements as defined below satisfies the conditions stated by Fréchet.

Def. 2. The sequence $\{p_n\}$ has the limit q if and only if for every $\mathfrak{R}$ such that $\mathfrak{R}^q$ there is a term of the sequence such that all following terms are in the class $\mathfrak{R}$. In symbols: †

* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII.

† The notation $\lim_{n=\infty} p_n = q$ is here replaced by the more convenient but equally suggestive notation $L_n p = q$. Note that a sequence may have an ideal element as limit.

$$L\,p_n = q \mathrel{.:\equiv:.} (\{p_n\} \cdot q) \ni (\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \cdot \supset \cdot p_n^{\mathfrak{R}}).$$

The conditions stated by Fréchet for the definition of the limit relation between sequence and element are implied by the following theorem:

THEOREM III. (a) *A sequence formed by repeating a single element has that element for limit.*

(b) *A sequence can not have two distinct limits.*

(c) *If a sequence* $\{p_n\}$ *has a limit* q, *then every subsequence* $\{p_{n_m}\}$ *such that* n_m *becomes infinite with* m *has the limit* q.

Proof: Proposition (a) is an immediate consequence of postulate (I).

(b): Suppose a sequence $\{p_n\}$ to have two distinct limits, q_1 and q_2. By postulate (V) we have

$$\exists\, (\mathfrak{R}_1^{q_1} \cdot \mathfrak{R}_2^{q_2}) \ni \overline{\exists}\, p \ni p^{\mathfrak{R}_1} \cdot p^{\mathfrak{R}_2}.$$

But by the supposition

$$\exists\, n_{\mathfrak{R}_1} \ni n > n_{\mathfrak{R}_1} \cdot \supset \cdot p_n^{\mathfrak{R}_1} \quad \text{and} \quad \exists\, n_{\mathfrak{R}_2} \ni n > n_{\mathfrak{R}_2} \cdot \supset \cdot p_n^{\mathfrak{R}_2}.$$

By considering p_n such that n is greater than both $n_{\mathfrak{R}_1}$ and $n_{\mathfrak{R}_2}$, we reach a contradiction.

(c): By hypothesis we have

$$\mathfrak{R}^q : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \cdot \supset \cdot p_n^{\mathfrak{R}},$$

and since n_m becomes infinite with m there is for every $n_{\mathfrak{R}}$ a number $m_{\mathfrak{R}}$ such that if m is greater than $m_{\mathfrak{R}}$, then n_m is greater than $n_{\mathfrak{R}}$; thus

$$\mathfrak{R}^q : \supset : \exists\, m_{\mathfrak{R}} \ni m > m_{\mathfrak{R}} \cdot \supset \cdot p_{n_m}^{\mathfrak{R}},$$

which is the required condition.

The three propositions of the theorem are equivalent to the properties of the Fréchet limit, except that he uses instead of (c) a less restrictive condition, obtained by adding to the hypothesis of (c) the restriction that the elements of the subsequence are taken in the same order as in the original sequence. In the proof just given we made use only of postulates (I) and (V), so that these two conditions on a system $(\mathfrak{P}; \mathfrak{U}; T)$ are sufficient for the development of a theory at least as extensive as that pertaining to the class (L) of Fréchet.*

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* Fréchet denotes by (L) what we should represent by $(\mathfrak{P}; L)$, where L is a relation between sequences of elements of $\mathfrak{P}$ and individual elements of $\mathfrak{P}$. Observe that the limit relation which we have defined differs in type from that of Fréchet to the extent that we include ideal limiting elements. One sees, however, that the presence of ideal elements does not interfere with the application to the present situation of the theorems proved by Fréchet on the basis of his limit relation and without the aid of his *écart* or his *voisinage*.

Theorem IV. *A necessary and sufficient condition that q shall be a limiting element of $\mathfrak{R}$ is that there exist a sequence $\{p_n\}$ of distinct elements of $\mathfrak{R}$ such that $L\, p_n = q$.*

It is necessary: By postulate (III) we have

$$\exists\ \{\mathfrak{R}_n\} \ni [(n \centerdot \supset \centerdot \mathfrak{R}_n^q) \centerdot (\overline{\mathfrak{R}}^q : \supset : \exists\ n_{\overline{\mathfrak{R}}} \ni n > n_{\overline{\mathfrak{R}}} \centerdot \supset \centerdot \mathfrak{R}_n^{\overline{\mathfrak{R}}})],$$

and, q being a limiting element of $\mathfrak{R}$, we have by definition 1,

$$n \centerdot \supset \centerdot \exists\ p_n \neq q \ni p_n^{\mathfrak{R}_n} \centerdot p_n^{\mathfrak{R}}.$$

The sequence $\{p_n\}$ thus secured is such that $L\, p_n = q$, and since the elements of the sequence constitute a class having the limiting element q, we see by theorem II, (c), that the number of distinct elements of the sequence is not finite. There exists, then, an infinite subsequence $\{p_{n_m}\}$ of distinct terms such that n_m becomes infinite with m, and which, by theorem III, (c), has the limit q.

It is sufficient: This proposition is a direct result of proposition (a) of theorem II.

Theorem IV shows that our definition of limiting element of a subclass is consistent with the definition employed by Fréchet. It may be noticed that in establishing these relations with the work of Fréchet, and the relations to the work of Riesz discussed in the previous section, no use has been made of postulate (IV); it is clear, therefore, that while we have sacrificed much in the matter of generality, we gain somewhat in the extent of the theory available for our system. We consider here certain properties of subclasses that are found useful in the next chapter.

Def. 3. The *derived class* of a subclass $\mathfrak{R}$ is the class of all limiting elements of $\mathfrak{R}$.

Def. 4. A subclass is *closed* if it contains its derived class.

Def. 5. A subclass $\mathfrak{R}$ is *compact* * if every infinite subclass of $\mathfrak{R}$ has at least one limiting element.

The propositions of the following theorem, which are given by Fréchet, are seen to be valid here, his proof of (d) being entirely applicable to the present situation, and the first three propositions being obvious deductions from the definition of compactness.

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* See Fréchet, *loc. cit.*, p. 6. Here, again, attention must be called to the fact that we recognize ideal limiting elements.

THEOREM V. (a) *Every subclass of a compact class is compact.*

(b) *If every subclass of $\mathfrak{R}$ is compact, then $\mathfrak{R}$ is compact.*

(c) *A class formed of a finite number of compact classes is compact.*

(d) *If every member of a sequence $\{\mathfrak{R}_n\}$ of subclasses of a compact class $\mathfrak{R}$ is closed, contains the succeeding member, and contains at least one element, then there is an element common to all classes of the sequence.*

A proposition somewhat different in content from this last, but permitting of a very similar proof, is stated by Riesz,* and may be stated here as follows:

THEOREM VI. *If every member of a sequence $\{\mathfrak{R}_n\}$ of infinite subclasses of a compact class $\mathfrak{R}$ contains the succeeding member, then the members of the sequence possess at least one common limiting element.*

An important proposition in the theory of point-sets and in the analogous theories † in the domain of general analysis is the following: "The derived class of every subclass is closed." The following theorem, in the proof ‡ of which we find the first use for postulate (IV), reduces to this proposition in case no ideal elements exist, *i. e.*, in case $\mathfrak{U}$ is the null class.

THEOREM VII. *If $\overline{\mathfrak{R}}_1$ is the class of all actual limiting elements of $\overline{\mathfrak{R}}$, then every limiting element of $\overline{\mathfrak{R}}_1$ is a limiting element of $\overline{\mathfrak{R}}$.*

Proof: By postulate (IV) we have for a given limiting element q of $\overline{\mathfrak{R}}_1$,

$$(1) \qquad \mathfrak{R}^q : \supset : \exists\, \mathfrak{R}_1^q \ni p^{\mathfrak{R}_1} . \supset . \exists\, \mathfrak{R}_2^{\mathfrak{R}} \ni \mathfrak{R}_2^p .$$

Now such a class $\mathfrak{R}_1$ must contain an element p of $\overline{\mathfrak{R}}_1$ distinct from q. Then there is a subclass $\mathfrak{R}_2$ of $\mathfrak{R}$ such that $\mathfrak{R}_2^p$, and p being a limiting element of $\overline{\mathfrak{R}}$,

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* *Loc. cit.*, p. 20.

† E. R. Hedrick, "On Properties of a Domain for which Any Derived Set is Closed," *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 289.

‡ Fréchet shows (*loc. cit.*, p. 15) that this proposition does not follow from the hypotheses he has made on the class (L). He secures this theorem only after the introduction of the notion of *voisinage*. The following example shows that the theorem is not a consequence of postulates (I), (II), (III) and (V), which, as we have shown, are together as strong as the postulates on the class (L) combined with the postulates of Riesz. We specify a system $(\mathfrak{P}; \mathfrak{U}; T)$ as follows: $\mathfrak{P}$ consists of an element p, a sequence $\{p_n\}$ of elements, and a double sequence $\{p_{mn}\}$ of elements. Two elements having different notations are distinct. $\mathfrak{U}$ is the null class. The relation $\mathfrak{R}^p$ holds if and only if $\mathfrak{R}$ consists of the element p together with all, excepting a finite number, of the elements of the sequence $\{p_n\}$. For a given n, the relation $\mathfrak{R}^{p_n}$ holds if and only if $\mathfrak{R}$ consists of the element p_n together with all, excepting a finite number, of the elements of the simple sequence $\{p_{mn}\}$. For a given m and n, $\mathfrak{R}^{p_{mn}}$ holds if and only if $\mathfrak{R}$ consists of the single element p_{mn}. This system satisfies postulates (I), (II), (III) and (V). For a given n the element p_n is the limit of the sequence $\{p_{mn}\}$; and the sequence $\{p_n\}$ has the limit p. The subclass $\mathfrak{R}$ consisting of the elements of the double sequence $\{p_{mn}\}$ has for its derived class the class $\mathfrak{R}_1$, which consists of the elements of the sequence $\{p_n\}$. The only limiting element of $\mathfrak{R}_1$ is p, which is not in $\mathfrak{R}_1$. Thus the derived class $\mathfrak{R}_1$ is not closed.

there is by theorem IV a sequence of distinct elements of $\overline{\mathfrak{R}}$ in $\mathfrak{R}_2$, and therefore in $\mathfrak{R}$. Clearly then we have

$$\mathfrak{R}^q \cdot \supset \cdot \exists\, p \neq q \ni p^{\overline{\mathfrak{R}}} \cdot p^{\mathfrak{R}};$$

that is, q is a limiting element of $\overline{\mathfrak{R}}$.

The proofs of the following propositions relative to a composite system should cause no difficulty.

THEOREM VIII. *If* $(\mathfrak{P}; \mathfrak{U}; T)$ *is the composite system of the systems* $(\mathfrak{P}^1; \mathfrak{U}^1; T^1)$, $(\mathfrak{P}^2; \mathfrak{U}^2; T^2)$, $\ldots\ldots$, $(\mathfrak{P}^r; \mathfrak{U}^r; T^r)$, *and if* $\mathfrak{R} = \mathfrak{R}^1\mathfrak{R}^2 \ldots\ldots \mathfrak{R}^r$, *and* $q = q^1 q^2 \ldots\ldots q^r$, *the following propositions hold:*

(a) *If, for every* n, $p_n = p_n^1 p_n^2 \ldots\ldots p_n^r$, *then* $L\,p_n = q$ *if and only if for* $i = 1, 2, \ldots\ldots, r$ *we have* $L\,p_n^i = q^i$.

(b) *The element* q *is a limiting element of* $\mathfrak{R}$ *if and only if every* q^i $(i = 1, 2, \ldots\ldots, r)$ *is contained in* $\mathfrak{R}^i$ *or is a limiting element of* $\mathfrak{R}^i$, *and at least one of the* q^i *is a limiting element of the corresponding* $\mathfrak{R}^i$.

(c) $\mathfrak{R}$ *is closed if and only if every* $\mathfrak{R}^i$ *is closed* $(i = 1, 2, \ldots\ldots, r)$.

(d) $\mathfrak{R}$ *is compact if and only if every* $\mathfrak{R}^i$ *is compact* $(i = 1, 2, \ldots\ldots, r)$.

(To be concluded in the April number.)

Iterated Limits in General Analysis.—*Continued.*

By Ralph E. Root.

CHAPTER III.

A Theory of Functions Based on Properties of a System $(\mathfrak{P}; \mathfrak{U}; T)$.

§ 8. *Introductory.*

In this chapter we are concerned with a definite system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, which is assumed to satisfy the five postulates of § 5. In § 10 and § 12, in order to provide for the features of iterated limits, special hypotheses are made with regard to the composite character of the system. Subclasses of $\overline{\mathfrak{P}}$ are, in general, denoted by $\mathfrak{R}$; but a definite one of these subclasses, denoted by $\mathfrak{P}$, receives special consideration, being the range of the independent variable in our theory of functions. The derived class of $\mathfrak{P}$ is denoted by $\mathfrak{L}$, and $\mathfrak{Q}$ is the least common superclass of $\mathfrak{P}$ and $\mathfrak{L}$. The letters p, q and l denote elements of the respective classes $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{L}$, while a general element of $\overline{\mathfrak{P}}$ is written $\bar{p}$, and $\bar{q}$ is an element of the class $\overline{\mathfrak{Q}} = \overline{\mathfrak{P}} + \mathfrak{U}$.

Functions are denoted by the letters θ, ϕ, μ, etc. The notion of function, in general, involves two classes, one called the range and the other the class to which the function-values belong. If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of elements, then a function θ on $\mathfrak{X}$ to $\mathfrak{Y}$ is a correspondence between elements of $\mathfrak{X}$ and subclasses of $\mathfrak{Y}$, whereby to every element x of $\mathfrak{X}$ there corresponds uniquely a subclass θ_x of $\mathfrak{Y}$. If, for every x, θ_x consists of a single element y of $\mathfrak{Y}$, then θ is a single-valued function on $\mathfrak{X}$ to $\mathfrak{Y}$. In the present chapter we consider only single-valued functions on a subclass of $\overline{\mathfrak{Q}}$ to $\hat{\mathfrak{A}}$, where $\hat{\mathfrak{A}}$ is the class of all real numbers with the ideal elements $+\infty$ and $-\infty$ adjoined. The notation $\mathfrak{A}$ stands for the class of all real numbers, and the letter a invariably represents a real number, while the letters e and d always denote positive real numbers.*

§ 9. *The Character of a Function in the Neighborhood of an Element.*

We take as the subject of discussion a definite function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$. In the consideration of the character of the function μ with respect to a particular limiting element l of the range, three symbols play an important rôle:

* The letters e and d replace the usual $\epsilon>0$ and $\delta>0$. There can be no doubt that convenience and economy are conserved by these and other special notations. We find sufficient precedent in the work of Professor E. H. Moore on "General Analysis."

(a) $\overline{\lim\limits_{p \to l}}\, \mu_p$: upper limit of μ_p as p approaches l.

(b) $\underline{\lim\limits_{p \to l}}\, \mu_p$: lower limit of μ_p as p approaches l.

(c) $\lim\limits_{p \to l} \mu_p$: limit of μ_p as p approaches l.

Following are the conditions under which these symbols may represent definite finite numbers:

Def. 1. The relation $\overline{\lim\limits_{p \to l}}\, \mu_p = a$ is equivalent to the following two conditions* on μ, l and a:

$$(1) \quad e : \supset : \exists\, \mathfrak{R}^l_e \ni p^{\mathfrak{R}_e} \neq l . \supset . \mu_p \leqq a + e,$$

$$(2) \quad e . \mathfrak{R}^l : \supset : \exists\, p^{\mathfrak{R}} \neq l \ni \mu_p \geqq a - e.$$

Def. 2. The relation $\underline{\lim\limits_{p \to l}}\, \mu_p = a$ is equivalent to the following two conditions on μ, l and a:

$$(1) \quad e : \supset : \exists\, \mathfrak{R}^l_e \ni p^{\mathfrak{R}_e} \neq l . \supset . \mu_p \geqq a - e,$$

$$(2) \quad e . \mathfrak{R}^l : \supset : \exists\, p^{\mathfrak{R}} \neq l \ni \mu_p \leqq a + e.$$

Def. 3. The relation $\lim\limits_{p \to l} \mu_p = a$ is equivalent to the following condition on μ, l and a:

$$e : \supset : \exists\, \mathfrak{R}^l_e \ni p^{\mathfrak{R}_e} \neq l . \supset . | \mu_p - a | \leqq e.$$

These definitions lead to the obvious theorem:

THEOREM I. *The limit of μ_p as p approaches a limiting element l is a number a if and only if the upper limit and the lower limit of μ_p as p approaches l are both equal to this same number a.* In symbols:

$$\lim_{p \to l} \mu_p = a : \backsim : \overline{\lim_{p \to l}}\, \mu_p = \underline{\lim_{p \to l}}\, \mu_p = a.$$

For a given limiting element l of $\mathfrak{P}$ we have also the following analogue of the Cauchy condition for convergence:

THEOREM II. *The following two conditions are necessary and sufficient for the existence of a finite limit of μ_p as p approaches l:*

(1) *If $\mathfrak{R}^l$ there exists a p in $\mathfrak{R}$ distinct from l such that μ_p is finite.*

(2) $e : \supset : \exists\, \mathfrak{R}^l_e \ni p_1^{\mathfrak{R}_e} \neq l . p_2^{\mathfrak{R}_e} \neq l . \supset . | \mu_{p_1} - \mu_{p_2} | \leqq e.$

Proof: That the conditions are necessary is quite obvious. As to their being sufficient, we observe that since l is a limiting element of $\mathfrak{P}$ we have a sequence $\{p_n\}$ of distinct elements of $\mathfrak{P}$ such that

* Condition (1) may be read: "For every e there exists a class $\mathfrak{R}_e$ such that $\mathfrak{R}^l_e$ ($\mathfrak{R}_e$ has the relation T to l) and such that for every p in $\mathfrak{R}_e$ distinct from l the function-value μ_p is less than or equal to $a+e$"; and (2) may be read: "For every e and every $\mathfrak{R}$ such that $\mathfrak{R}^l$ there exists a p in $\mathfrak{R}$ distinct from l such that the function-value μ_p is greater than or equal to $a-e$." The conditions in definitions 2 and 3 may be read in similar fashion.

$$(1) \qquad \mathfrak{R}^l : \supset : \exists\, n_{\mathfrak{R}} \ni n > n_{\mathfrak{R}} \,.\supset.\, p_n^{\mathfrak{R}}.$$

If for a given e we consider the class $\mathfrak{R}_e$ required to exist by condition (2) of the theorem, we see that

$$\exists\, n_e \ni n_1 > n_e \,.\, n_2 > n_e \,.\supset.\, | \mu_{p_{n_1}} - \mu_{p_{n_2}} | \leqq e.$$

By condition (1) of the theorem the terms of the sequence $\{\mu_{p_n}\}$, after a finite number of terms, are all real numbers; therefore the sequence is convergent to a finite limit by the condition just written. Let this limit be a, then

$$(2) \qquad e : \supset : \exists\, n_e \ni n > n_e \,.\supset.\, | \mu_{p_n} - a | \leqq \frac{e}{2},$$

and by condition (2) of the theorem we have for this same e

$$(3) \qquad \exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l \,.\, p_n^{\mathfrak{R}_e} \neq l \,.\supset.\, | \mu_p - \mu_{p_n} | \leqq \frac{e}{2}.$$

Since, by (1), this conclusion is fulfilled for some value of n, we see that, by (2) and (3),

$$p^{\mathfrak{R}_e} \neq l \,.\supset.\, | \mu_p - a | \leqq e;$$

that is, the limit of μ_p as p approaches l is a.

THEOREM III. *If for a given l there exist a number a and a class $\mathfrak{R}$ such that $\mathfrak{R}^l$ and such that, for every p in R, $|\mu_p| \leqq a$, then there exist numbers $\bar{a}$ and $\underline{a}$ such that*

$$\overline{\lim_{p \to l}}\, \mu_p = \bar{a} \quad \textit{and} \quad \underline{\lim_{p \to l}}\, \mu_p = \underline{a}.$$

Proof: By use of postulate (III) we secure a sequence $\{\mathfrak{R}_n\}$ such that for every n we have $\mathfrak{R}_n^l$ and $\mathfrak{R}_n^{\mathfrak{R}}$, and further, such that each term of the sequence is contained in the preceding term. For every n let $\bar{a}_n$ be the least upper bound of μ_p, where p belongs to $\mathfrak{R}_n$ and is distinct from l, and let $\underline{a}_n$ be the greatest lower bound of μ_p with the same restrictions on p; then the sequences $\{\bar{a}_n\}$ and $\{\underline{a}_n\}$ are respectively decreasing and increasing monotonic sequences of real numbers. Since $\bar{a}_n \geqq \underline{a}_n$, both sequences converge. Let $\bar{a}$ and $\underline{a}$ be the limits of these sequences, then it is clear that they are respectively the upper limit and the lower limit of μ_p as p approaches l.

In the following definition, in which we employ the symbol, $\equiv$, of definitional equivalence, we indicate the conditions under which the limit, upper limit and lower limit of μ_p as p approaches a limiting element l may be infinite elements of the class $\mathfrak{A}$.

Def. 4. With respect to an arbitrary limiting element l of $\mathfrak{P}$ the following are definitional identities:

(a) $\overline{\lim\limits_{p\to l}}\,\mu_p = +\infty : \equiv : a\,.\,\mathfrak{R}^l\,.\supset.\,\exists\, p^{\mathfrak{R}} \neq l \ni \mu_p > a,$

(b) $\underline{\lim\limits_{p\to l}}\,\mu_p = -\infty : \equiv : a\,.\,\mathfrak{R}^l\,.\supset.\,\exists\, p^{\mathfrak{R}} \neq l \ni \mu_p < a,$

(c) $\underline{\lim\limits_{p\to l}}\,\mu_p = +\infty \;.:\equiv:.\; \lim\limits_{p\to l}\mu_p = +\infty \;.:\equiv:.\; a :\supset: \exists\, \mathfrak{R}^l_a \ni p^{\mathfrak{R}_a} \neq l\,.\supset.\,\mu_p > a,$

(d) $\overline{\lim\limits_{p\to l}}\,\mu_p = -\infty \;.:\equiv:.\; \lim\limits_{p\to l}\mu_p = -\infty \;.:\equiv:.\; a :\supset: \exists\, \mathfrak{R}^l_a \ni p^{\mathfrak{R}_a} \neq l\,.\supset.\,\mu_p < a.$

THEOREM IV. *If* $\{p_n\}$ *is a sequence such that* $Lp_n = l$, *then*

$$\underline{\lim_{p\to l}}\,\mu_p \leqq \underline{\lim_{n\to\infty}}\,\mu_{p_n} \leqq \overline{\lim_{n\to\infty}}\,\mu_{p_n} \leqq \overline{\lim_{p\to l}}\,\mu_p .$$

These four symbols clearly always represent definite elements of the class $\hat{\mathfrak{A}}$; *i. e.*, "the quantities always exist, finite or infinite." The proof of the theorem follows immediately from theorem III and the foregoing definitions, with some use of well-known properties of sequences of real numbers.

THEOREM V. *For a given* l *there exist sequences* $\{p_n\}$ *of distinct elements such that* $Lp_n = l$ *satisfying each of the conditions:*

$$\text{(a)} \quad \lim_{n\to\infty}\mu_{p_n} = \overline{\lim_{p\to l}}\,\mu_p,$$

$$\text{(b)} \quad \lim_{n\to\infty}\mu_{p_n} = \underline{\lim_{p\to l}}\,\mu_p.$$

Proof of (a): First, suppose that $\overline{\lim\limits_{p\to l}}\,\mu_p = +\infty$. Consider a sequence $\{a_n\}$ of real numbers such that $\lim\limits_{n\to\infty} a_n = +\infty$, and a sequence $\{\mathfrak{R}_n)$ as required to exist for l by postulate (III). We have by definition 4, (a),

$$n\,.\supset.\,\exists\, p_n^{\mathfrak{R}_n} \neq l \ni \mu_{p_n} > a_n .$$

A sequence $\{p_n\}$ so secured has the desired properties, namely, $L\,p_n = l$ and $\lim\limits_{n\to\infty}\mu_{p_n} = +\infty$.

Next, suppose that $\overline{\lim\limits_{p\to l}}\,\mu_p = a$. Consider a sequence $\{e_n\}$ of positive real numbers such that $\lim\limits_{n\to\infty} e_n = 0$. By condition (1) of definition 1,

$$(1) \qquad n :\supset: \exists\, \mathfrak{R}^l_{e_n} \ni p^{\mathfrak{R}_{e_n}} \neq l\,.\supset.\,\mu_p \leqq a + e_n .$$

Take a sequence $\{\mathfrak{R}_n\}$ such that for every n $\mathfrak{R}^l_n$ and $\mathfrak{R}_n^{\mathfrak{R}_{e_n}}$, and such that if $\mathfrak{R}^l$ the classes $\mathfrak{R}_n$, after a finite number, are all contained in $\mathfrak{R}$, all of which may be done by means of postulate (III). By condition (2) of definition 1 we have

$$(2) \qquad n :\supset: \exists\, p_n^{\mathfrak{R}_n} \neq l \ni \mu_{p_n} \geqq a - e_n .$$

Clearly, $Lp_n = l$, and since p_n is an element of $\mathfrak{R}_{e_n}$, we have by (1) and (2)

$$n :\supset: |\,\mu_{p_n} - a\,| \leqq 2\,e_n ,$$

and, the limit of e_n being zero, it is seen that the limit of μ_{p_n} is a, so that the sequence $\{p_n\}$ is of the kind desired.

In case $\overline{\lim\limits_{p \to l}}\, \mu_p = -\infty$ the proof is entirely obvious, and an entirely analogous proof is available for the condition (b).

THEOREM VI. (a) *For a given limiting element* l, $\lim\limits_{p \to l} \mu_p = a$ *if and only if for every sequence* $\{p_n\}$ *of distinct elements such that* $L\, p_n = l$ *it is true that* $\lim\limits_{n \to \infty} \mu_{p_n} = a$.

(b) *Proposition* (a) *still holds if* a *be replaced by* $+\infty$ *or by* $-\infty$.

This theorem* is an immediate consequence of Theorems IV and V.

§ 10. *Iterated Limits at an Element of a Composite Range.*

Let the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ be the composite of two systems, $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$, and let $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, where $\mathfrak{P}'$ and $\mathfrak{P}''$ are subclasses of $\overline{\mathfrak{P}}'$ and $\overline{\mathfrak{P}}''$ respectively. Let $\mathfrak{L}'$ and $\mathfrak{L}''$ be the derived classes of $\mathfrak{P}'$ and $\mathfrak{P}''$, and let $\mathfrak{Q}' = \mathfrak{P}' + \mathfrak{L}'$ and $\mathfrak{Q}'' = \mathfrak{P}'' + \mathfrak{L}''$; then we have for the derived class of $\mathfrak{P}$ the class $\mathfrak{L} = \mathfrak{L}'\mathfrak{Q}'' + \mathfrak{Q}'\mathfrak{L}''$, while $\mathfrak{Q} = \mathfrak{P} + \mathfrak{L} = \mathfrak{Q}'\mathfrak{Q}''$.

We consider again a definite function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$. For every p' the symbol $\mu_{p'}$ represents a function on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$, having for every p'' the function-value $\mu_{p'p''}$. Similarly, for every p'' $\mu_{p''}$ is a function on $\mathfrak{P}'$ to $\hat{\mathfrak{A}}$. The symbols

$$\overline{\lim_{p' \to l'}}\, \mu_{p'p''}, \quad \underline{\lim_{p' \to l'}}\, \mu_{p'p''}, \quad \overline{\lim_{p'' \to l''}}\, \mu_{p'p''}, \quad \underline{\lim_{p'' \to l''}}\, \mu_{p'p''}$$

represent definite functions, the first two being defined on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ and the last two on $\mathfrak{P}'$ to $\hat{\mathfrak{A}}$.

THEOREM VII. *For a given limiting element* $l = l'l''$ *we have*

$$\text{(a)} \quad \underline{\lim_{p'' \to l''}}\, \underline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \underline{\lim_{p'' \to l''}}\, \overline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \overline{\lim_{p'' \to l''}}\, \overline{\lim_{p' \to l'}}\, \mu_{p'p''},$$

$$\text{(b)} \quad \underline{\lim_{p'' \to l''}}\, \underline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \overline{\lim_{p'' \to l''}}\, \underline{\lim_{p' \to l'}}\, \mu_{p'p''} \leqq \overline{\lim_{p'' \to l''}}\, \overline{\lim_{p' \to l'}}\, \mu_{p'p''}.$$

It is clear that the symbols indicated always represent definite elements of the class $\hat{\mathfrak{A}}$, and the inequalities follow easily from theorem IV.

THEOREM VIII. *If* $l = l'l''$, *then there exists a sequence* $\{p_n\}$ *of distinct elements such that*

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* Compare Alfred Pringsheim, "Zur Theorie der zweifach unendlichen Zahlenfolgen," *Mathematische Annalen*, Vol. LIII (1900), p. 301; also Franz London, "Ueber Doppelfolgen und Doppelreihen," same volume, p. 330. Analogies between the theorems of § 9 and § 10 of the present paper and theorems by Pringsheim and London in the papers here cited, analogies extending in some instances to the very details of the proofs, are so apparent that they call for no special notice. A method of specializing the present general theory to secure a theory of multiple sequences is shown in § 14.

$$L p_n = l \textit{ and } \lim_{n \to \alpha} \mu_{p_n} = \overline{\lim_{p'' \to l''}}\ \overline{\lim_{p' \to l'}}\ \mu_{p' p''},$$

and the proposition remains true if the iterated upper limit be replaced by any of the three quantities

$$\underline{\lim_{p'' \to l''}}\ \overline{\lim_{p' \to l'}}\ \mu_{p' p''}, \quad \overline{\lim_{p'' \to l''}}\ \underline{\lim_{p' \to l'}}\ \mu_{p' p''}, \quad \underline{\lim_{p'' \to l''}}\ \underline{\lim_{p' \to l'}}\ \mu_{p' p''}.$$

We indicate the proof for the case of the iterated upper limit when this is a finite number a. Let θ be a function on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that for every p'' $\theta_p{}'' = \overline{\lim_{p' \to l'}}\ \mu_{p' p''}$; then $\overline{\lim_{p'' \to l''}}\ \theta_{p''} = a$, and by theorem V there is a sequence $\{p''_n\}$ of distinct elements such that $L p''_n = l''$ and such that

$$(1) \qquad e : \supset : \exists\, n_e \ni n > n_e \,.\supset .\, | \theta_{p''_n} - a | \leqq \frac{e}{2}.$$

We may regard the sequence $\{p''_n\}$ as chosen so that $\theta_{p''_n}$ is finite for every n; then again, by theorem V, there exists for every n a sequence $\{p'_{m n}\}$ of distinct elements such that $L p'_{m n} = l'$ and such that $\lim\limits_{m \to \infty} \mu_{p'_{mn} p''_n} = \theta_{p''_n}$. Take a sequence $\{e_n\}$ such that the limit of e_n is zero. We have

$$(2) \qquad n : \supset : \exists\, m_{e_n} \ni m > m_{e_n} \,.\supset .\, | \mu_{p'_{mn} p''_n} - \theta_{p''_n} | \leqq e_n.$$

Consider a sequence $\{\mathfrak{R}'_n\}$ such as is required to exist for l' by postulate (III). Take a sequence $\{m_n\}$ such that for every n $p'_{m_n n}$ is in $\mathfrak{R}'_n$ and $m_n > m_{e_n}$; then clearly $L p'_{m_n n} = l'$ and by (2) we have

$$(3) \qquad n \,.\supset \qquad {}_{m_n n} {}_{p''_n} - \theta_{p''_n} | \leqq e_n.$$

For a given e we may take $\bar{n}_e$ so that $e_n \leqq \frac{e}{2}$ for $n > \bar{n}_e$; therefore we see by (1) and (2) that $\lim\limits_{n \to \infty} \mu_{p'_{m_n n} p''_n} = a$. Take a sequence $\{p_n\}$ such that for all values of n $p_n = p'_{m_n n}\, p''_n$; then by theorem VIII, (a), of Chapter II, we have $L p_n = l$, and also $\lim\limits_{n \to \infty} \mu_{p_n} = a$.

The proofs of the remaining cases offer no new difficulties, and are omitted in the interests of brevity.

Theorem IX. *If* $l = l' l''$, *then*

(a) $\underline{\lim_{p \to l}}\ \mu_p \leqq \underline{\lim_{p'' \to l''}}\ \underline{\lim_{p' \to l'}}\ \mu_{p' p''} \leqq \overline{\lim_{p'' \to l''}}\ \overline{\lim_{p' \to l'}}\ \mu_{p' p''} \leqq \overline{\lim_{p \to l}}\ \mu_p,$

(b) *If the limit of* μ_p *as* p *approaches* l *exists,** *then*

$$\lim_{p'' \to l''} \overline{\lim_{p' \to l'}}\ \mu_{p' p''} = \lim_{p'' \to l''} \underline{\lim_{p' \to l'}}\ \mu_{p' p''} = \lim_{p \to l} \mu_p.$$

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* The term "exists" is employed in the usual sense, to indicate that the limit is defined, *i. e.*, represents a definite real number or $+\infty$ or $-\infty$. The theorem may be regarded as comparing iterated limits with multiple limits; we can not, however, adopt with consistency any special notation to indicate that a limit is "multiple," since no class is assumed to be linear, and the limits studied in § 9 may be regarded as multiple limits of any order desired.

The propositions of this theorem are easy deductions from theorems V, VI, VII and VIII.

We have just derived, in terms of iterated limits, a necessary condition for the existence of the limit of μ_p as p approaches a limiting element l. To derive sufficient conditions on the iterated limits for the existence of the limit is a matter calling for additional restrictions. For this purpose it is convenient to introduce certain uniformity features. We indicate that a condition is satisfied uniformly with respect to the elements of a class by placing the notation for the class in parentheses following the symbol for the condition. Thus, if $\overline{\mathfrak{R}}''$ is a subclass of $\mathfrak{P}''$, and θ is a function on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$, finite for every p'' in $\overline{\mathfrak{R}}''$, and such that $\lim\limits_{p'\to l'} \mu_{p'p''} = \theta_{p''}$ for every p'', then we may indicate that the limit function θ is approached uniformly on $\overline{\mathfrak{R}}''$ as in the following definition:

Def. 5. $\lim\limits_{p'\to l'} \mu_{p'} = \theta(\overline{\mathfrak{R}}'') .:\equiv:. e : \supset : \exists\, \mathfrak{R}_e'^{\,l'} \ni p'^{\mathfrak{R}_e'} \neq l' . p''^{\overline{\mathfrak{R}}''} . \supset . |\mu_{p'p''} - \theta_{p''}| \leqq e.$

It will be observed that the uniformity of the condition consists in the existence for a given e of a single $\mathfrak{R}_e'$ effective for every p'' in $\overline{\mathfrak{R}}''$. Similarly we write:

Def. 6. (a) $\lim\limits_{p'\to l'} \mu_{p'} = +\infty\ (\overline{\mathfrak{R}}'') .:\equiv:. a : \supset : \exists\, \mathfrak{R}_a'^{\,l'} \ni p'^{\mathfrak{R}_a'} \neq l' . p''^{\overline{\mathfrak{R}}''} . \supset . \mu_{p'p''} > a,$

(b) $\lim\limits_{p'\to l'} \mu_{p'} = -\infty\ (\overline{\mathfrak{R}}'') .:\equiv:. a : \supset : \exists\, \mathfrak{R}_a'^{\,l'} \ni p'^{\mathfrak{R}_a'} \neq l' . p''^{\overline{\mathfrak{R}}''} . \supset . \mu_{p'p''} < a.$

THEOREM X. *If $l = l'l''$ and $\mathfrak{R}''$ is a class such that $\mathfrak{R}''^{\,l''}$, and if $\overline{\mathfrak{R}}''$ is the greatest common subclass of $\mathfrak{R}''$ and $\mathfrak{P}''$, then the following propositions hold:*

(a) $\lim\limits_{p'\to l'} \mu_{p'} = \theta(\overline{\mathfrak{R}}'') . \supset . \overline{\lim\limits_{p\to l}}\, \mu_p = \overline{\lim\limits_{p''\to l''}}\, \theta_{p''} . \underline{\lim\limits_{p\to l}}\, \mu_p = \underline{\lim\limits_{p''\to l''}}\, \theta_{p''},$

(b) $\lim\limits_{p'\to l'} \mu_{p'} = +\infty\ (\overline{\mathfrak{R}}'') . \supset . \lim\limits_{p\to l} \mu_p = +\infty,$

(c) $\lim\limits_{p'\to l'} \mu_{p'} = -\infty\ (\overline{\mathfrak{R}}'') . \supset . \lim\limits_{p\to l} \mu_p = -\infty;$

provided that in case l' is a p' we have in the respective cases the additional hypotheses

(a) $\underline{\lim\limits_{p''\to l''}}\, \theta_{p''} \leqq \underline{\lim\limits_{p''\to l''}}\, \mu_{l'p''}$ *and* $\overline{\lim\limits_{p''\to l''}}\, \mu_{l'p''} \leqq \overline{\lim\limits_{p''\to l''}}\, \theta_{p''},$

(b) $\lim\limits_{p''\to l''} \mu_{l'p''} = +\infty,$

(c) $\lim\limits_{p''\to l''} \mu_{l'p''} = -\infty;$

and in case l'' is a p'', in (a) *the further hypothesis*

$$\underline{\lim\limits_{p''\to l''}}\, \theta_{p''} \leqq \theta_{l''} \leqq \overline{\lim\limits_{p''\to l''}}\, \theta_{p''}.$$

We prove proposition (a), supposing that l' and l'' are improper limiting elements of $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively. Suppose first that $\overline{\lim\limits_{p''\to l''}}\, \theta_{p''} = a$; then for a given e we have

$$(1) \qquad \exists\, \mathfrak{R}_e''^{\,l''} \ni p''^{\mathfrak{R}_e''} . \supset . \theta_{p''} \leqq a + \frac{e}{2},$$

$$(2) \qquad \mathfrak{R}_1''^{\,l''} . \supset . \exists\, p''^{\mathfrak{R}_1'} \ni \theta_{p''} \geqq a - \frac{e}{2},$$

and by definition 5,

$$(3) \qquad \exists\, \mathfrak{R}_e'^{\,l'} \ni p'^{\mathfrak{R}_e'} . p''^{\bar{\mathfrak{R}}''} . \supset . | \mu_{p'p''} - \theta_{p''} | \leqq \frac{e}{2}.$$

Let $\mathfrak{R}_2''$ be a common subclass of $\mathfrak{R}_e''$ and $\mathfrak{R}''$, such that $\mathfrak{R}_2''^{\,l''}$, since such classes exist by postulate (III); then by (1) and (3) we have

$$p'^{\mathfrak{R}_2''} . \supset . \theta_{p''} \leqq a + \frac{e}{2},$$

$$p'^{\mathfrak{R}_e'} . p''^{\mathfrak{R}_2''} . \supset . | \mu_{p'p''} - \theta_{p''} | \leqq \frac{e}{2};$$

therefore, if $\mathfrak{R}_e = \mathfrak{R}_e' \mathfrak{R}_2''$, we see that

$$(4) \qquad p^{\mathfrak{R}_e} . \supset . \mu_p \leqq a + e.$$

Any class $\mathfrak{R}$ such that $\mathfrak{R}^l$ must be of the form $\mathfrak{R} = \mathfrak{R}_3' \mathfrak{R}_3''$, where $\mathfrak{R}_3'^{\,l'}$ and $\mathfrak{R}_3''^{\,l''}$. Since $\mathfrak{R}_3'$ and $\mathfrak{R}_e'$ have elements in common, and since $\mathfrak{R}_3''$ and $\mathfrak{R}''$ have a common subclass which fulfils the hypothesis of (2), we have from (2) and (3)

$$(5) \qquad \exists\, p^{\mathfrak{R}} \ni \mu_p \geqq a - e.$$

We see, then, from (4) and (5), that $\overline{\lim\limits_{p\to l}}\, \mu_p = a$.

Now suppose that $\overline{\lim\limits_{p''\to l''}}\, \theta_{p''} = +\infty$; then for a given a we have

$$(6) \qquad \mathfrak{R}_4''^{\,l''} . \supset . \exists\, p''^{\mathfrak{R}_4} \ni \theta_{p''} > a + 1,$$

and, taking $e \leqq 1$, we have by (3),

$$p'^{\mathfrak{R}_e'} . p''^{\mathfrak{R}''} . \supset . | \mu_{p'p''} - \theta_{p''} | \leqq 1.$$

Let $\mathfrak{R}^l$ and let $\mathfrak{R} = \mathfrak{R}_5' \mathfrak{R}_5''$, then since $\mathfrak{R}_5'$ and $\mathfrak{R}_e'$ have elements in common, and since $\mathfrak{R}_5''$ and $\mathfrak{R}''$ have a common subclass which fulfils the hypothesis of (6), there is a p in $\mathfrak{R}$ such that $\mu_p > a$, and thus $\overline{\lim\limits_{p\to l}}\, \mu_p = +\infty$.

The proof in case $\overline{\lim\limits_{p''\to l''}}\, \theta_{p''} = -\infty$ is not difficult. Under the same hypotheses with respect to l' and l'' we easily see that $\underline{\lim\limits_{p''\to l''}}\, \theta_{p''} = \underline{\lim\limits_{p\to l}}\, \mu_p$.

If we remove the restrictions on l' and l'', the additional hypotheses then available easily lead to the desired conclusion.

The proofs of (b) and (c), which offer no new difficulties, are omitted.

We obtain conclusions similar to those of theorem X under more mild hypotheses as follows:

THEOREM XI. *If* $l = l'\,l''$ *and if* $\bar{\theta}$ *and* $\underline{\theta}$ *are functions on* $\mathfrak{P}''$ *to* $\hat{\mathfrak{A}}$, *then*

(a) *A sufficient condition for the relation* $\overline{\lim\limits_{p \to l}}\, \mu_p \leqq \overline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''}$ *is*

$$\exists\, \mathfrak{N}''^{l''} \ni [e :\supset: \exists\, \mathfrak{N}'^{l'}_e \ni p'^{\mathfrak{N}'_e} \,.\, p''^{\mathfrak{N}''} \,.\!\supset\!.\, \mu_{p'p''} \leqq \bar{\theta}_{p''} + e],$$

(b) *A sufficient condition for the relation* $\underline{\lim\limits_{p \to l}}\, \mu_p \geqq \underline{\lim\limits_{p'' \to l''}}\, \underline{\theta}_{p''}$ *is*

$$\exists\, \mathfrak{N}''^{l''} \ni [e :\supset: \exists\, \mathfrak{N}'^{l'}_e \ni p'^{\mathfrak{N}'_e} \,.\, p''^{\mathfrak{N}''} \,.\!\supset\!.\, \mu_{p'p''} \geqq \underline{\theta}_{p''} - e];$$

provided that if l'' *is a* p'' *we have for* (a) *and* (b) *the respective conditions,* $\overline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''} \geqq \bar{\theta}_{l''}$ *and* $\underline{\lim\limits_{p'' \to l''}}\, \underline{\theta}_{p''} \leqq \underline{\theta}_{l''}$.

We observe that if $\bar{\theta}_{p''}$ is finite for every p'' in $\mathfrak{N}''$, and if $\bar{\theta} = \underline{\theta}$, then the combined hypotheses of (a) and (b) are equivalent to the hypothesis of (a) of theorem X, while the combined conclusions are equivalent to the conclusion of X, (a) only in case $\overline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''} = \underline{\lim\limits_{p'' \to l''}}\, \bar{\theta}_{p''} = \overline{\lim\limits_{p'' \to l''}}\, \theta_{p''} = \underline{\lim\limits_{p'' \to l''}}\, \theta_{p''}$. If for every p'' in $\mathfrak{N}''$ we suppose $\bar{\theta}_{p''} = -\infty$, then (a) is equivalent to theorem X, (c), both in hypothesis and in conclusion, and similarly X, (b) is a corollary of XI, (b).

§ 11. *Continuity of Functions.*

We return now to the general situation considered in § 9. Making use of the notations there employed, we turn attention to questions of the continuity of the function μ on the range $\mathfrak{P}$.

Def. 7. A function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ is *continuous* on $\mathfrak{P}$ if and only if μ_p is finite for every p, and for every proper limiting element l of $\mathfrak{P}$ $\lim\limits_{p \to l} \mu_p = \mu_l$.

The definition here given is analogous to the usual definition of continuity of a function of a real variable. With the present postulates on our system $(\mathfrak{P}; \mathfrak{U}; T)$ we are not able to define an analogue of uniform continuity on a range; but by requiring, in addition to the conditions of definition 7, that there shall exist a finite limit for the function μ at every improper limiting element of $\mathfrak{P}$, we obtain a form of continuity that in most applications is equivalent to uniform continuity. We call this extensible continuity, and a function having this property is said to be extensibly continuous. We have then:

Def. 8. μ is *extensibly continuous* on $\mathfrak{P}$ if and only if it is continuous on $\mathfrak{P}$ and for every improper limiting element l of $\mathfrak{P}$ there exists an a_l such that $\lim\limits_{p \to l} \mu_p = a_l$.

The following theorem is obvious:

THEOREM XII. *If $\mathfrak{P}$ is closed, then μ is continuous on $\mathfrak{P}$ if and only if μ is extensibly continuous on $\mathfrak{P}$.*

With every function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ there are two associated functions, ϕ and ψ, called respectively the upper and lower limiting functions of μ on $\mathfrak{P}$.

Def. 9. ϕ, the upper limiting function of μ on $\mathfrak{P}$, is a function on the range $\mathfrak{Q}$ such that, for every l, $\phi_l = \overline{\lim\limits_{p \to l}}\, \mu_p$ and, for every p not in $\mathfrak{L}$, $\phi_p = \mu_p$.

Def. 10. ψ, the lower limiting function of μ on $\mathfrak{P}$, is a function on the range $\mathfrak{Q}$ such that, for every l, $\psi_l = \underline{\lim\limits_{p \to l}}\, \mu_p$ and, for every p not in $\mathfrak{L}$, $\psi_p = \mu_p$.

The functions ϕ and ψ are then functions on $\mathfrak{Q}$ to $\hat{\mathfrak{A}}$. They lead to greater economy in the statement of propositions on the function μ.

THEOREM XIII. (a) *μ is continuous on $\mathfrak{P}$ if and only if for every element p the function-values ϕ_p, ψ_p and μ_p are equal and finite.*

(b) *μ is extensibly continuous on $\mathfrak{P}$ if and only if for every q the function-values ϕ_q and ψ_q are equal and finite, and for every p they are equal to μ_p.*

(c) *If μ is extensibly continuous on $\mathfrak{P}$ and $\overline{\mathfrak{R}}$ is the greatest common subclass of $\overline{\mathfrak{P}}$ and $\mathfrak{Q}$, then ϕ is extensibly continuous on $\overline{\mathfrak{R}}$ and for every u in $\mathfrak{Q}$* $\lim\limits_{\bar{r} \to u} \phi_{\bar{r}} = \phi_u$.

The truth of (a) and (b) is obvious. As to (c), it should be remarked that since $\overline{\mathfrak{R}}$ is a subclass of $\mathfrak{Q}$, ϕ is defined on $\overline{\mathfrak{R}}$, and since $\overline{\mathfrak{R}}$ is a subclass of $\overline{\mathfrak{P}}$, continuity and extensible continuity are defined for functions on $\overline{\mathfrak{R}}$ to $\hat{\mathfrak{A}}$. Further, in view of theorem II, (b), of § 6, and theorem VII, of § 7, we see that every limiting element of $\overline{\mathfrak{R}}$ is a limiting element of $\mathfrak{P}$. Since by (b) ϕ is finite for every $\bar{r}$ (element of $\overline{\mathfrak{R}}$), it remains, for the proof of (c), merely to show that for every l we have $\lim\limits_{\bar{r} \to l} \phi_{\bar{r}} = \phi_l$. Since μ is extensibly continuous on $\mathfrak{P}$, we have for every l, $\lim\limits_{p \to l} \mu_p = \phi_l$; that is,

$$(1) \qquad e \,.\, l : \supset : \exists\, \mathfrak{R}_e^l \ni p^{\mathfrak{R}_e} \neq l \,.\, \supset .\, |\mu_p - \phi_l| \leqq \frac{e}{2}.$$

By postulate (IV) we see that

$$(2) \qquad \exists\, \mathfrak{R}_1^l \ni \bar{p}^{\mathfrak{R}_1} .\, \supset .\, \exists\, \mathfrak{R}_2^{\bar{p}} \ni \mathfrak{R}_2^{\mathfrak{R}_e}.$$

We wish now to show that

$$(3) \qquad \bar{r}^{\mathfrak{R}_1} \neq l \,.\, \supset .\, |\phi_{\bar{r}} - \phi_l| \leqq e.$$

If $\bar{r}$ is a p, this follows from the fact that $\mathfrak{R}_1$ is necessarily a subclass of $\mathfrak{R}_e$. If $\bar{r}$ is not a p, it is an improper limiting element of $\mathfrak{P}$, and we have by an application of (1),

(4) $$\exists\, \overline{\mathfrak{R}}_e' \ni p^{\overline{\mathfrak{R}}_e} \,.\supset.\, |\phi_{\overline{r}} - \mu_p| \leqq \frac{e}{2}.$$

By use of (2) and postulates (III) and (V) we may show that $\overline{\mathfrak{R}}_e$ contains an element p of $\mathfrak{R}_e$ distinct from l; then (3) follows from (1) and (4).

THEOREM XIV. (a) *If $\mathfrak{P}$ is compact and μ is extensibly continuous on $\mathfrak{P}$, then μ is bounded on $\mathfrak{P}$, by finite bounds.*

(b) *If $\mathfrak{P}$ is compact, then there exist q_1 and q_2 such that the least upper bound of μ on $\mathfrak{P}$ is either ϕ_{q_1} or μ_{q_1}, and the greatest lower bound of μ on $\mathfrak{P}$ is either ψ_{q_2} or μ_{q_2}.*

(c) *If $\mathfrak{P}$ is compact and closed and μ is continuous on $\mathfrak{P}$, then there exist p_1 and p_2 such that the least upper bound and the greatest lower bound of μ on $\mathfrak{P}$ are respectively μ_{p_1} and μ_{p_2}.*

Proof of (a): Suppose that μ is not bounded from $+\infty$, and consider a sequence $\{a_n\}$ such that $\lim\limits_{n\to\infty} a_n = +\infty$. We have then

(1) $$n \,.\supset.\, \exists\, p_n \ni \mu_{p_n} > a_n.$$

Since $\mathfrak{P}$ is compact, and since the number of distinct elements in the sequence $\{p_n\}$ can not be finite, there exists a subsequence $\{p_{n_m}\}$ of distinct elements of the sequence $\{p_n\}$ which has some element l as limit. Since μ is extensibly continuous, there is a number a_l such that

(2) $$e :\supset: \exists\, m_e \ni m > m_e \,.\supset.\, |\mu_{p_{n_m}} - a_l| \leqq e.$$

Now we may take an n_1 such that for $n > n_1$ we have $a_n > a_l + e$, and since there exist values of m greater than m_e such that n_m is greater than n_1, we see that (1) and (2) are contradictory. Thus μ is bounded from $+\infty$, and in similar manner we may show that μ is bounded from $-\infty$.

Proof of (b): The least upper bound of μ on $\mathfrak{P}$ may be a finite number a or $+\infty$. In either case there exists an element p such that μ_p is this least upper bound, or there is a sequence $\{p_n\}$ such that the limit of μ_{p_n} is this least upper bound. If the former is true, then $q_1 = p$ meets the requirements of the theorem; if the latter is true, then there is a subsequence $\{p_{n_m}\}$ of distinct elements of $\{p_n\}$ which has a limit l, and clearly we have ϕ_l as the least upper bound of μ on $\mathfrak{P}$. Similarly q_2 exists, fulfilling the conditions of the theorem.

Proof of (c): Since $\mathfrak{P}$ is closed, every q is a p, and since μ is continuous, $\phi_p' = \mu_p = \psi_p$ for every p; therefore (c) is a corollary of (b).

§ 12. *Functions on a Composite Range.*

Returning to the special case when the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is the composite of the two systems $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$, and using the notations $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{L}'$ and $\mathfrak{P}''$, $\mathfrak{Q}''$, $\mathfrak{L}''$, as in § 10, we discuss the character of a function μ

on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ with regard to continuity and related properties. Use is made also of the upper and lower limiting functions, ϕ and ψ, of μ on $\mathfrak{P}$. We notice that for every q' $\phi_{p'}$ and $\psi_{q'}$ are definite functions on $\mathfrak{Q}''$, and for every q'' $\phi_{p''}$ and $\psi_{q''}$ are definite functions on $\mathfrak{Q}'$.

THEOREM XV. (a) *If l' is such that for every l of the form $l = l'\,p''$ ϕ_l is finite and $\lim\limits_{p\to l} \mu_p = \phi_l$, then $\phi_{l'}$ is continuous on $\mathfrak{P}''$.*

(b) *If l' is such that for every l of the form $l = l'\,q''$ ϕ_l is finite and $\lim\limits_{p\to l} \mu_p = \phi_l$, then $\phi_{l'}$ is extensibly continuous on $\mathfrak{P}''$ and for every l'' $\lim\limits_{p''\to l''} \phi_{l'\,p''} = \phi_{l'\,l''}$.*

(c) *If μ is continuous on $\mathfrak{P}$, then for every p'' $\mu_{p''}$ is continuous on $\mathfrak{P}'$ and for every p' $\mu_{p'}$ is continuous on $\mathfrak{P}''$.*

(d) *If μ is extensibly continuous on $\mathfrak{P}$, then for every p'' $\mu_{p''}$ is extensibly continuous on $\mathfrak{P}'$ and for every p' $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$.*

The propositions of this theorem are easy deductions from theorems I and IX, (b).

In the following theorem we employ the notation for uniform approach to a limit that was introduced in definition 5.

THEOREM XVI. (a) *If l' is such that in every $\mathfrak{R}'$ such that $\mathfrak{R}'^{\,l'}$ there is a p' distinct from l' such that $\mu_{p'}$ is continuous on $\mathfrak{P}''$, and if there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p'\to l'} \mu_{p'} = \theta\,(\mathfrak{P}'')$, then θ is continuous on $\mathfrak{P}''$.*

(b) *If l' is such that in every $\mathfrak{R}'$ such that $\mathfrak{R}'^{\,l'}$ there is a p' distinct from l' such that $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$, and if there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p'\to l'} \mu_{p'} = \theta\,(\mathfrak{P}'')$, then θ is extensibly continuous on $\mathfrak{P}''$.*

(c) *With the additional hypothesis that l' is an improper limiting element of $\mathfrak{P}'$, or, in case l' is a p', that $\mu_{l'p''} = \theta_{p''}$ for every p'', we have for both* (a) *and* (b) *the additional conclusion that for every p'' $\theta_{p''} = \phi_{l'p''} = \psi_{l'p''}$, and for* (b) *the further conclusion that for every l'' $\lim\limits_{p''\to l''} \theta_{p''} = \phi_{l'l''} = \psi_{l'l''}$.*

We prove first the lemma:

LEMMA. *If $l = l'\,l''$ and $\mathfrak{R}''^{\,l''}$ and $\overline{\mathfrak{R}}''$ is the greatest common subclass of $\mathfrak{P}''$ and $\mathfrak{R}''$, and if for every $\mathfrak{R}'$ such that $\mathfrak{R}'^{\,l'}$ there is a p' distinct from l' in $\mathfrak{R}'$ and a number a such that $\lim\limits_{p''\to l''} \mu_{p'p''} = a$, and if there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p'\to l'} \mu_{p'} = \theta\,(\overline{\mathfrak{R}}'')$, then there is an a_1 such that $\lim\limits_{p''\to l''} \theta_{p''} = a_1$.*

By the uniform approach to the function θ, we have for a given e,

$$(1) \qquad \exists\, \mathfrak{R}_e'^{\,l'} \ni p'^{\,\mathfrak{R}'_e} \neq l' \cdot p''^{\,\overline{\mathfrak{R}}''} \cdot \supset \cdot \,|\mu_{p'p''} - \theta_{p''}| \leq \frac{e}{3};$$

and by the remaining hypothesis on μ there is a p_1' in $\mathfrak{R}_e'$ such that, taking account of theorem I, we have

$$(2) \qquad \exists\, \mathfrak{R}_1''\, l'' \ni p_1''\,\mathfrak{R}_1'' \neq l'' \,.\, p_2''\,\mathfrak{R}_1'' \neq l'' \,.\supset.\, | \mu_{p_1' p_1''} - \mu_{p_1' p_2''} | \leqq \frac{e}{3}.$$

By postulate (III) it may be shown that there exists a common subclass $\mathfrak{R}_2''$ of $\mathfrak{R}_1''$ and $\mathfrak{R}''$ such that $\mathfrak{R}_2''\, l''$. If p_1'' and p_2'' are any two elements of $\mathfrak{R}_2''$ distinct from l'', we have by (1) and (2) the three conditions

$$| \theta_{p_1''} - \mu_{p_1' p_1''} | \leqq \frac{e}{3}, \quad | \mu_{p_1' p_2''} - \theta_{p_2''} | \leqq \frac{e}{3} \quad \text{and} \quad | \mu_{p_1' p_1''} - \mu_{p_1' p_2''} | \leqq \frac{e}{3},$$

from which we obtain $| \theta_{p_1''} - \theta_{p_2''} | \leqq e$. Since condition (1) of theorem II is obviously fulfilled, we conclude that there exists an a_1 such that $\lim\limits_{p'' \to l''} \theta_{p''} = a_1$.

From the lemma and theorem X, (a), the present theorem should now be evident.

The language "μ is continuous on $\mathfrak{P}'$" might conveniently be used to indicate that for every p'' the function $\mu_{p''}$ is continuous on $\mathfrak{P}'$; and this manner of speaking is especially advantageous if the continuity on $\mathfrak{P}'$ is uniform on $\mathfrak{P}''$, *i. e.*, uniform with respect to p''. Thus we have

Def. 11. (a) μ is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ if and only if for every p the function value μ_p is finite, and for every proper limiting element l' of $\mathfrak{P}'$ $\lim\limits_{p' \to l'} \mu_{p'} = \mu_{l'}\ (\mathfrak{P}'')$.

(b) μ is extensibly continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ if and only if μ is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$, and for every improper limiting element l' of $\mathfrak{P}'$ there exists a function θ on $\mathfrak{P}''$ to $\hat{\mathfrak{A}}$ such that $\lim\limits_{p' \to l'} \mu_{p'} = \theta\ (\mathfrak{P}'')$.

The following theorem is a result of an easy application of the propositions of theorem XVI.

THEOREM XVII. (a) *If μ is continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ and if for every p' $\mu_{p'}$ is continuous on $\mathfrak{P}''$, then μ is continuous on $\mathfrak{P}$.*

(b) *If μ is extensibly continuous on $\mathfrak{P}'$ uniformly on $\mathfrak{P}''$ and if for every p' $\mu_{p'}$ is extensibly continuous on $\mathfrak{P}''$, then μ is extensibly continuous on $\mathfrak{P}$.*

The following theorem, the notations of which may easily be interpreted by analogy with those previously defined, is not without interest, and is found convenient in some applications that follow.

THEOREM XVIII. *If $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is the composite of three systems, $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$, $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ and $(\overline{\mathfrak{P}}'''; \mathfrak{U}'''; T''')$, and if μ is defined on $\mathfrak{P} \doteq \mathfrak{P}'\,\mathfrak{P}''\,\mathfrak{P}'''$, and $l = l'\, l''\, l'''$, then the following propositions hold:*

(a) *If there exists a function* θ *on* $\mathfrak{P}'\mathfrak{P}''$ *to* $\mathfrak{A}$ *such that* $\lim\limits_{p'''\to l'''} \mu_{p'''} = \theta(\mathfrak{P}'\mathfrak{P}'')$, *and if there exists a function* α *on* $\mathfrak{P}'''$ *to* $\mathfrak{A}$ *such that for every* p''' $\lim\limits_{p''\to l''} \lim\limits_{p'\to l'} \mu_{p'p''p'''} = \alpha_{p'''}$, *then there exists a number* a *such that* $\lim\limits_{p''\to l''} \lim\limits_{p'\to l'} \theta_{p'p''} = \lim\limits_{p'''\to l'''} \alpha_{p'''} = a$.

(b) *If, in addition to the hypotheses of* (a), *there exists a function* ξ *on* $\mathfrak{P}''\mathfrak{P}'''$ *to* $\mathfrak{A}$ *such that for every* p''' $\lim\limits_{p'\to l'} \mu_{p'p'''} = \xi_{p'''}(\mathfrak{P}'')$, *then there exists a function* γ *on* $\mathfrak{P}''$ *to* $\mathfrak{A}$ *such that* $\lim\limits_{p'\to l'} \theta_{p'} = \gamma(\mathfrak{P}'')$.

Proof: By the first hypothesis in (a) we see that

$$(1) \qquad p'' . \supset . \lim_{p'''\to l'''} \mu_{p''p'''} = \theta_{p''}(\mathfrak{P}'),$$

and by the second hypothesis we know that there exists a function ξ on $\mathfrak{P}''\mathfrak{P}'''$ such that

$$(2) \qquad p'' . p''' . \supset . \lim_{p'\to l'} \mu_{p'p''p'''} = \xi_{p''p'''};$$

therefore, applying the lemma to theorem XVI and theorems X, (a) and IX, (b), we see that there exists a γ on $\mathfrak{P}''$ such that

$$(3) \qquad p'' . \supset . \lim_{p'\to l'} \theta_{p'p''} = \lim_{p'''\to l'''} \xi_{p''p'''} = \gamma_{p''}.$$

Now, for a given e, the first hypothesis in (a), (2) and (3) give respectively the conditions

$$(4) \qquad \exists\, \mathfrak{R}_e'''^{\,l'''} \ni p' . p'' . p'''^{\mathfrak{R}_e'''} \neq l''' . \supset . |\mu_{p'p''p'''} - \theta_{p'p''}| \leqq \frac{e}{3},$$

$$(5) \qquad p'' . p''' : \supset : \exists\, \mathfrak{R}'^{\,l'}_{e\,p''p'''} \ni p'^{\mathfrak{R}'_{e\,p'p''p'''}} \neq l' . \supset . |\xi_{p''p'''} - \mu_{p'p''p'''}| \leqq \frac{e}{3},$$

$$(6) \qquad p'' : \supset : \exists\, \mathfrak{R}'^{\,l'}_{e\,p''} \ni p'^{\mathfrak{R}'_{e\,p''}} \neq l' . \supset . |\theta_{p'p''} - \gamma_{p''}| \leqq \frac{e}{3}.$$

Since for every p'' and p''' the two classes $\mathfrak{R}'_{e\,p''p'''}$ and $\mathfrak{R}'_{e\,p''}$ have a common p' distinct from l', we see from (4), (5) and (6) that $\lim\limits_{p'''\to l'''} \xi_{p'''} = \gamma(\mathfrak{P}'')$, and since by the second hypothesis $\lim\limits_{p''\to l''} \xi_{p''p'''} = \alpha_{p'''}$ for every p''', we have the conclusion of (a) by another application of theorems X, (a), IX, (b) and the lemma to XVI.

As to proposition (b), it remains to prove that the approach of $\theta_{p'}$ to γ is uniform on $\mathfrak{P}''$. This follows from (4) and the following two conditions,

which come respectively from the special hypothesis of (b) and the fact that $\lim\limits_{p''' \to l'''} \xi_{p'''} = \gamma(\mathfrak{P}'')$,

$$p''' : \supset : \exists\, \mathfrak{R}'^{\,l'}_{e\,p'''} \ni p'^{\mathfrak{R}_{e\,p'''}} \neq l' \,.\, p'' \,.\, \supset \,.\, |\mu_{p'\,p''\,p'''} - \xi_{p''\,p'''}| \leqq \frac{e}{3},$$

$$\exists\, \overline{\mathfrak{R}}'''^{\,l'''}_{e} \ni p'' \,.\, p'''^{\overline{\mathfrak{R}}'''_e} \neq l''' \,.\, \supset \,.\, |\xi_{p''\,p'''} - \gamma_{p''}| \leqq \frac{e}{3};$$

for, since $\overline{\mathfrak{R}}'''_e$ and $\mathfrak{R}'''_e$ have a common p''' distinct from l''', we have

$$p'^{\mathfrak{R}'_{e\,p'''}} \neq l' \,.\, p'' \,.\, \supset \,.\, |\theta_{p'\,p''} - \gamma_{p''}| \leqq e,$$

CHAPTER IV.

APPLICATIONS OF THE GENERAL THEORY BY DIRECT SPECIALIZATION.

§ 13. *Introductory.*

In developing the theory of Chapters I, II and III it has not been necessary to specify the character of the elements under consideration, and the nature of the conditions postulated is such as to provide great latitude in the matter of applications. Special theories are obtained by particular determination either of a system $(\mathfrak{P}; R)$ which satisfies the postulates of § 2, thus giving rise to a system $(\mathfrak{P}; \mathfrak{U}; T)$ of the required character, or directly of a system $(\mathfrak{P}; \mathfrak{U}; T)$ which satisfies the postulates of § 5. In the present chapter we suggest, by means of chosen examples, certain methods of procedure to secure these special theories. The first instances used, viz., multiple sequences and functions of real variables, are chosen not because of any novelty of form or content of the results reached, but rather because of the interest that may be attached to the manner in which various familiar theorems, usually treated as independent, emerge as special cases of the same general theorem. The remaining examples are in the domain of general analysis, and are chosen to show the availability of the present method in certain fields already shown to be fruitful of interesting and useful theories.

§ 14. *Multiple Sequences.*

We may specify a system $(\mathfrak{P}; R)$ as follows: The class $\mathfrak{P}$ is the class of all positive integers; the relation $\mathfrak{R}_1 R\, \mathfrak{R}_2$ holds if and only if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are equal and consist of a single element, or there exist two positive integers, n_1 and n_2, such that $\mathfrak{R}_1$ consists of all integers greater than n_1 and $\mathfrak{R}_2$ consists of all integers greater than n_2.

This system clearly satisfies the postulates of § 2, and the resulting system $(\mathfrak{P}; \mathfrak{U}; T)$ therefore satisfies the postulates of § 5. This latter system is said to be of type A_1, and is as follows: The class $\mathfrak{P}$ is the class of all positive integers, $\mathfrak{U}$ is a singular class having only the element ∞; the relation $\mathfrak{R}^n$ holds if and only if $\mathfrak{R}$ consists of the single element n, and the relation $\mathfrak{R}^\infty$ holds if and only if $\mathfrak{R}$ consists of all integers greater than some given integer. The composite system of r systems of type A_1 is a system of type A_r.

To obtain a theory of multiple sequences, consider the special case when the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ of Chapter III is of the type A_r, and the class $\mathfrak{P}$ coincides with $\overline{\mathfrak{P}}$. A function μ on $\mathfrak{P}$ to $\hat{\mathfrak{A}}$ then gives an r-fold sequence of function-values, every one of which is a real number or $+\infty$ or $-\infty$. Since the nature of the range in this instance renders it unnecessary to place in evidence the notation for limiting element, and since it is desired to emphasize the character of the limits as multiple limits, it is expedient to adopt notation which places all the variables concerned in evidence. Accordingly the notation $\lim\limits_{(n^1 \ldots . n^r)} \mu_{n^1 \ldots . n^r}$ is used to indicate the limit of the function-value $\mu_{n^1 \ldots . n^r}$ as the variables $n^1 \ldots . n^r$ simultaneously increase without limit. Similarly, the notations $\overline{\lim\limits_{(n^1 \ldots . n^r)}} \mu_{n^1 \ldots . n^r}$ and $\underline{\lim\limits_{(n^1 \ldots . n^r)}} \mu_{n^1 \ldots . n^r}$ indicate respectively the upper and lower limits under the same conditions. Explicit definition of these symbols in the light of definitions 1, 2, 3 and 4 of Chapter III should cause no difficulty. For example, $\overline{\lim\limits_{(n^1 \ldots . n^r)}} \mu_{n^1 \ldots . n^r} = a$ is equivalent to the conditions: (a) For every e there exists $n_e^1 \ldots . n_e^r$ such that if $n^i > n_e^i$ $(i = 1, 2, \ldots , r)$, then $\mu_{n^1 \ldots . n^r} \leqq a + e$; and (b) For every e and every $n^1 \ldots . n^r$ there exist $n_1^1 \ldots . n_1^r$ such that $n_1^i > n^i$ $(i = 1, 2, \ldots , r)$ and such that $\mu_{n_1^1 \ldots . n_1^r} \geqq a - e$.

Among the contributions of § 9 to the theory of multiple sequences are the following, which we record in the form of a theorem.

Theorem I. (a) *The limit of the multiple sequence* $\{\mu_{n^1 \ldots . n^r}\}$ *is a finite number* a *if and only if the upper and lower limits of the multiple sequence are both equal to* a.

(b) *The limit of the multiple sequence* $\{\mu_{n^1 \ldots . n^r}\}$ *exists and is finite if and only if for every* $n^1 \ldots . n^r$ *there exist* $n_1^1 \ldots . n_1^r$ *such that* $n_1^i > n^i$ $(i = 1, 2, \ldots , r)$ *and such that* $\mu_{n_1^1 \ldots . n_1^r}$ *is finite, and for every* e *there exist* $n_e^1 \ldots . n_e^r$ *such that, if* $n_1^i > n_e^i$ *and* $n_2^i > n_e^i$ $(i = 1, 2, \ldots , r)$, *then* $|\mu_{n_1^1 \ldots . n_1^r} - \mu_{n_2^1 \ldots . n_2^r}| \leqq e$.

(c) *If there exist* $n_1^1 \ldots . n_1^r$ *such that, for* $n^i > n_1^i$ $(i = 1, 2, \ldots , r)$, $\mu_{n^1 \ldots . n^r}$ *is finitely bounded, then there exist numbers* $\bar{a}$ *and* $\underline{a}$ *such that*

$$\overline{\lim_{(n^1\dots n^r)}}\ \mu_{n^1\dots n^r} = \bar{a} \quad \textit{and} \quad \underline{\lim_{(n^1\dots n^r)}}\ \mu_{n^1\dots n^r} = \underline{a}.$$

(d) *If the simple sequence* $\{\mu_{n^1_m\dots n^r_m}\}$ *is such that* $\lim\limits_{m\to\infty} n^i_m = \infty$ $(i = 1, 2, \dots, r)$, *then*

$$\underline{\lim_{(n^1\dots n^r)}}\ \mu_{n^1\dots n^r} \leqq \underline{\lim}_{m\to\infty}\ \mu_{n^1_m\dots n^r_m} \leqq \overline{\lim}_{m\to\infty}\ \mu_{n^1_m\dots n^r_m} \leqq \overline{\lim_{(n^1\dots n^r)}}\ \mu_{n^1\dots n^r}.$$

(e) *There exists a simple sequence of the kind described in* (d) *having for limit* $\overline{\lim\limits_{(n^1\dots n^r)}}\ \mu_{n^1\dots n^r}$, *and also one having for limit* $\underline{\lim\limits_{(n^1\dots n^r)}}\ \mu_{n^1\dots n^r}$.

(f) *The limit of the multiple sequence exists, finite or infinite, if and only if all simple sequences of the kind described in* (d) *have the same limit; and the limit of the multiple sequence is the common limit of the simple sequences.*

In § 10 we may consider the systems $(\mathfrak{P}'; \mathfrak{U}'; T')$ and $(\mathfrak{P}''; \mathfrak{U}''; T'')$ to be of types A_{r_1} and A_{r_2} respectively, where $r_1 + r_2 = r$; then our hypothesis with respect to $(\mathfrak{P}; \mathfrak{U}; T)$ is fulfilled. The real force of the theorems on iterated limits is here realized only by repeated application of the principles established, a process made available by the persistence, under composition of systems, of the conditions specified in our postulates. We may conveniently use the notation* $\overline{\lim\limits_{(n^1\dots n^s)\dots(n^t\dots n^r)}}\ \mu_{n^1\dots n^r}$ to denote the result of taking the upper limits as the variables $n^1\dots n^r$ tend to infinity in groups, the group $n^t\dots n^r$ passing to the limit first, etc. Analogous notations, easy of interpretation, may be used for limits of other types. With a little reflection the theorems of § 10 are seen to yield the following results:

THEOREM II. (a) *For every expression of the type*

$$\lim_{\overline{(n^1\dots n^s)}\dots\underline{(n^t\dots n^r)}}\ \mu_{n^1\dots n^r},$$

where the grouping of the variables and the arrangement of upper and lower dashes are entirely arbitrary, there exists a simple sequence $\{\mu_{n^1_m\dots n^r_m}\}$ *such that* $\lim\limits_{m\to\infty} n^i_m = \infty$ $(i = 1, 2, \dots, r)$ *having the given expression for limit.*

(b) *An expression of the type mentioned in* (a) *is not less than the expression obtained from it by replacing any number of upper dashes by lower dashes; or by the subdivision of any group that has an upper dash and giving the subgroups either upper or lower dashes; or by combining any number of adjacent groups and giving the combined group a lower dash. In particular,*

* Compare Bromwich and Hardy, "Some Extensions to Multiple Series of Abel's Theorem on the Continuity of Power Series," *Proceedings of the London Mathematical Society*, Series 2, Vol. II, p. 161.

*if the limit of the multiple sequence exists, then every expression of the type discussed is equal to the limit of the multiple sequence.**

(c) *If there is an s-fold sequence* $\{\theta_{n^1\ldots n^s}\}$ *such that for* $n^i > n_1^i$ $(i = 1, 2, \ldots, r)$ $\theta_{n^1\ldots n^r}$ *is finite, and such that*

$$\lim_{(n^{s+1}\ldots n^r)} \mu_{n^{s+1}\ldots n^r} = \theta(\mathfrak{N}^1\ldots\mathfrak{N}^s),$$

where $\mathfrak{N}^i$ *consists of all* n^i *greater than* n_1^i $(i = 1, 2, \ldots, r)$, *then*

$$\overline{\lim_{(n^1\ldots n^r)}}\, \mu_{n^1\ldots n^r} = \overline{\lim_{(n^1\ldots n^s)}}\, \theta_{n^1\ldots n^s} \quad \text{and} \quad \underline{\lim_{(n^1\ldots n^r)}}\, \mu_{n^1\ldots n^r} = \underline{\lim_{(n^1\ldots n^s)}}\, \theta_{n^1\ldots n^s};$$

and if for similarly chosen $\mathfrak{N}^1\ldots\mathfrak{N}^s$ *we have*

$$\lim_{(n^{s+1}\ldots n^r)} \mu_{n^{s+1}\ldots n^r} = +\infty\,(\mathfrak{N}^1\ldots\mathfrak{N}^s),$$

then we also have

$$\lim_{(n^1\ldots n^r)} \mu_{n^1\ldots n^r} = +\infty;$$

and this latter statement remains true if $+\infty$ *be replaced by* $-\infty$.

The theorems of § 11 and § 12 are clearly applicable to multiple sequences, although in some cases the results are trivial, and in some cases are identical with results already obtained from § 9 and § 10. We note here the fact that if the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is of type A_1, then an extensibly continuous function on $\overline{\mathfrak{P}}$ gives a convergent sequence of finite terms, while analogous statements hold for systems of type A_r. If we assume the system to be of type A_r and consider μ defined on the class $\mathfrak{P} = \overline{\mathfrak{P}}$, then the lemma to theorem XVI, § 12, may be interpreted as follows:

Theorem III. *If the s-fold sequence* $\{\theta_{n^1\ldots n^s}\}$ *is such that*

$$\lim_{(n^{s+1}\ldots n^r)} \mu_{n^{s+1}\ldots n^r} = \theta(\mathfrak{P}^1\ldots\mathfrak{P}^r),$$

and if for every $n^{s+1}\ldots n^r$ *there are values* $n_1^{s+1}\ldots n_1^r$ *such that* $n_1^i > n^i$ $(i = s+1, \ldots, r)$ *and such that* $\mu_{n_1^{s+1}\ldots n_1^r}$ *is a convergent s-fold sequence, then* $\{\theta_{n^1\ldots n^s}\}$ *is a convergent s-fold sequence.*

§ 15. *Functions of Real Variables.*

To obtain applications of the general theory to functions of a real variable we might, as in the previous section, specify a system $(\mathfrak{P}; R)$ in which $\mathfrak{P}$ should be the class of real numbers and R should be so defined as to secure ideal elements corresponding to $+\infty$ and $-\infty$. For the sake of simplicity, however, we proceed at once to the specification of a system $(\mathfrak{P}; \mathfrak{U}; T)$. $\mathfrak{P}$ is

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* Compare the note by G. H. Hardy, *Proceedings of the London Mathematical Society*, Series 2, Vol. II, p. 190.

the class of all real numbers; $\mathfrak{U}$ consists of two elements, $+\infty$ and $-\infty$; a class $\mathfrak{R}$ has the relation T to a given element p_1 if and only if there is a number d such that $\mathfrak{R}$ consists of all elements p such that $|p-p_1| \leqq d$; $\mathfrak{R}$ has the relation T to $+\infty$ if and only if there is a number a such that $\mathfrak{R}$ consists of all elements p such that $p > a$; and $\mathfrak{R}$ has the relation T to $-\infty$ if and only if there is a number a such that $\mathfrak{R}$ consists of all elements p such that $p < a$.

Such a system, which obviously fulfils our postulates, is designated as a system of type B_1, and the composite of r such systems is a system of type B_r. Attention should be called to the fact that with this special determination of the system $(\mathfrak{P}; \mathfrak{U}; T)$ the definitions and theorems in Chapter II relative to limiting elements, and, pertaining to properties of subclasses of $\mathfrak{P}$, are in accordance with the usual treatment of these features of the range of a real variable.

It may be seen, without detailed discussion here, that in this instance the theory developed in Chapter III is a theory of multiple and iterated limits and continuity of functions of several real variables. The definitions and terminology employed render the interpretations of the various theorems immediate, except for the fact that the term "extensible continuity" has not been in use to denote a property of a function of a real variable. We show in theorem V that this property, for functions on a limited number set, is equivalent to the property "uniform continuity." For convenience in the proof of theorem V we prove first the following theorem:*

THEOREM IV. *If the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is of type B_r, then every subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ is compact.*

Proof: In view of theorems V, (a) and VII, (d) of § 7, it is sufficient to consider the special case when $r=1$ and $\mathfrak{P} = \overline{\mathfrak{P}}$. Let $\{p_n\}$ be a sequence of distinct elements of $\mathfrak{P}$. If any limited subclass of $\mathfrak{P}$ contains an infinite subsequence of $\{p_n\}$, then by a well-known property of the number system this subsequence gives rise to at least one limiting element. If no limited subclass of $\mathfrak{P}$ contains such a subsequence, then, considering a sequence $\{a_m\}$ of real numbers such that $\lim\limits_{m \to \infty} a_m = +\infty$, we see that for every m there is an n_m such that $p_{n_m} > a_m$ or $p_{n_m} < -a_m$. Clearly at least one of the infinite ideal elements is then a limit of a subsequence of $\{p_n\}$.

THEOREM V. *If the system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$ is of type B_r, and if μ is defined on the subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$, then we have the propositions:*

* There is a difference in the force of the term "compact" as employed here and as employed by Fréchet (*Rendiconti del Circolo Matematico di Palermo*, Vol. XXII, p. 6), due to the fact that we recognize ideal limiting elements while Fréchet does not.

(a) *If μ is extensibly continuous on $\mathfrak{P}$, then μ is uniformly continuous on $\mathfrak{P}$.*

(b) *If $\mathfrak{P}$ is limited and μ is uniformly continuous on $\mathfrak{P}$, then μ is extensibly continuous on $\mathfrak{P}$.*

Proof of (a): By the definition of extensible continuity we see that for a given limiting element p of $\mathfrak{P}$ and for an arbitrary positive number e we have

$$(1) \qquad \exists\, d_{ep} \ni p = p^1 \ldots\ldots p^r \cdot p_1 = p_1^1 \ldots\ldots p_1^r \cdot |p_1^i - p^i| \leqq d_{ep} \cdot \supset \cdot |\mu_{p_1} - \mu_p| \leqq e \qquad (i = 1, 2, \ldots\ldots, r);$$

while for an element p that is not a limiting element of $\mathfrak{P}$ we have

$$(2) \qquad \exists\, d_p \ni p = p^1 \ldots\ldots p^r \cdot p_1 = p_1^1 \ldots\ldots p_1^r \cdot |p_1^i - p^i| \leqq d_p \cdot \supset \cdot p_1 = p \qquad (i = 1, 2, \ldots\ldots, r).$$

Consider a function δ on $\mathfrak{P}$ defined as follows: For every limiting element p of $\mathfrak{P}$ let δ_p be one-half of the least upper bound of the set of values effective as d_{ep} in (1). For every element p that is not a limiting element of $\mathfrak{P}$ let δ_p be one-half of the least upper bound of the set of values effective as d_e in (2). This function δ is positive for every p, and we proceed to show that the greatest lower bound of δ on $\mathfrak{P}$ is positive. Let a_e denote this greatest lower bound of δ on $\mathfrak{P}$, for the value of e in question, then since $\mathfrak{P}$ is compact by theorem IV, we see by theorem XIV, (b), of § 10 that either there is a p such that $\delta_p = a_e$ or there is a limiting element l of $\mathfrak{P}$ such that $\underline{\lim}_{p \to l}\, \delta_p = a_e$. In the former case a_e is clearly positive. In the latter case we have by the extensible continuity of μ on $\mathfrak{P}$, applying theorem II of § 9,

$$\exists\, \mathfrak{R}_e^l \ni p_1^{\mathfrak{R}_e} \cdot p_2^{\mathfrak{R}_e} \cdot \supset \cdot |\mu_{p_1} - \mu_{p_2}| \leqq e.$$

Now we clearly have $l = q^1 \ldots\ldots q^r$ and $\mathfrak{R}_e = \mathfrak{R}^1 \ldots\ldots \mathfrak{R}^r$, where the relation $\mathfrak{R}^i q^i$ holds for $i = 1, 2, \ldots\ldots, r$; therefore each of the $\mathfrak{R}^i$ must be of one of the three forms: (a) all $\bar{p}^i$ such that $|\bar{p}^i - q^i| \leqq d^i$; (b) all $\bar{p}^i$ such that $\bar{p}^i > \bar{a}^i$; (c) all $\bar{p}^i$ such that $\bar{p}^i < \underline{a}^i$. Consider now a class $\mathfrak{R}_1 = \mathfrak{R}_1^1 \ldots\ldots \mathfrak{R}_1^r$, such that $\mathfrak{R}_1^l$, defined as follows: If $\mathfrak{R}^i$ is of form (a), then $\mathfrak{R}_1^i$ consists of all $\bar{p}^i$ such that $|\bar{p}^i - q^i| \leqq d$; if $\mathfrak{R}^i$ is of form (b), then $\mathfrak{R}_1^i$ consists of all $\bar{p}^i$ such that $\bar{p}^i > \bar{a}^i + d$; if $\mathfrak{R}^i$ is of form (c), then $\mathfrak{R}_1^i$ consists of all $\bar{p}^i$ such that $\bar{p}^i < \underline{a}^i - d$. The number d is one-half of the least of the d^i, in case any of the $\mathfrak{R}^i$ are of form (a), and otherwise d is unity. For a given $\mathfrak{R}_e$, then, $\mathfrak{R}_1$ is a definite class, and for every p in $\mathfrak{R}_1$ we have $\delta_p \geqq \frac{1}{2} d$. Clearly $\underline{\lim}_{p \to l}\, \delta_p \geqq \frac{1}{2} d$, so that a_e is positive. Now referring to (1) and (2) we see that if $p_1 = p_1^1 \ldots\ldots p_1^r$, and $p_2 = p_2^1 \ldots\ldots p_2^r$, and $|p_1^i - p_2^i| \leqq a_e$, then $|\mu_{p_1} - \mu_{p_2}| \leqq e$; that is, μ is uniformly continuous on $\mathfrak{P}$.

Proof of (b): Since uniform continuity on $\mathfrak{P}$ implies the convergence of μ_p as p approaches any proper limiting element of $\mathfrak{P}$, and since, $\mathfrak{P}$ being limited, every limiting element of $\mathfrak{P}$ is finite, it is sufficient to show that μ is convergent at every finite improper limiting element of $\mathfrak{P}$. By hypothesis we have, if $p_1 = p_1^1 \dots p_1^r$ and $p_2 = p_2^1 \dots p_2^r$,

$$(3) \qquad e : \supset : \exists d_e \ni |p_1^i - p_2^i| \leqq d_e \, (i = 1, 2, \dots, r) . \supset . |\mu_{p_1} - \mu_{p_2}| \leqq e.$$

If l is an improper limiting element of $\mathfrak{P}$, then for a given e we may take $\mathfrak{R}_e = \mathfrak{R}_e^1 \dots \mathfrak{R}_e^r$, where $\mathfrak{R}_e^i = [\text{all } \bar{p}^i \ni |\bar{p}^i - q^i| \leqq d_e/2] \; (i = 1, 2, \dots, r)$, where $l = q^1 \dots q^r$, and obviously, if p_3 and p_4 are both in $\mathfrak{R}_e$, we have by (3) $|\mu_{p_3} - \mu_{p_4}| \leqq e$. Thus, by theorem II of § 9, μ is convergent at l.

Interesting results are obtained if, in § 10 and § 12, we take one of the component systems to be of type A_r and the other of type B_r. We notice here a few special cases.

In theorem IX, (b), of § 10 let $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ be of type A_1 and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ of type B_1. Let $\mathfrak{P}' = \overline{\mathfrak{P}}'$, but let $\mathfrak{P}''$ be an arbitrary subclass of $\overline{\mathfrak{P}}''$. l' is necessarily the ideal element ∞, but we take l'' as a proper limiting element of $\mathfrak{P}''$, this class being assumed to have such a limiting element. If we replace the notation p' by n and p'' by x, and set $l'' = x_0$, the theorem yields the following:

THEOREM VI. *If* $\{\xi_n(x)\}$ *is a sequence of functions defined for every* x *of the set* $\mathfrak{P}''$, *and if* $\overline{\lim_{n\to\infty}} \xi_n(x) = \bar{\xi}(x)$ *and* $\underline{\lim_{n\to\infty}} \xi_n(x) = \underline{\xi}(x)$, *and if for every* e *there exist* n_e *and* d_e *such that* $|\xi_n(x) - a| \leqq e$ *for* $n > n_e$ *and* x *such that* $|x - x_0| \leqq d_e$, *then we have*

$$\lim_{x\to x_0} \bar{\xi}(x) = \bar{\xi}(x_0) = \lim_{x\to x_0} \underline{\xi}(x) = \underline{\xi}(x_0) = a.$$

A sequence of functions satisfying for every e the condition of the hypothesis of this theorem is here designated as a sequence "totally* convergent at x_0." By similar specialization we obtain from theorem X, (a), of § 10 the following:

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* If there is an interval $x_0 - d$ to $x_0 + d$ such that, for every x in the interval, $\lim_{n\to\infty} \xi_n(x) = \xi(x)$, and if for every e there is an n_e and a d_e such that for $n > n_e$ and $x_0 - d_e \leqq x \leqq x_0 + d$ we have the condition $|\xi_n(x) - \xi(x)| \leqq e$, then the sequence may be called "uniformly convergent at x_0," by analogy with the use of this term in the theory of series of functions (W. H. Young, *Proceedings of the London Math. Society*, Series 2, Vol. I, p. 90; also Vol. VI, p. 29). In case the limit function $\xi(x)$ exists for all values of x in an interval $x_0 - d$ to $x_0 + d$, total convergence at x_0 implies uniform convergence at x_0; and in case the number of functions of the sequence that are continuous at x_0 is not finite, uniform convergence at x_0 implies total convergence at x_0; but for an unconditioned sequence of functions total convergence at a point and uniform convergence at a point are independent properties.

THEOREM VII. *If a sequence of functions is uniformly convergent on a set consisting of all elements x in the interval from $x_0 - d$ to $x_0 + d$ and if the limit function is continuous at x_0, then the sequence is totally convergent at x_0.*

And from the same theorem, but taking $(\overline{\mathfrak{P}}'; \mathfrak{U}'; T')$ to be of type B_1, and $(\overline{\mathfrak{P}}''; \mathfrak{U}''; T'')$ of type A_1, we have

THEOREM VIII. *If the functions of a sequence are equally* continuous at x_0 and if the sequence is convergent at x_0, then the sequence is totally convergent at x_0.*

By taking account of theorem V of the present section we have the following two theorems resulting from theorems XV and XVI of § 12:

THEOREM IX. *If $\{\xi_n(x)\}$ is a sequence of functions defined on $\mathfrak{P}''$, and if the sequence is convergent for every x in $\mathfrak{P}''$, then*

(a) *If the sequence is totally convergent at every proper limiting element of $\mathfrak{P}''$, the limit function is continuous on $\mathfrak{P}''$.*

(b) *If the sequence is totally convergent at every limiting element of $\mathfrak{P}''$, the limit function is uniformly continuous on $\mathfrak{P}''$.*

THEOREM X. *If $\{\xi_n(x)\}$ is a sequence of functions defined on the limited set $\mathfrak{P}''$, and if the sequence is uniformly convergent on $\mathfrak{P}''$, then*

(a) *If for every term of the sequence there is a subsequent term that is continuous on $\mathfrak{P}''$, then the limit function is continuous on $\mathfrak{P}''$.*

(b) *If for every term of the sequence there is a subsequent term that is uniformly continuous on $\mathfrak{P}''$, then the limit function is uniformly continuous on $\mathfrak{P}''$.*

The remaining two theorems of this section are seen to follow from theorem XVII of § 12, if we take account of theorem V of the present section and remember that in the general theorems the situation is symmetrical with respect to the two component systems.

THEOREM XI. (a) *If $\{\xi_n(x)\}$ is a sequence of functions continuous on $\mathfrak{P}'$, and if the sequence is uniformly convergent on $\mathfrak{P}''$, then at every proper limiting element of $\mathfrak{P}''$ the sequence is totally convergent.*

(b) *If $\mathfrak{P}''$ is limited and the functions of the sequence are uniformly continuous on $\mathfrak{P}''$, and if the sequence is uniformly convergent on $\mathfrak{P}''$, then at every limiting element of $\mathfrak{P}''$ the sequence is totally convergent.*†

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* The functions of a sequence are equally continuous at x_0 if for every e there is a d_e such that for every n and for $|x - x_0| \leq d_e$ we have $|\xi_n(x) - \xi_n(x_0)| \leq e$. For a discussion of the term see Fréchet, *loc. cit.*, p. 11.

† Theorems XI, (a), and IX, (a), together give the well-known theorem: "A uniformly convergent sequence of continuous functions has a continuous function for limit." Propositions (b) of these two theorems give the corresponding theorem for a sequence of uniformly continuous functions. We notice that these theorems are corollaries of the two propositions of theorem X.

THEOREM XII. (a) *If the functions of the sequence* $\{\xi_n(x)\}$ *are equally continuous on* $\mathfrak{P}''$, *and if the sequence is convergent at every* x *in* $\mathfrak{P}''$, *then at every proper limiting element of* $\mathfrak{P}''$ *the sequence is totally convergent, and the limit function is continuous on* $\mathfrak{P}''$.

(b) *If* $\mathfrak{P}''$ *is limited and the functions of the sequence are equally uniformly continuous* on* $\mathfrak{P}''$, *and if the sequence is convergent at every* x *in* $\mathfrak{P}''$, *then at every limiting element of* $\mathfrak{P}''$ *the sequence is totally convergent and the limit function is uniformly continuous on* $\mathfrak{P}''$.

§ 16. *An R Relation in Terms of* K_1.

The K_1 relation used by Professor E. H. Moore in Part II of his memoir on "General Analysis" may be characterized as a relation on the composite class $\mathfrak{P}\,\mathfrak{J}$, where $\mathfrak{P}$ is a class of elements of any kind whatever, and $\mathfrak{J}$ is the class of positive integers.† In other words, K_1 may be considered defined for a class $\mathfrak{P}$ if a criterion exists by which it may be determined for every element p and integer m whether K_{pm} or $^-K_{pm}$; *i. e.*, whether the relation K_1 holds or does not hold for p and m. While Professor Moore defines certain properties for this K_1 relation, he does not permanently condition the relation by any fixed properties or postulates.

We define a relation R in terms of K_1 as follows: The relation $\mathfrak{R}_1 R\, \mathfrak{R}_2$ holds if and only if $\mathfrak{R}_1$ and $\mathfrak{R}_2$ both consist of the same single element p, or there exist two integers m_1 and m_2 such that $\mathfrak{R}_1$ consists of all elements p such that K_{pm_1} and $\mathfrak{R}_2$ consists of all elements p such that K_{pm_2}.

In order that the system $(\mathfrak{P}; R)$ obtained in this way shall satisfy the seven postulates of § 2, it is necessary that some restrictions be placed on the K_1 relation. The following conditions are found to be sufficient:

(1) For every m there exists a p such that K_{pm}.
(2) For every p there exists an m for which K_{pm} does not hold.
(3) If $m_1 < m_2$ and if K_{pm_2}, then K_{pm_1}.

Assuming that these conditions are fulfilled, we see that a class v_p consists of a single class $\mathfrak{R}$ which contains the single element p. One ideal element exists, consisting of all classes $\mathfrak{R}$ of the type $\mathfrak{R} = [\text{all } p \ni K_{pm}]$, there being a class $\mathfrak{R}$ of this class of classes corresponding to each integer m.

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* The functions of the sequence are "equally uniformly continuous" on P'' if, for every e, there is a d_e such that, if $|x_1 - x_2| \leqq d_e$, then the relation $|\xi_n(x_1) - \xi_n(x_2)| \leqq e$ holds for every n.

† E. H. Moore, "Introduction to a Form of General Analysis," p. 126.

§ 17. *Application to a System* $(\mathfrak{P}; K_2)$.

In this section we consider a system $(\mathfrak{P}; K_2)$, where $\mathfrak{P}$ is an arbitrary class of elements and K_2 is a relation on $\mathfrak{P}\mathfrak{P}\mathfrak{J}$. That is, we suppose a criterion provided by which we are able to say for every p_1, p_2 and m whether or not the relation $K_{p_1 p_2 m}$ holds. The notation K_2 is used by Professor Moore for a relation of this type,* and as in the case of the relation K_1, the relation is conditioned by various hypotheses to secure desired results in the theorems in which it is involved, but no permanent postulates or conditions are adopted.

For our purposes it is convenient to postulate the following conditions on the system† $(\mathfrak{P}; K_2)$:

(1) The relation K_{ppm} holds for every p and m.

(2) If $K_{p_1 p_2 m}$, then $K_{p_2 p_1 m}$.

(3) If $m_1 < m_2$ and if $K_{p_1 p_2 m_2}$, then $K_{p_1 p_2 m_1}$.

(4) For every m there exists an m_1 such that if $K_{p_2 p_1 m_1}$ and $K_{p_1 p m_1}$, then $K_{p_2 p m}$.

(5) If p_1 and p_2 are such that $K_{p_1 p_2 m}$ holds for every m, then $p_1 = p_2$.

Still further restrictions‡ on the system $(\mathfrak{P}; K_2)$ would be required to enable us to derive from it a system $(\mathfrak{P}; R)$ which would fulfil our postulates. These assumptions, however, furnish sufficient basis for a special definition of ideal elements, and for the determination of a system§ $(\mathfrak{P}; \mathfrak{U}; T)$ which fulfils the postulates of § 5.

An ideal element of the system $(\mathfrak{P}; K_2)$ is a class s of sequences $\{p_n\}$ of elements of $\mathfrak{P}$, having the following properties:

1. If $\{p_n\}$ is a sequence belonging to the class s, then for every m there exists an n_m such that, if n_1 and n_2 are both greater than n_m, the relation $K_{p_{n_1} p_{n_2} m}$ holds.

2. If $\{p_n\}$ and $\{\hat{p}_n\}$ are sequences belonging to the class s, then for every m there exists an n_m such that, if n_1 and n_2 are both greater than n_m, the relation $K_{p_{n_1} \hat{p}_{n_2} m}$ holds.

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* *Loc. cit.*, p. 126.

† A system of the type here indicated forms the basis of T. H. Hildebrandt's "Contribution to the Foundations of Fréchet's Calcul Fonctionnel" (AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV, p. 237). He gives a "complete existential theory" for eight properties of the system, including the first three and the fifth of the properties postulated here. The first three are among the properties considered by E. H. Moore, *loc. cit.*, p. 127.

‡ A sufficient additional restriction would be the following assumption: If $p_1 \neq p_2$, then for every m there is a p such that $K_{p_1 p m}$ holds but $K_{p_2 p m}$ does not hold.

§ A system $(\mathfrak{P}; \mathfrak{U}; T)$ in which $\mathfrak{U}$ is the null class may be derived by omitting this definition of ideal elements. In this case assumption (4) may be made less restrictive by permitting m_1 to depend on p as well as on m.

3. For every p there exists a sequence $\{p_n\}$ of the class s and an integer m such that for every n there is an n_1 greater than n such that $K_{p n_1 p m}$ does not hold.

4. Any class s_1 which has properties 1, 2 and 3 and contains s, must coincide with s.

Let $\mathfrak{U}$ denote the class of all ideal elements arising by this definition and let $\mathfrak{Q} = \mathfrak{P} + \mathfrak{U}$, then we may extend the definition of K_2, to make it a relation on $\mathfrak{P}\mathfrak{Q}\mathfrak{J}$, as follows: The relation K_{pum} holds if and only if there is a sequence $\{p_n\}$ of the class s such that for every n the relation $K_{pp_n m}$ holds.

To complete the specification of a system $(\mathfrak{P}; \mathfrak{U}; T)$ it remains to define a relation T for the classes $\mathfrak{P}$ and $\mathfrak{U}$. Let the relation $\mathfrak{R}^q$ hold if and only if there is an integer m such that $\mathfrak{R}$ consists of all elements p for which the relation K_{pqm} holds. That this system $(\mathfrak{P}; \mathfrak{U}; T)$ satisfies the postulates of § 5 easily follows from five conditions which are easily deduced from the assumptions on the system $(\mathfrak{P}; K_2)$ and the special definition of ideal elements. These five conditions, the first of which is identical with the first assumption, may be written:

$(\bar{1})$ The relation K_{ppm} holds for every p and m.

$(\bar{2})$ For every u and m there exists a p such that K_{pum}.

$(\bar{3})$ If $m_1 < m_2$ and if K_{pqm_2}, then K_{pqm_1}.

$(\bar{4})$ For every m there exists an m_1 such that if $K_{p_2 p_1 m_1}$ and $K_{p_1 q m_1}$, then $K_{p_2 q m}$.

$(\bar{5})$ If $q_1 \neq q_2$ there exists an m such that no element p can fulfil both relations $K_{p q_1 m}$ and $K_{p q_2 m}$.

The theory of Chapters II and III is available for any system $(\mathfrak{P}; K_2)$ which satisfies the assumptions of this section, by the mediation of the associated special system $(\mathfrak{P}; \mathfrak{U}; T)$. Here, as in the case of a real variable, there is a close relation between the properties "uniform continuity" and "extensible continuity." If $(\overline{\mathfrak{P}}; K_2)$ is a system which fulfils the foregoing assumptions, and μ is a function defined on a subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$, then μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists an m_e such that if $K_{p_1 p_2 m_e}$ then $|\mu_{p_1} - \mu_{p_2}| \leqq e$.

THEOREM XIII. *If* $(\overline{\mathfrak{P}}; K_2)$ *is a system fulfilling assumptions* (1) *to* (5), *then a function* μ *is uniformly continuous on a compact subclass* $\mathfrak{P}$ *of* $\overline{\mathfrak{P}}$ *if and only if, with reference to the associated system* $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, μ *is extensibly continuous on* $\mathfrak{P}$.

The proof of this theorem may be made similar to that of theorem V, § 15.

§ 18. *The Fréchet Voisinage.*

In his thesis M. Fréchet* denotes by (V) a class of elements for which there is defined, fulfilling certain conditions which he specifies, the notion of "voisinage." We may present his assumptions accurately, but in form adapted to our purposes, as follows: With the class $\mathfrak{P}$ of undefined elements is associated a function V, forming the system $(\mathfrak{P}; V)$. The function V is defined on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$; *i. e.*, it assigns to every pair of elements, p_1 and p_2, a real number, which is denoted by $(p_1 p_2)$. The conditions postulated for this system $(\mathfrak{P}; V)$ are:

(1) For every two elements p_1 and p_1 we have $(p_1 p_2) = (p_2 p_1) \geqq 0$.

(2) If $(p_1 p_2) = 0$, then $p_1 = p_2$.

(3) If $p_1 = p_2$, then $(p_1 p_2) = 0$.

(4) There exists a function $\phi(d)$ such that $\lim \phi(d) = 0$ and such that if $(p_1 p_2) \leqq d$ and $(p_2 p_3) \leqq d$, then $(p_1 p_3) \leqq \phi(d)$.

Proceeding now as in the case of a system $(\mathfrak{P}; K_2)$, we give attention to the introduction of ideal elements.† An ideal element of the system $(\mathfrak{P}; V)$ is a class s of sequences $\{p_n\}$ of elements of the class $\mathfrak{P}$ which fulfils the following conditions:

1. If $\{p_n\}$ is a sequence belonging to the class S, then for every d there exists an n_d such that, if n_1 and n_2 are both greater than n_d, then $(p_{n_1} p_{n_2}) \leqq d$.

2. If $\{p_n\}$ and $\{\hat{p}_n\}$ are sequences belonging to s, then for every d there exists an n_d such that, if n_1 and n_2 are both greater than n_d, then $(p_{n_1} \hat{p}_{n_2}) \leqq d$.

3. For every p there exists a sequence $\{p_n\}$ of the class s and a positive number d such that for every n there is an n_1 greater than n such that $(p_{n_1} p) > d$.

4. Any class s_1 which satisfies conditions 1, 2 and 3 and contains s must coincide with s.

We extend the definition of V so that it is a function on $\mathfrak{P}\mathfrak{Q}$, where, as before, $\mathfrak{Q}$ is the class $\mathfrak{P}$ with ideal elements u adjoined, by assigning to $(p\,u)$ the value d_1 of the greatest lower bound of the set of numbers d for each of which there exists a sequence $\{p_n\}$ of the class u such that $(p p_n) \leqq d$ for every n.

In terms of this extended function V a relation T is specified: The relation $\mathfrak{R}^q$ holds if and only if there exists a d such that $\mathfrak{R}$ consists of all elements p for which $(p\,q) \leqq d$. It may be shown without difficulty that the system

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* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII, p. 17.

† Here also we might specify a system $(\mathfrak{P}; R)$ in which R should be defined in terms of V in such manner as to fulfil our postulates, by the adoption of an additional condition on $(\mathfrak{P}; V)$. A condition effective for this purpose would be: If $p_1 \neq p_2$, then for every d there exists a p such that $(p\,p_1) \leqq d$, while $(p\,p_2) > d$.

$(\mathfrak{P};\mathfrak{U};T)$, now definitely determined by the system $(\mathfrak{P};V)$, satisfies the postulates of § 5. As to the significance of the general theory of Chapters II and III, in this special case, we give attention here only to a feature of extensible continuity. Let $(\overline{\mathfrak{P}};V)$ be a system satisfying conditions (1) to (4) and let μ be a function defined on the subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$. The function μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists a d_e such that if $(p_1 p_2) \leqq d_e$, then $|\mu_{p_1} - \mu_{p_2}| \leqq e$.

In strict analogy with theorem XIII of § 17 we have

THEOREM XIV. *If* $(\overline{\mathfrak{P}};V)$ *is a system satisfying conditions* (1) *to* (4), *then a function* μ *is uniformly continuous on a compact subclass* $\mathfrak{P}$ *of* $\overline{\mathfrak{P}}$ *if and only if, with reference to the associated system* $(\overline{\mathfrak{P}};\mathfrak{U};T)$, μ *is extensibly continuous on* $\mathfrak{P}$.

§ 10. *A Class of Functions as Range of the Independent Variable.*

By the mediation of our definition of a system $(\mathfrak{P};\mathfrak{U};T)$ in terms of a system $(\mathfrak{P};V)$ our general theory is available for any class $\mathfrak{P}$ of elements for which there is defined a *voisinage*, or an *écart*, which may be considered as a special voisinage. Among these classes $\mathfrak{P}$, is the class of all real-valued, single-valued functions that are uniformly continuous on a given interval of the real number system.* Our theory is equally applicable, however, to a class $\mathfrak{P}$ consisting of all real-valued, single-valued functions on a range absolutely unconditioned.

Let $\mathfrak{K}$ denote a class of elements k, concerning which no hypotheses are required. We consider a system† $(\mathfrak{P};\mathfrak{U};T)$ in which $\mathfrak{P}$ is the class of all single-valued functions on $\mathfrak{K}$ to $\mathfrak{A}$, $\mathfrak{U}$ is the null class, and T is defined relative to a particular function σ on $\mathfrak{K}$ to $\mathfrak{A}$. For a given function σ the relation T is specified as follows: The relation $\mathfrak{R}^p$ holds if and only if there exists a positive number e such that $\mathfrak{R}$ consists of all functions p_1 such that for every k $|p_{1k} - p_k| \leqq e|\sigma_k|$. This system obviously satisfies the five postulates of § 5. Since in this special instance the class $\mathfrak{U}$ is not arbitrary, and since the system involves the arbitrary class $\mathfrak{K}$ and the arbitrary function σ, the notation $(\mathfrak{P};T;\mathfrak{K};\sigma)$ is adopted for a system of this special type.

A necessary and sufficient condition that a sequence $\{p_n\}$ shall have the limit p, by definition 2, § 7, is: For every e there exists an n_e such that, if $n > n_e$, then $|p_{nk} - p_k| \leqq e|\sigma_k|$ for every k. If this condition is fulfilled, the

* M. Fréchet, *loc. cit.*, p. 36.

† We might just as readily set up a system $(\mathfrak{P};R)$ which would give rise to this system $(\mathfrak{P};\mathfrak{U};T)$. by the process explained in § 5, but the present plan is more direct.

sequence $\{p_n\}$ of functions is said to approach the function p *relatively uniformly,** the relativity being with respect to the *scale function* σ. Theorem IV of § 7 now shows that a subclass $\mathfrak{N}$ of $\mathfrak{P}$ is closed (definition 4, § 7) if and only if every sequence of functions belonging to $\mathfrak{N}$ that converges relatively uniformly with respect to σ converges to a limit function† that is in $\mathfrak{N}$. For example, the class of all functions constant on $\mathfrak{K}$ is closed; or, the class of all functions p such that $a_1 \leqq p_k \leqq a_2$ for every k is closed.

If we make the special hypothesis that $\mathfrak{K}$ is a subclass of a class $\overline{\mathfrak{P}}$ for which there is defined a relation R so that $(\overline{\mathfrak{P}}; R)$ satisfies the postulates of § 2, then, through the associated system of the type $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, the theory of Chapter III is applicable to functions p defined on $\mathfrak{K}$. With this restriction on $\mathfrak{K}$ and the hypothesis that σ is bounded on $\mathfrak{K}$, we have the following examples of closed subclasses $\mathfrak{N}$ of $\mathfrak{P}$:

1. The class of all functions p that are convergent at a given limiting element of $\mathfrak{K}$.

2. The class of all functions p that are convergent to a given limit at a given limiting element of $\mathfrak{K}$.

3. The class of all functions p that are continuous on $\mathfrak{K}$.

4. The class of all functions p that are extensibly continuous on $\mathfrak{K}$.

With the further hypothesis that $(\overline{\mathfrak{P}}; R)$ is the composite of two systems, $(\overline{\mathfrak{P}}'; R')$ and $(\overline{\mathfrak{P}}''; R'')$, and that $\mathfrak{K} = \mathfrak{K}'\mathfrak{K}''$, where $\mathfrak{K}'$ and $\mathfrak{K}''$ are subclasses respectively of $\overline{\mathfrak{P}}'$ and $\overline{\mathfrak{P}}''$, we have the further examples of closed subclasses of $\mathfrak{P}$:

5. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$, $\lim\limits_{k'' \to l''} \lim\limits_{k' \to l'} p_{k'k''}$ exists and is finite.

6. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$ and a given number a, $\lim\limits_{k'' \to l''} \lim\limits_{k' \to l'} p_{k'k''} = a$.

7. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$, there is (for each p) a function α on $\mathfrak{K}''$, convergent at l'', such that $\lim\limits_{k' \to l'} p_{k'} = \alpha\ (\mathfrak{K}'')$.

8. The subclass of 7 containing every function p for which the corresponding α has the limit a at l''.

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* E. H. Moore, *loc. cit.*, p. 29; also *Atti del IV Congresso Internazionale dei Matematici* (Rome, 1908), Vol. II, p. 101.

† A class of functions that is closed in this sense, for a given σ, has the closure property $C_{\mathfrak{S}}$. "closed as to $\mathfrak{S}$," used by E. H. Moore ("General Analysis," p. 37; and *Atti, etc.*, p. 101), where $\mathfrak{S}$, the scale class, contains the single function σ. If there exist positive real numbers a_1 and a_2 such that $a_1 \leqq |\sigma_k| \leqq a_2$ for every k, then closure as to σ is equivalent to closure under extension by adjoining the limits of all uniformly convergent sequences of functions of the class. Important instances in which this equivalence is not effective are furnished by classes of functions defined for positive integers and giving rise to absolutely convergent series (E. H. Moore, "General Analysis," p. 38, as to $\mathfrak{M}^{\mathrm{III}}{}_{*}$; and *Atti, etc.*, p. 102, as to $\mathfrak{M}^{\mathrm{III}}$).

9. The class of all functions p such that for every k'' $p_{k''}$ is continuous on $\mathfrak{K}'$.

10. The class of all functions p such that for every k'' $p_{k''}$ is extensibly continuous on $\mathfrak{K}'$.

11. The class of all functions p continuous on $\mathfrak{K}'$ uniformly on $\mathfrak{K}''$.

12. The class of all functions p extensibly continuous on $\mathfrak{K}'$ uniformly on $\mathfrak{K}''$.

The proofs that these classes are closed* may be made to depend on the general theorems of Chapter III in the same way that the theorems on sequences of functions in § 15 are deduced from them.

Dropping now the special hypotheses on $\mathfrak{K}$ and σ, we observe that theorem† VII of § 7 reduces in this case to the familiar proposition, "Every derived class is closed."

Consider a system $(\overline{\mathfrak{P}}; T; \mathfrak{K}; \sigma)$ as defined above, where $\mathfrak{K}$ and σ are arbitrary, and consider a function μ defined on a subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$, finite for every p. The function μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists a d_e such that, if $|p_{1k}-p_{2k}| \leqq d_e |\sigma_k|$ for every k, then $|\mu_{p_1}-\mu_{p_2}| \leqq e$. We have again the relation between uniform continuity and extensible continuity:

THEOREM XV. *If $\mathfrak{P}$ is a compact subclass of a class $\overline{\mathfrak{P}}$ pertaining to a system $(\overline{\mathfrak{P}}; T; \mathfrak{K}; \sigma)$, then a function μ is uniformly continuous on $\mathfrak{P}$ if and only if μ is extensibly continuous on $\mathfrak{P}$.*

NOTE.—The investigations leading to the present paper were completed in June, 1911, and the manuscript left the hands of the author in April, 1912. These facts are offered in explanation of what might otherwise appear to be unwarranted disregard of certain recent contributions to the literature of this field. We have added foot-note references to papers by E. R. Hedrick, *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 285–294, and T. H. Hildebrandt, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 237–290. At this point we should mention a recent paper by E. V. Huntington, on "A Set of Postulates for Abstract Geometry, Expressed in Terms of the Simple Relation of Inclusion," *Mathematische Annalen*, Band 73 (1913), pp. 522–559, which obviously has a strong bearing in the field of the present paper. Mention may also be made of a more recent paper by the present writer, "Limits in Terms of Order, with the Example of Limiting Element not Approachable by a Sequence," *Transactions of the American Mathematical Society*, Vol. XV (1914), pp. 51–71, which pertains to the same general field, and in which relationships of various systems of postulates receive further attention.

ANNAPOLIS, MD., *February*, 1914.

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* These classes, 1–12, have also the property "self-closure" defined by E. H. Moore, *Atti, etc.*, Vol. II, p. 102, and designated simply as "closure" in "General Analysis," p. 37. Other properties of general reference, *i. e.*, properties defined for classes of real-valued functions in general, and therefore applicable to the classes here enumerated, are the five *dominance properties*, D, D_0, D'_0, D_1 and D_2 ("General Analysis," p. 39). Classes 1, 3, 5, 7, 9, 10, 11 and 12 have properties D_0 and D_1; classes 2, 6 and 8 have property D_1, and, in case the given number a which enters in the definition of each class is positive or zero, they have property D_0, and property D'_0 in case a is zero; and the class 4 has properties D, D_0, D_1 and D_2.

† It is worthy of note that this general theorem, as applied in the special case now under consideration, is, by an application of theorem II, (b), of § 6, equivalent to a special case of theorem III, p. 52, of Professor Moore's memoir on "General Analysis." The special feature is, obviously, in the reduction of the scale class $\mathfrak{S}$ to the single function σ.

§ 18. *The Fréchet Voisinage.*

In his thesis M. Fréchet* denotes by (V) a class of elements for which there is defined, fulfilling certain conditions which he specifies, the notion of "voisinage." We may present his assumptions accurately, but in form adapted to our purposes, as follows: With the class $\mathfrak{P}$ of undefined elements is associated a function V, forming the system $(\mathfrak{P}; V)$. The function V is defined on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$; *i. e.*, it assigns to every pair of elements, p_1 and p_2, a real number, which is denoted by $(p_1 p_2)$. The conditions postulated for this system $(\mathfrak{P}; V)$ are:

(1) For every two elements p_1 and p_1 we have $(p_1 p_2) = (p_2 p_1) \geqq 0$.

(2) If $(p_1 p_2) = 0$, then $p_1 = p_2$.

(3) If $p_1 = p_2$, then $(p_1 p_2) = 0$.

(4) There exists a function $\phi(d)$ such that $\lim \phi(d) = 0$ and such that if $(p_1 p_2) \leqq d$ and $(p_2 p_3) \leqq d$, then $(p_1 p_3) \leqq \phi(d)$.

Proceeding now as in the case of a system $(\mathfrak{P}; K_2)$, we give attention to the introduction of ideal elements.† An ideal element of the system $(\mathfrak{P}; V)$ is a class s of sequences $\}p_n\{$ of elements of the class $\mathfrak{P}$ which fulfils the following conditions:

1. If $\}p_n\{$ is a sequence belonging to the class S, then for every d there exists an n_d such that, if n_1 and n_2 are both greater than n_d, then $(p_{n_1} p_{n_2}) \leqq d$.

2. If $\}p_n\{$ and $\}\hat{p}_n\{$ are sequences belonging to s, then for every d there exists an n_d such that, if n_1 and n_2 are both greater than n_d, then $(p_{n_1} \hat{p}_{n_2}) \leqq d$.

3. For every p there exists a sequence $\}p_n\{$ of the class s and a positive number d such that for every n there is an n_1 greater than n such that $(p_{n_1} p) > d$.

4. Any class s_1 which satisfies conditions 1, 2 and 3 and contains s must coincide with s.

We extend the definition of V so that it is a function on $\mathfrak{P}\mathfrak{Q}$, where, as before, $\mathfrak{Q}$ is the class $\mathfrak{P}$ with ideal elements u adjoined, by assigning to $(p\,u)$ the value d_1 of the greatest lower bound of the set of numbers d for each of which there exists a sequence $\}p_n\{$ of the class u such that $(p p_n) \leqq d$ for every n.

In terms of this extended function V a relation T is specified: The relation $\mathfrak{R}^q$ holds if and only if there exists a d such that $\mathfrak{R}$ consists of all elements p for which $(p\,q) \leqq d$. It may be shown without difficulty that the system

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* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII, p. 17.

† Here also we might specify a system $(\mathfrak{P}; R)$ in which R should be defined in terms of V in such manner as to fulfil our postulates, by the adoption of an additional condition on $(\mathfrak{P}; V)$. A condition effective for this purpose would be: If $p_1 \neq p_2$, then for every d there exists a p such that $(p\,p_1) \leqq d$, while $(p\,p_2) > d$.

$(\mathfrak{P}; \mathfrak{U}; T)$, now definitely determined by the system $(\mathfrak{P}; V)$, satisfies the postulates of § 5. As to the significance of the general theory of Chapters II and III, in this special case, we give attention here only to a feature of extensible continuity. Let $(\overline{\mathfrak{P}}; V)$ be a system satisfying conditions (1) to (4) and let μ be a function defined on the subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$. The function μ is uniformly continuous on $\mathfrak{P}$ if and only if for every e there exists a d_e such that if $(p_1 p_2) \leqq d_e$, then $|\mu_{p_1} - \mu_{p_2}| \leqq e$.

In strict analogy with theorem XIII of § 17 we have

THEOREM XIV. *If $(\overline{\mathfrak{P}}; V)$ is a system satisfying conditions (1) to (4), then a function μ is uniformly continuous on a compact subclass $\mathfrak{P}$ of $\overline{\mathfrak{P}}$ if and only if, with reference to the associated system $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, μ is extensibly continuous on $\mathfrak{P}$.*

§ 10. *A Class of Functions as Range of the Independent Variable.*

By the mediation of our definition of a system $(\mathfrak{P}; \mathfrak{U}; T)$ in terms of a system $(\mathfrak{P}; V)$ our general theory is available for any class $\mathfrak{P}$ of elements for which there is defined a *voisinage*, or an *écart*, which may be considered as a special voisinage. Among these classes $\mathfrak{P}$, is the class of all real-valued, single-valued functions that are uniformly continuous on a given interval of the real number system.* Our theory is equally applicable, however, to a class $\mathfrak{P}$ consisting of all real-valued, single-valued functions on a range absolutely unconditioned.

Let $\mathfrak{K}$ denote a class of elements k, concerning which no hypotheses are required. We consider a system† $(\mathfrak{P}; \mathfrak{U}; T)$ in which $\mathfrak{P}$ is the class of all single-valued functions on $\mathfrak{K}$ to $\mathfrak{A}$, $\mathfrak{U}$ is the null class, and T is defined relative to a particular function σ on $\mathfrak{K}$ to $\mathfrak{A}$. For a given function σ the relation T is specified as follows: The relation $\mathfrak{R}^p$ holds if and only if there exists a positive number e such that $\mathfrak{R}$ consists of all functions p_1 such that for every k $|p_{1k} - p_k| \leqq e|\sigma_k|$. This system obviously satisfies the five postulates of § 5. Since in this special instance the class $\mathfrak{U}$ is not arbitrary, and since the system involves the arbitrary class $\mathfrak{K}$ and the arbitrary function σ, the notation $(\mathfrak{P}; T; \mathfrak{K}; \sigma)$ is adopted for a system of this special type.

A necessary and sufficient condition that a sequence $\{p_n\}$ shall have the limit p, by definition 2, § 7, is: For every e there exists an n_e such that, if $n > n_e$, then $|p_{nk} - p_k| \leqq e|\sigma_k|$ for every k. If this condition is fulfilled, the

* M. Fréchet, *loc. cit.*, p. 36.

† We might just as readily set up a system $(\mathfrak{P}; R)$ which would give rise to this system $(\mathfrak{P}; \mathfrak{U}; T)$ by the process explained in § 5, but the present plan is more direct.

sequence $\{p_n\}$ of functions is said to approach the function p *relatively uniformly,** the relativity being with respect to the *scale function* σ. Theorem IV of § 7 now shows that a subclass $\mathfrak{N}$ of $\mathfrak{P}$ is closed (definition 4, § 7) if and only if every sequence of functions belonging to $\mathfrak{N}$ that converges relatively uniformly with respect to σ converges to a limit function† that is in $\mathfrak{N}$. For example, the class of all functions constant on $\mathfrak{K}$ is closed; or, the class of all functions p such that $a_1 \leqq p_k \leqq a_2$ for every k is closed.

If we make the special hypothesis that $\mathfrak{K}$ is a subclass of a class $\overline{\mathfrak{P}}$ for which there is defined a relation R so that $(\overline{\mathfrak{P}}; R)$ satisfies the postulates of § 2, then, through the associated system of the type $(\overline{\mathfrak{P}}; \mathfrak{U}; T)$, the theory of Chapter III is applicable to functions p defined on $\mathfrak{K}$. With this restriction on $\mathfrak{K}$ and the hypothesis that σ is bounded on $\mathfrak{K}$, we have the following examples of closed subclasses $\mathfrak{N}$ of $\mathfrak{P}$:

1. The class of all functions p that are convergent at a given limiting element of $\mathfrak{K}$.

2. The class of all functions p that are convergent to a given limit at a given limiting element of $\mathfrak{K}$.

3. The class of all functions p that are continuous on $\mathfrak{K}$.

4. The class of all functions p that are extensibly continuous on $\mathfrak{K}$.

With the further hypothesis that $(\overline{\mathfrak{P}}; R)$ is the composite of two systems, $(\overline{\mathfrak{P}}'; R')$ and $(\overline{\mathfrak{P}}''; R'')$, and that $\mathfrak{K} = \mathfrak{K}'\mathfrak{K}''$, where $\mathfrak{K}'$ and $\mathfrak{K}''$ are subclasses respectively of $\overline{\mathfrak{P}}'$ and $\overline{\mathfrak{P}}''$, we have the further examples of closed subclasses of $\mathfrak{P}$:

5. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$, $\lim\limits_{k'' \to l''} \lim\limits_{k' \to l'} p_{k'k''}$ exists and is finite.

6. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$ and a given number a, $\lim\limits_{k'' \to l''} \lim\limits_{k' \to l'} p_{k'k''} = a$.

7. The class of all functions p such that, for a given limiting element $l = l'l''$ of $\mathfrak{K}$, there is (for each p) a function α on $\mathfrak{K}''$, convergent at l'', such that $\tilde{\lim\limits_{k' \to l'}} p_{k'} = \alpha\ (\mathfrak{K}'')$.

8. The subclass of 7 containing every function p for which the corresponding α has the limit a at l''.

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Simply Transitive Primitive Groups Whose Maximal Subgroup Contains a Transitive Constituent of Order p^2, or pq, or a Transitive Constituent of Degree 5.

By Elizabeth Ruth Bennett.

If the maximal subgroup G_1 of degree $n-1$ of a simply transitive primitive group G of degree n contains a transitive constituent of degree 3, it is known that it must also contain another transitive constituent of degree 3 or 6.* Similar restrictions on G_1 have been determined when G_1 contains a transitive constituent of degree 4.† We proceed to consider restrictions that may be placed on the degree of G or on the transitive constituents of G_1 when G_1 contains a transitive constituent of order p^2 or pq, or a transitive constituent of degree 5.

Theorem I. *If a transitive constituent of G_1 is of order p^2, the degree of G is q^a, q a prime. When k, the number of transitive constituents of order p^2 in G_1, is odd, $k\equiv 3$, mod. 4, and $q=2$. When $k\equiv 2$, mod. 4, q is a prime of form $4n+3$ and a is an odd number.*

All groups of order p^2 are abelian and can be represented transitively only in regular form. Therefore the order of G_1 can not exceed p^2 and G_1 must be formed from the simple isomorphism of groups of degree and order p^2.‡ G is then of class $n-1$ and of degree q^a, q a prime.§

The following relation must exist where k represents the number of transitive constituents of order p^2 in G_1:

$$kp^2+1=q^a. \tag{I}$$

By considering (I) with respect to modulus 4, the remainder of the theorem is evident.

Theorem II. *If a transitive constituent of G_1 is of order pq, p and q primes, $p>q$, and the order of G_1 is pq, G_1 is formed from establishing a*

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* Miller, American Journal of Mathematics, Vol. XXXV, p. 7.

† Bennett, American Journal of Mathematics, Vol. XXXIV, pp. 8, 9.

‡ Miller, *Bulletin American Mathematical Society*, Vol. VI, p. 104.

§ Frobenius, *Berliner Sitzungsberichte* (1902), pp. 455–459.

simple isomorphism between groups of order pq. When the constituent of order pq is abelian, G is of degree r^a, r a prime, and of class $n-1$. If the order of G_1 exceeds pq, G_1 must contain a transitive constituent of degree pq whose order is greater than pq, and, in case the constituent of order pq is of degree p, the order of G_1 must be $q^a p$.

When the order of G_1 is pq, G_1 is formed from the simple isomorphism of groups of order pq, for the order of G_1 is not divisible by the square of a prime number.*

A group of order pq can be represented transitively only on pq or p letters, and, in case the group of order pq is abelian, the representation must be on pq letters. Therefore, if the order of G_1 is pq and the constituent of order pq is abelian, G is of class $n-1$ and degree r^a, r a prime. When the order of G_1 exceeds pq and the transitive constituent of order pq is regular, then G_1 must contain an additional transitive constituent of degree pq whose order exceeds pq.† When the constituent of order pq is of degree p, the subgroup leaving a letter of the constituent of degree p fixed is composed of transitive constituents of order and degree q. G_1 must then contain a transitive constituent whose degree is pq and whose order is greater than pq.‡ If the constituent of order pq is of degree p, the order of G_1 must be $q^a p$,† for the order of G_1 is not divisible by p^2.

COROLLARY I. *If G_1 contains a dihedral group of prime degree p as a transitive constituent, the order of G_1 is $2^a p$.*

THEOREM III. *When G_1 contains k transitive constituents of order 5, k an odd number, the degree of G is $2^{4\beta}$ and $k \equiv 3$, mod. 4. If $k \equiv 2$, mod. 4, the degree of G is q^a, q a prime of form $4n+3$ and a an odd number.*

Since G_1 contains a transitive constituent of order 5, the order of G_1 is 5 and G is of class $n-1$. The degree of G is then q^a, q a prime. If the number of transitive constituents of order 5 is odd, $5k+1$ is even or $5k+1=2^a$ and $k \equiv 3$, mod. 4. Since 2 is a primitive root of 5, in order that the above equation be satisfied $a=4\beta$. If $k \equiv 2$, mod. 4, $5k+1 \equiv 3$, mod. 4, and in order that $q^a \equiv 3$, mod. 4, a must be odd and q a prime of form $4n+3$.

THEOREM IV. *If G_1 contains the semi-metacyclic group of degree 5 as a transitive constituent, G_1 must contain another transitive constituent of degree 10 or 5 and the order of G_1 must be $2^a \cdot 5$.*

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* Miller, *Proceedings London Mathematical Society*, Vol. XXVIII, p. 536.

† Reitz, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI, p. 9.

‡ Bennett, *loc. cit.*, p. 6.

When the order of G_1 is 10, G_1 can be formed only from the simple isomorphism of groups of degrees 5 and 10. If the order of G_1 exceeds 10, from Theorem II G_1 must contain a transitive constituent of degree 10, and the order of G_1 is $2^{\alpha}\cdot 5$.

THEOREM V. *If G_1 contains the alternating group of degree 5 as a transitive constituent and the order of G_1 is 60, G_1 is formed from the simple isomorphism of groups whose degree can be only 60, 30, 20, 15, 12, 10, 6 and 5. If the order of G_1 exceeds 60, G_1 must contain a transitive constituent of degree 20. The order of G_1 is $2^{\alpha}\cdot 3^{\beta}\cdot 5$.*

When the order of G_1 is 60, since the alternating group of degree 5 in simple, the order of the other transitive constituents of G_1 must also be 60. The group of order 60 can be represented only on 60, 30, 20, 15, 12, 10, 6 and 5 letters; therefore, only transitive constituents of such degrees may occur when G_1 is of order 60. If the order of G_1 exceeds 60, G_1 must contain a transitive constituent of degree 20, for the subgroup leaving fixed a letter of the alternating group of degree 5 is a primitive group.* Since the order of G_1 is not divisible by 5^2, the order of G is $2^{\alpha}\cdot 3^{\beta}\cdot 5$.

A theorem concerning the symmetric group of degree 5 may be stated which differs from Theorem V only with regard to the possible representations.

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* Bennett, *loc. cit.*, p. 6.

An Extension of Green's Theorem.

By Ida Barney.

§ 1. *Rectifiable Green Fields.*

In the usual proof of Green's theorem the functions must be continuous, and have at each point in the field of integration partial derivatives of the first order which are integrable over the given field. The field itself is assumed to have as a boundary a rectifiable curve, which is cut only a limited number of times by lines parallel to the axes. Other proofs* of this theorem have been given, in which the conditions on the functions are not so narrow, but all require the field of integration to satisfy the condition mentioned above. The proof of Green's theorem given in this paper makes the theorem apply to a much larger class of functions than has been possible before, and also permits the field of integration to be cut an infinite number of times by each one of a certain set of parallels.

For the sake of clearness, the definition of a line integral over any rectifiable curve will be given, together with some other geometric definitions.

Let $x = X(t)$, $y = Y(t)$ be one-valued continuous functions of t in the interval $\mathfrak{A} = (\alpha < \beta)$. As t ranges over $\mathfrak{A}$ the point x, y will describe a continuous curve C_{ab}. If such a curve has no double points, it will be called a *Jordan curve.* A continuous closed curve without double points will then be a closed Jordan curve. It has been proved that the necessary and sufficient condition for C to be rectifiable, *i. e.*, to have length, is that $X(t)$ and $Y(t)$ have limited variation.†

Let P' and P'' correspond to $t = t'$ and $t = t''$ on the curve C. If $t' < t''$, then we say P' precedes P'' and write $P' < P''$.

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* M. B. Porter, "Concerning Green's Theorem," *Annals of Math.*, Ser. 2, Vol. VII (1905), p. 1.
A. Pringsheim, "Zur Theorie des Doppel-Integrals des Green'schen Integralsatzes," *Sitzungsberichte der k. b. Akademie d. Wissenschaften zu München*, Vol. XXIX (1899), p. 49.
Heffter, "Zur Theorie der reellen Curvenintegrale," *Göttinger Nachrichten* (1902), p. 115.
J. Thomae, "Einleitung in die Theorie der bestimmten Integrale."
W. F. Osgood, "Lehrbuch der Funktionentheorie."
C. Jordan, "Cours d'Analyse," Vol. I.

† Pierpont, "The Theory of Functions of Real Variables," Vol. II, p. 583. Hereafter this will be referred to as Lectures, Vol. II.

As t ranges from α to β, C is described in the positive direction.

If P' on C corresponds to $t=t'$, and at t' either $X(t)$ or $Y(t)$ has a proper extreme,* then the curve C will be said to have a *peak* at P'.

Let C_{ab} be a rectifiable Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ ranging over } \mathfrak{A} = (\alpha < \beta).$$

A function $f(x, y)$ is defined for each point of C_{ab} and limited over C_{ab}.

Let Δ be a division of $\mathfrak{A}$ of norm δ into subintervals $\delta_1, \delta_2, \ldots, \delta_m$. Then to Δ corresponds a division D of C_{ab} of norm d into arcs $l_1, l_2, \ldots, l_m$. As C is continuous, $d \doteq 0$ with δ.

Let $a = (x^0, y^0), (x^1, y^1), \ldots, (x^m, y^m) = b$ be the end points of arcs $l_1, \ldots, l_m$. Let v_ι be any point on l_ι. Then

$$\int_{C_{ab}} f(x, y)\, dx = \lim_{\delta=0} \Sigma f(v_\iota)(x^\iota - x^{\iota-1}),$$

$$\int_{C_{ab}} f(x, y)\, dy = \lim_{\delta=0} \Sigma f(v_\iota)(y^\iota - y^{\iota-1}),$$

when these limits exist.

Functions for which both limits exist will be called integrable functions over the curve C_{ab}. A sufficient condition for a limited function $f(x, y)$ to be integrable is that $\lim_{\delta=0} \Sigma\, \omega_\iota l_\iota = 0$, where $\omega_\iota = \operatorname{osc} f$ over arc l_ι of C_{ab}.

A set of parallels to the axes will be called a *pantactic* set, if any interval of either the x- or the y-axis is cut by at least one of these parallels.

The field of integration considered in this section will satisfy the following conditions:

1°. The boundary C is a closed rectifiable Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta).$$

2°. $X(t)$ and $Y(t)$ are functions such that, if A is a discrete set in $\mathfrak{A}$, the image of A given by $X(t)$ or by $Y(t)$ is also discrete.

3°. The points in $\mathfrak{A}$ corresponding to peaks on C form a discrete set.

4°. C has only a finite number of segments parallel to the axes.

5°. C is cut by each one of a pantactic set of parallels in only a finite number of points.

We call such a field a *Green field* and denote it by $\mathfrak{G}$.

We will denote by $\mathfrak{P}_x$ a pantactic set of parallels to the x-axis each of which cuts C only a finite number of times, and does not pass through peaks of C. $\mathfrak{P}_y$ will denote a similar set parallel to the y-axis. We set $\mathfrak{P} = \mathfrak{P}_x + \mathfrak{P}_y$. That such a pantactic set exists, follows from 2°, 3° and 5°, since by 2° and 3° the projections on the axes of the peaks on C are discrete.

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* Lectures, Vol. II, p. 521.

It will be convenient at times to let $\mathfrak{P}_x$, $\mathfrak{P}_y$ denote only the points of $\mathfrak{G}$ on these parallels; in so doing no ambiguity will arise.

A *normal* division D of norm d of a Green field $\mathfrak{G}$ is a division of $\mathfrak{G}$ made by a finite number of parallels of $\mathfrak{P}_x$ and of $\mathfrak{P}_y$. Since $\mathfrak{P}$ is pantactic, d may be made to approach zero.

THEOREM I. *Let C be the boundary of a Green field $\mathfrak{G}$. Let $x=\alpha$, a parallel of $\mathfrak{P}_x$, cut C in the points whose ordinates are $a_1 < a_2 < \ldots . < a_{2n}$. If $x < \alpha$ for points on C immediately preceding $P_\iota = (\alpha, a_\iota)$; then $x > \alpha$ for points on C immediately preceding $P_{\iota+1} = (\alpha, a_{\iota+1})$, and conversely.*

We take $P' = (x', y')$ on C preceding P_ι, so that:

1°. Between P' and P_ι, C does not cut $x=\alpha$.

2°. Dist $(P' P_\iota) < \varepsilon$. (1)

The theorem will be proved for the case where $x' < \alpha$. A similar proof holds where $x' > \alpha$.

Now let $P'' = (x'', y'')$ be a point on C so that $P_{\iota+1} < P''$, and between $P_{\iota+1}$ and P'' there is no point of C on $x=\alpha$. Also let dist $(P_{\iota+1} P_\iota) < \varepsilon$. (2)

Suppose that $x < \alpha$ for points immediately preceding $P_{\iota+1}$. Then $x'' > \alpha$ for P''.

Let $C_{\iota,\iota+1}$ denote the part of C from a_ι to $a_{\iota+1}$, which does not contain the points P' and P''.

$C_{\iota,\iota+1}$, together with the interval $(a_\iota a_{\iota+1})$, forms a closed Jordan curve Γ, and divides the plane into two precincts,* R and L. (3)

It will be proved later that P' lies in one of these precincts and P'' in the other. For the moment this will be assumed. P' and P'' can be joined by an arc of C, which does not include $C_{\iota,\iota+1}$. Call this arc C'. C' must intersect Γ, as P' is in one precinct and P'' in the other. C has no double points, so C' can not intersect $C_{\iota,\iota+1}$ and must cut the interval $(a_\iota a_{\iota+1})$. Therefore $a_{\iota+1}$ is not the next point to a_ι. This is contrary to the hypothesis. Thus $x > \alpha$ for points immediately preceding $P_{\iota+1}$.

To prove P' and P'' are in different precincts, we take $Q' > P_\iota$ and $Q'' < P_{\iota+1}$, so that:

1°. Dist $(Q' P_\iota)$ and dist $(Q'' P_{\iota+1}) < \varepsilon'$.

2°. C does not intersect $x=\alpha$ between P_ι and Q' or between Q'' and $P_{\iota+1}$. About $P_{\iota+1}$ and P_ι circles ζ_ι and $\zeta_{\iota+1}$, of radius ρ, can be drawn, so small that they will contain no points of $C_{\iota,\iota+1}$ except such as lie between P_ι and Q' or

* Lectures, Vol. II, p. 612.

between Q'' and $P_{\iota+1}$. Within $(a_\iota a_{\iota+1})$ take an interval $\lambda = (a_\iota + \eta, a_{\iota+1} - \eta)$, $\eta < \rho$. Then

$$\mu = \text{dist } (\lambda\, C_{\iota,\iota+1}) \text{ is } > 0. \tag{4}$$

Starting with $a_\iota + \eta$ as the first center, draw a series of equal overlapping circles $c_1, c_2, \ldots.$ with centers $m_1, m_2, \ldots.$ on λ, with radius $r < \mu$. Finally, each point of λ will lie in at least one circle. The number of circles required will be finite. Let it be m.

$a_\iota + \eta$ is a frontier point of Γ; therefore c_1 contains points of both R and L. Since c_1 contains no point of $C_{\iota,\iota+1}$ as $r < \mu$, any two points of c_1 for which $x > \alpha$ can be joined by a curve not cutting Γ. Thus all such points belong to one precinct. The same statement is true for points of c_1, where $x < \alpha$. A point in c_1 as M for which $x > \alpha$, and a point in c_1 as N for which $x < \alpha$, can not belong to the same precinct, because then all points of c_1 would belong to the same precinct. Of the two precincts L, R introduced above, let L denote that precinct to which the points in c_1 belong for which $x < \alpha$. (5)

Similar reasoning can be applied to each circle, since the centers are frontier points of Γ. This reasoning proves that in any circle all the points for which $x < \alpha$ belong to one precinct, and all those where $x > \alpha$ to the other. (6)

Some points of c_1 lie in ζ_ι, since $\eta < \rho$. Let us take a set of points $p_0, p_1, \ldots, p_m$ for which $x < \alpha$, and such that p_0 is in ζ_ι and c_1, p_1 in c_1 and c_2, etc. Finally, p_m is in c_m and $\zeta_{\iota+1}$. From (5), p_0 and p_1 belong to L. From (6), p_1 and p_2 belong to the same precinct. Thus p_0, p_1, p_2 are all in L.

Continuing in this way, we can prove that all the p points belong to L. (7)

Take a corresponding set $q_0, q_1, \ldots, q_m$ for which $x > \alpha$. Reasoning on these points as we did on the others, we can prove that they belong to R. (8)

In ζ_ι there is no part of $C_{\iota,\iota+1}$ for which $x < \alpha$; therefore all points in ζ_ι for which $x < \alpha$ belong to one precinct. From (7), p_0 belongs to L; thus all points in ζ_ι for which $x < \alpha$ belong to L.

From (1), P' is in ζ_ι if $\varepsilon < \rho$, and $x' < \alpha$ by hypothesis. Therefore P' belongs to L.

Similarly all points in $\zeta_{\iota+1}$ for which $x > \alpha$ belong to one precinct. From (8), q_m belongs to R; therefore all points in $\zeta_{\iota+1}$ where $x > \alpha$ belong to R.

From (2), P'' is in $\zeta_{\iota+1}$ if $\varepsilon < \rho$, and we assumed that $x'' > \alpha$. Thus P'' belongs to R.

Therefore P' and P'' are in different precincts.

Remark. A similar theorem can be proved for $y = \beta$, any parallel of $\mathfrak{P}_y$.

We can now assume that the positive direction on C is such that, if (α, a_1)

is the first point of C on $x = \alpha$, a parallel of $\mathfrak{P}_x$, then $x < \alpha$ for points on C immediately preceding (α, a_1). For if this were not the case, the variable t could be replaced by a new variable $-t$.

THEOREM II. *Let C be the boundary of a Green field $\mathfrak{G}$. Let $x = \alpha$ and $y = \beta$ be parallels of $\mathfrak{P}$. Let $y = \beta$ cut C in $Q_1, Q_2, \ldots, Q_{2m}$, reckoned from left to right. Then $y > \beta$ for points on C immediately preceding Q_1.*

Let S be a square containing $\mathfrak{G}$, and bounded by $x=A$, $x=B$, $y=E$, $y=F$. $A < B$, $E < F$. On C, we take points $V = (v_1, v_2) < Q_1$ and $W = (w_1, w_2) > Q_{2m}$, so that:

1°. Dist $(V Q_1)$ and dist $(W Q_{2m}) < \varepsilon$. (1)

2°. C does not intersect $y = \beta$ between V and Q_1 or between Q_{2m} and W. Suppose $y < \beta$ for points on C immediately preceding Q_1. Then Theorem I shows that $y > \beta$ for points immediately preceding Q_{2m}. Then

$$w_2 < \beta \text{ and } v_2 < \beta. \tag{2}$$

Let $C_{1,2m}$ be the part of C from Q_1 to Q_{2m} not including V or W.

We now take $x = \alpha$ so that $x = \alpha$ cuts $y = \beta$ between the points Q_1 and Q_{2m}, and also $v_1 < \alpha < w_1$. (3)

Let λ_1 be the segment of $y = \beta$ from Q_1 to $x = A$. Let λ_{2m} be the segment of $y = \beta$ from Q_{2m} to $x = B$. $C_{1,2m} + \lambda_1 + \lambda_{2m}$ and the boundary of S where $y < \beta$ make up the boundary of a precinct H.

Let a_r be the point of $C_{1,2m}$ on $x = \alpha$ nearest $y = E$. Let the segment of $x = \alpha$ from a_r to $y = E$ be τ. τ divides H into two precincts. That precinct which includes λ_1 in its frontier we will denote by L, and the other by R. Then V is in L and W in R.

To prove this, let us take P_1 and P_{2m}, two points on $C_{1,2m}$, so that $Q_1 < P_1$ and $P_{2m} < Q_{2m}$. Moreover, between Q_1 and P_1, P_{2m} and Q_{2m} there shall be no point of $C_{1,2m}$ on $y = \beta$. Let $C^*_{1,2m}$ denote the part of $C_{1,2m}$ between P_1 and P_{2m}. Then

$$d_1 = \text{dist}\,(\lambda_1 C^*_{1,2m}),\quad d_2 = \text{dist}\,(\lambda_2 C^*_{1,2m}) > 0.$$

About Q_1 and Q_{2m} draw circles c_1, c_2 with radius $r_1 < d_1$, $r_2 < d_{2m}$. If $\varepsilon < r$, V lies in c_1 and W in c_2, from (1).

Since c_1 contains no points of $C^*_{1,2m}$, all points of c_1 where $y < \beta$ must belong to L. But $v_2 < \beta$, so V lies in L. Similarly we can prove that W lies in R.

V and W can be joined by an arc of C not including $C_{1,2m}$. Call this arc C_{vw}. C_{vw} must intersect the boundary of L. C_{vw} can not cut the boundary of S, or $C_{1,2m}$, or λ_1, or λ_{2m}; so it must intersect τ at some point a_s. C_{vw} may

cut $x=\alpha$ in other points below a_s, but in order to return to V in L, C_{vw} will have to cut $x=\alpha$ an odd number of times. Then Theorem I proves that $x>\alpha$ for points immediately preceding (α, a_1). This is contrary to the definition of the positive direction on the curve C. Thus the theorem follows.

It is now necessary to introduce the idea of limited fluctuation.*

Let P be one of the parallels of $\mathfrak{P}_x$. Let D be a division of P of norm d. Let $\omega_\iota = \operatorname{osc} f$ over the ι-th interval on P. Suppose now

$$\max \Sigma \omega_\iota < F, \text{ for any } P \text{ of } \mathfrak{P}_x. \tag{1}$$

Let a be a discrete set on P. Let $\omega_a = \Sigma \omega_\iota$ over those intervals of D containing points of a. Let

$$\omega_a < M \bar{a}_s \text{ for any line of } \mathfrak{P}_x, \tag{2}$$

where M is independent of P.

When (1) and (2) are satisfied, $f(x, y)$ has limited fluctuation with respect to y on $\mathfrak{P}_x$.

A similar definition holds for limited fluctuation with respect to x on $\mathfrak{P}_y$.

Let the curve C be the boundary of a Green field $\mathfrak{G}$. We effect a division D of norm d of C by taking points

$$C_1, C_2, \ldots. \tag{1}$$

on it which satisfy the following conditions:

If C has no segments parallel to the y-axis, we will take the points (1) so that they lie on $x=\alpha_1$, $x=\alpha_2$, ..., parallels of $\mathfrak{P}_x$, and such that each arc $C_\iota C_{\iota+1}$ has length $< d$.

If C has segments parallel to the y-axis, we may suppose their lengths are all $< d$. For if the length of any such segment is $\geq d$, we may subdivide it The end points of these parallel segments also form a part of the division D.

Let the parallels $x=\alpha_\iota$ cut C in the points whose ordinates are

$$a_{\iota,1} < a_{\iota,2} < \ldots. < a_{\iota,2n}.$$

We now prove Theorem III.

THEOREM III. *Let $f(x,y)$ be limited and integrable over the boundary C of a Green field, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then*

$$\lim_{d=0} \sum_{\iota j} [f(\alpha_\iota, a_{\iota,2j}) - f(\alpha_\iota, a_{\iota,2j-1})] \Delta x_\iota = -\int_C f(x, y)\, dx, \tag{1}$$

where $\Delta x_\iota = \alpha_{\iota+1} - \alpha_\iota$.

Using the notation in the beginning of this paper,

$$\Sigma f(v_\iota)(x^\iota - x^{\iota-1}) \doteq \int_C f(x, y)\, dx. \tag{2}$$

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* A similar classification of functions is given in Lectures, Vol. II, p. 634.

For the sake of brevity let us set $\delta_\iota = x^\iota - x^{\iota-1}$. Three classes of points are to be considered.

(A) *Where $a_{\iota,j}$ and $a_{\iota,j+1}$ are consecutive points of D.* Here $\delta_\iota = 0$, so $\Sigma f(v_\iota)\delta_\iota$ is not affected by such terms. However, in general,

$$f(\alpha_\iota, a_{\iota,j}) \neq f(\alpha_\iota, a_{\iota,j+1}).$$

Let $\sum_A = \sum_{\iota j} [f(\alpha_\iota, a_{\iota,2j}) - f(\alpha_\iota, a_{\iota,2j-1})]\,\Delta x_\iota$ for terms belonging to points of class (A). Then

$$|\sum_A| \leq P \sum_A \Delta x_\iota, \tag{3}$$

where $P = \max \Sigma$ osc f over any line of $\mathfrak{P}_x$.

P is finite, as $f(x, y)$ has limited fluctuation. Since the intervals $\sum_A \Delta x_\iota$ contain the projection of the peaks of C, and this projection is a discrete set,

$$\sum_A \Delta x_\iota < \varepsilon/2P, \qquad d < d'.$$

From (3),

$$|\sum_A| < \varepsilon/2, \qquad d < d'. \tag{4}$$

(B) *Points lying on vertical segments of C.* Let m be the number of such segments. In the sum in (2) there are at most $2m$ terms affected by these points. Denote their sum by $\sum_B f(v_\iota)\delta_\iota$. Then

$$|\sum_B f(v_\iota)\delta_\iota| < \varepsilon/4, \qquad d < d''. \tag{5}$$

Let the sum of the corresponding terms in (1) be $\sum_B$. This contains at most $2m$ terms. Therefore

$$|\sum_B| < \varepsilon/4, \qquad d < d'''. \tag{6}$$

(C) *Where two consecutive division points of D lie on two consecutive parallels $x = \alpha_\iota$, $x = \alpha_{\iota+1}$.* The positive direction on C is such that $x < \alpha$ for points on C immediately preceding $(\alpha_\iota, a_{\iota,1})$. Theorem I proves the same is true for $(\alpha_\iota, a_{\iota,2j-1})$, and also proves that $x > \alpha$ for points on C immediately preceding $(\alpha_\iota, a_{\iota,2j})$. Therefore,

$$f(\alpha_\iota, a_{\iota,2j})\,\Delta x_\iota = -f(v_\iota)\,\delta_\iota,$$
$$f(\alpha_\iota, a_{\iota,2j-1})\,\Delta x_\iota = f(v_\iota)\,\delta_\iota.$$

Thus

$$\begin{aligned} |-\Sigma f(v_\iota)\delta_\iota - \sum_{\iota j}[f(\alpha_\iota, a_{\iota,2j}) - f(\alpha_\iota, a_{\iota,2j-1})]\,\Delta x_\iota| &\leq |\sum_A| + |\sum_B| + |\sum_B f(v_\iota)\delta_\iota| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}, \text{ from (4), (5), (6),} \\ &< \varepsilon, \qquad \delta < \delta_0, \end{aligned} \tag{7}$$

where $\delta', \delta'', \delta''' \geq \delta_0$.

From (2), (7) comes (1).

Remark. If x and y are interchanged in the preceding theorem, and if Theorem II is used, (1) becomes

$$\lim_{d=0} \sum_{\iota j} [f(b_{j,2\iota}, \beta_j) - f(b_{j,2\iota-1}, \beta_j)] \Delta y_j = \int_C f(x,y)\, dy.$$

In the Green field $\mathfrak{G}$, we take a discrete set A. Some cells of a normal division D of norm d of $\mathfrak{G}$ will contain points of A. We will call this sum A_d. A_d will in general consist of one or more columns of cells parallel to the y-axis. The boundaries of these columns will be made up of lines parallel to the axes and parts of C. Let B_d denote the parts of these boundaries which belong to C or are parallel to the x-axis. That is, B_d is the boundary of A_d exclusive of the parts parallel to the y-axis.

For the parallel $x = \alpha_\iota$ of D which cuts B_d, denote the points of intersection by $\zeta_{\iota,1} < \zeta_{\iota,2} < \ldots. < \zeta_{\iota,2m}$. In the next theorem, Theorem IV, the ζ's will have the meaning defined above.

THEOREM IV. *Let $f(x,y)$ be limited and integrable over the boundary C of a Green field $\mathfrak{G}$, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then*

$$\sum_{\iota j} [f(\alpha_\iota, \zeta_{\iota,2j-1}) - f(\alpha_\iota, \zeta_{\iota,2j})] \Delta x_\iota \doteq 0 \text{ as } d \doteq 0. \tag{1}$$

Let $\sum_\iota \Delta' x_\iota$ be the sum of the projections on the x-axis of columns of cells belonging to A_d, where $\sum_j |\zeta_{\iota,j} - \zeta_{\iota,j+1}| >$ some positive number L. Then

$$\lim_{d=0} \sum_\iota \Delta' x_\iota = 0 \text{ for any fixed } L. \tag{2}$$

For suppose the contrary; *i. e.*,

$$\sum_\iota \Delta' x_\iota > M > 0 \text{ for some } L.$$

Then area $A_d \geq ML > 0$. This is contrary to the hypothesis that A is discrete.

Let $\mathfrak{G}_x$ be the projection of $\mathfrak{G}$ on the x-axis. Divide the columns of cells of A_d into two classes according as

$$\text{(a)} \quad \sum_j |f(\alpha_\iota, \zeta_{\iota,2j-1}) - f(\alpha_\iota, \zeta_{\iota,2j})| \leq \varepsilon/2\mathfrak{G}_x, \qquad d < d'; \tag{3}$$

$$\text{(b)} \quad \sum_j |f(\alpha_\iota, \zeta_{\iota,2j-1}) - f(\alpha_\iota, \zeta_{\iota,2j})| > \varepsilon/2\mathfrak{G}_x, \qquad d < d'. \tag{4}$$

Since $f(x,y)$ has limited fluctuation, for columns of (b) $\sum_j |\zeta_{\iota,j} - \zeta_{\iota,j+1}| >$ some number L. For such columns of cells, from (2),

$$\sum_\iota \Delta' x_\iota < \varepsilon/2P, \qquad d < d'', \tag{5}$$

where $P = \max \sum \operatorname{osc} f$ for any line of $\mathfrak{P}_x$. P is finite, as $f(x,y)$ has limited

fluctuation. Then the sum in (1) breaks up into two sums $\sum_a$ and $\sum_b$:

$$|\sum_a [f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})]\Delta x_\iota| < \mathfrak{G}_x \max \sum_j |f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})|$$

$$< \varepsilon/2, \qquad d < d_0 \text{ from (3)},$$

$$|\sum_b [f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})]\Delta x_\iota| < P \sum_\iota{}' \Delta x_\iota < \varepsilon/2, \qquad d < d_0 \text{ from (4)},$$

where $d', d'' > d_0$. Therefore

$$|\sum_{\iota j} [f(\alpha_\iota, \zeta_{\iota,2j}) - f(\alpha_\iota, \zeta_{\iota,2j-1})]\Delta x_\iota| < |\sum_a| + |\sum_b| < \varepsilon, \qquad d < d_0.$$

Remark. In the preceding theorem x and y may obviously be interchanged.

THEOREM V. *Let $\mathfrak{B}$ be a pantactic* set in the Green field $\mathfrak{G}$. Let A be a discrete set in $\mathfrak{G} - \mathfrak{B}$. Let D be a normal division of $\mathfrak{G}$ of norm d. Let $d_1, \ldots, d_m$ be cells of $\mathfrak{G}$ containing no points of A. Let $f(x, y)$ be limited and integrable over $\mathfrak{B}$. Then*

$$\lim_{d=0} \Sigma f(v_\iota) d_\iota = \int_{\mathfrak{B}} f(x, y)\, d\mathfrak{B}. \tag{1}$$

Let $e_1, \ldots, e_n$ denote the cells of $\mathfrak{G}$ made by D. Let $\delta_1, \ldots, \delta_r$ denote the cells of $\mathfrak{G}$ containing points of A. Then

$$\Sigma e_\iota = \Sigma d_\iota + \Sigma \delta_\iota.$$

By definition of an integral,

$$|\int_{\mathfrak{B}} f(x, y)\, d\mathfrak{B} - \Sigma f(v_\iota) e_\iota| < \varepsilon/2, \qquad d < d'. \tag{2}$$

Also

$$|\Sigma f(v_\iota) e_\iota - \Sigma f(v_\iota) d_\iota| < F \Sigma \delta_\iota, \tag{3}$$

where $|f| < F$ in $\mathfrak{B}$. Since A is discrete,

$$|\Sigma \delta_\iota| < \varepsilon/2F, \qquad d < d''. \tag{4}$$

From (2), (3), (4),

$$|\int_{\mathfrak{B}} f(x, y)\, d\mathfrak{B} - \Sigma f(v_\iota) d_\iota| < \varepsilon, \qquad d < d_0,$$

where $d_0 < d'$, and $d_0 < d''$.

THEOREM VI. *Let $\mathfrak{B}$ be pantactic on each parallel of $\mathfrak{P}_x$, and $\mathfrak{C}$ on each parallel of $\mathfrak{P}_y$, in a Green field $\mathfrak{G}$. Let $B = \mathfrak{P}_x - \mathfrak{B}$ and $C = \mathfrak{P}_y - \mathfrak{C}$ be discrete in two-way space.*

Then $\Delta = \mathfrak{D}v(\mathfrak{B}, \mathfrak{C})$ is pantactic in $\mathfrak{G}$.

Let D be a rectangular division of $\mathfrak{G}$. Let d be a cell of D. As $B + C$ is discrete, and as $\mathfrak{B}, \mathfrak{C}$ are pantactic, there exists in d a rectangle δ which contains only points of $\mathfrak{P}_x$ belonging to $\mathfrak{B}$ and points of $\mathfrak{P}_y$ belonging to $\mathfrak{C}$.

* Pantactic is used here in the sense defined in Lectures, Vol. II, p. 325.

Let $\mathfrak{d}_x$ be the part of one of the parallels of $\mathfrak{P}_x$ in $\mathfrak{d}$. Every point of $\mathfrak{d}_x$ belongs to $\mathfrak{B}$. The points on $\mathfrak{d}_x$ through which pass lines of $\mathfrak{P}_y$ are pantactic. Moreover, each point on $\mathfrak{P}_y$ in $\mathfrak{d}$ belongs to $\mathfrak{C}$. Therefore, points of Δ which lie on $\mathfrak{d}_x$ are pantactic relative to $\mathfrak{d}_x$. Thus $\mathfrak{d}$ and also d contain points of Δ. Since d is any cell of $\mathfrak{G}$, Δ is pantactic relative to $\mathfrak{G}$.

THEOREM VII. *Let $f(x,y)$ be limited and integrable over the boundary C of a Green field $\mathfrak{G}$, and have limited fluctuations on $\mathfrak{P}_x$ with respect to y. Let $\mathfrak{B}$ be the set on $\mathfrak{P}_x$ where $f_2 = \dfrac{\partial f}{\partial y}$ exists. Let $\mathfrak{B}$ be pantactic on each parallel of $\mathfrak{P}_x$ and let $B = \mathfrak{P}_x - \mathfrak{B}$ be a discrete set in $\mathfrak{G}$. Let f_2 be limited and integrable over $\mathfrak{B}$. Then*

$$\int_C f(x,y)\,dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\,d\mathfrak{B}. \tag{1}$$

Let D be a normal division of $\mathfrak{G}$ of norm d. Let $\mathfrak{A}_d$ be the cells of D where f_2 exists on $\mathfrak{P}_x$, *i. e.*, containing no points of B. Let d_j be a cell of $\mathfrak{A}_d$ one of whose sides lies on $x = \alpha_\iota$ and has end points (α_ι, ν), (α_ι, μ). Since f_2 exists for each point of (ν, μ) on $x = \alpha_\iota$ by hypothesis, the law of the mean may be applied, and we have

$$f(\alpha_\iota, \mu) - f(\alpha_\iota, \nu) = f_2(\alpha_\iota, k_j)(\mu - \nu), \qquad \nu \leq k_j \leq \mu. \tag{2}$$

If we do this for one side of each cell of $\mathfrak{A}_d$, taking always the sides parallel to the y-axis, and add all the equations thus obtained, some of the terms on the left will cancel in pairs. For example, if (α_ι, η) is a point within $\mathfrak{A}_d$, but at the corner of a cell d_j, it will be both a ν point and a μ point. Considering d_j we have

$$f(\alpha_\iota, r) - f(\alpha_\iota, \eta) = f_2(\alpha_\iota, k_j)(r - \eta), \qquad \eta \leq k_j \leq r.$$

Considering d_{j-1} we have

$$f(\alpha_\iota, \eta) - f(\alpha_\iota, \lambda) = f_2(\alpha_\iota, k_{j-1})(\eta - \lambda), \qquad \lambda \leq k_{j-1} \leq \eta.$$

When we add these two equations, $f(\alpha_\iota \eta)$ drops out. In the sum obtained by adding equations of type (2), only terms involving points on the boundary of $\mathfrak{A}_d$ will remain on the left. We call these ξ points, and denote those on $x = \alpha_\iota$ by

$$\xi_{\iota,1} < \xi_{\iota,2} < \dots < \xi_{\iota,2r}. \tag{3}$$

These points are finite in number, as D is a normal division. Adding equations of type (2) and multiplying by Δx_ι, we get

$$\sum_{\iota j} [f(\alpha_\iota, \xi_{\iota,2j}) - f(\alpha_\iota, \xi_{\iota,2j-1})]\,\Delta x_\iota = \sum_{\iota j} f_2(\alpha_\iota, k_{\iota,j})\,\Delta x_\iota \Delta y_j. \tag{4}$$

Let $x = \alpha_\iota$ cut C in the points $a_{\iota,1} < a_{\iota,2} < \dots < a_{\iota,2n}$. (5)

Let B_d be that part of the boundary of $\mathfrak{G} - \mathfrak{A}_d$ which is not parallel to the y-axis.

Let $x=\alpha_\iota$ cut B_d in $\zeta_{\iota,1}<\zeta_{\iota,2}<\ldots\ldots<\zeta_{\iota,2m}$. (6)

The ξ points are either ζ or a points; *i. e.*, either on C or on B_d.

Suppose $\xi_{\iota,k}=a_{\iota,s}$. Then both k and s are odd or both even. Let k be even. Then, from (3), points near $\xi_{\iota,k}$ but $<\xi_{\iota,k}$ on $x=\alpha_\iota$ belong to $\mathfrak{A}_d$; and those near $\xi_{\iota,k}$ but $>\xi_{\iota,k}$ on $x=\alpha_\iota$ do not. As $\xi_{\iota,k}$ is on C, these latter points do not belong to $\mathfrak{G}$, and the former do. Thus, from (5), s must be even. Similarly, s can be proved odd, when k is odd.

Suppose now $\xi_{\iota,k}=\zeta_{\iota,s}$. If k is even, s is odd and conversely. Let k be even. The statement above holds in regard to points belonging to $\mathfrak{A}_d$. Since $\xi_{\iota,k}$ is not on C, points above $\xi_{\iota,k}$ on $x=\alpha_\iota$ must belong to $\mathfrak{G}-\mathfrak{A}_d$. Then, from (6), s is odd.

Thus the left side of (4) may be broken up into two sums, one containing terms involving ζ points and the other containing terms involving a points. We have, therefore,

$$\sum_{\iota j}[f(\alpha_\iota,a_{\iota,2j})-f(\alpha_\iota,a_{\iota,2j-1})]\Delta x_\iota+\sum_{\iota j}[f(\alpha_\iota,\zeta_{\iota,2j-1})-f(\alpha_\iota,\zeta_{\iota,2j})]\Delta x_\iota=\sum_1+\sum_2. \quad (7)$$

If there exists a point θ which is both an a point and a ζ point, it is not on the boundary of $\mathfrak{A}_d$; so the term $f(\alpha_\iota,\theta)$ does not appear in (4).

Let $\theta=a_{\iota,s}=\zeta_{\iota,r}$. Reasoning similar to that used for the case where $\xi_{\iota,k}=a_{\iota,s}$ shows that s and r are both even or both odd. Therefore, in (7) the term involving $f(\alpha_\iota,\theta)$ enters twice, but with opposite signs; so (7) equals the left side of (4), no matter how many points like θ there may be. Therefore,

$$\sum_1+\sum_2=\sum_{\iota j}f_2(\alpha_\iota,k_{\iota,j})\,\Delta x_\iota\Delta y_j. \quad (8)$$

(7) contains now all the terms involving a points for lines $x=\alpha_\iota$ of D and all ζ points. By Theorem III,

$$\sum_1\doteq-\int_C f(x,y)\,dx. \quad (9)$$

By Theorem IV,

$$\sum_2\doteq 0. \quad (10)$$

By Theorem V,

$$\sum_{\iota j}f_2(\alpha_\iota,k_{\iota,j})\Delta x_\iota\Delta y_j\doteq\int_{\mathfrak{B}}\frac{\partial f}{\partial y}\,d\mathfrak{B}. \quad (11)$$

The theorem follows from (8), (9), (10), (11).

Remark. If in the preceding theorem and demonstration x and y are interchanged and Theorem II is used, then (1) becomes

$$\int_C g(x,y)\,dy=\int_{\mathfrak{C}}\frac{\partial g}{\partial x}\,d\mathfrak{C}. \quad (12)$$

We have, as a corollary of the foregoing,

THEOREM VIII. *Let* $\Delta = \mathfrak{D} v (\mathfrak{B}, \mathfrak{C})$. *Then, from* (1) *and* (12) *and Theorem* VI,

$$\int_C \{ f(x, y)\, dx + g(x, y)\, dy \} = \int_\Delta \left\{ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\} d\Delta. \tag{13}$$

The relation (13) is Green's theorem proved for a very general class of functions over a field which may be cut, by each one of a certain set of parallels, in an infinite number of points. This set may contain an infinite number of lines. The functions do not need to be continuous with respect to one variable on all lines in the field, but only on a certain set of lines $\mathfrak{P}_x$ or $\mathfrak{P}_y$. The value of the function on lines not belonging to this set is of no consequence, provided the function is integrable over the boundary of the field. The derivative need not exist for every point on the lines $\mathfrak{P}_x$ or $\mathfrak{P}_y$, but must exist for a certain set, over which it is integrable. The derivative may exist for points in the field not on $\mathfrak{P}_x$ or $\mathfrak{P}_y$. When this is true, the derivative may not be integrable over the whole field for which it exists.

§ 2. *Nonrectifiable Green Fields.*

Up to the present, line integrals have been defined only for rectifiable curves. If now we look at the relation (12) obtained in Theorem VIII, we see the right side has a meaning whether the boundary of $\mathfrak{G}$ is rectifiable or not. Let us see, then, if it is not possible to extend our definition of a line integral so that the left side of this relation has a meaning when the curve C is not rectifiable. This can be done for a class of curves defined as follows:

1°. C_{ab} is a Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta).$$

2°. A is a discrete set in $\mathfrak{A}$ having I as its image on C_{ab}.

3°. C_{ab} is rectifiable except in the vicinity of points of I.

By 3° is meant the following: Let Δ be a division of $\mathfrak{A}$ of norm δ. As A is discrete, if $\delta < \delta_0$ there are intervals containing no points of A. Let their sum be $\mathfrak{A}_d$, and let C_δ be the part of C_{ab} corresponding to $\mathfrak{A}_\delta$. Then C_δ is rectifiable for each $\delta < \delta_0$, but C_{ab} is not rectifiable.

An example of such a curve is

$$\begin{aligned} x = t, \quad y &= t \sin \frac{1}{t}, \qquad 0 < t \leq 1, \\ &= 0, \qquad\qquad t = 0. \end{aligned}$$

This curve is rectifiable except for the point $t = 0$.

The set I will be called the singular points of the nonrectifiable curve C_{ab}.

Let $f(x, y)$ be defined and limited over C_{ab}. If $f(x, y)$ is integrable over C_δ for each $\delta < \delta_0$, $f(x, y)$ is said to be regular over C_{ab}.

Let $f(x, y)$ be regular over the nonrectifiable curve C. If

$$\lim_{\delta=0} \int_{C_\delta} f(x, y)\, dx \text{ and } \lim_{\delta=0} \int_{C_\delta} f(x, y)\, dy$$

exist, these limits are denoted by

$$\int_{C_{ab}} f(x, y)\, dx \text{ and } \int_{C_{ab}} f(x, y)\, dy.$$

If both limits exist, $f(x, y)$ is said to be integrable over C_{ab}.

A field $\mathfrak{G}$ whose boundary C satisfies the following conditions will be called a *nonrectifiable Green field.*

1°. C is a closed Jordan curve defined by

$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta),$$

and rectifiable except for a set I whose images on the axes are discrete.

2°. For any C_δ the corresponding functions $X_\delta(t)$ and $Y_\delta(t)$ are such that, if A is a discrete set in $\mathfrak{A}_\delta$, the image of A given by $X_\delta(t)$ or by $Y_\delta(t)$ is discrete.

3°. The points at which either $X(t)$ or $Y(t)$ has a proper extreme form a discrete set in $\mathfrak{A}$.

4°. C has only a finite number of segments parallel to the axes.

5°. C is cut by each one of a pantactic set of parallels in only a finite number of points.

Let $f(x,y)$ be regular over the nonrectifiable boundary C of a Green field $\mathfrak{G}$, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then $f(x,y)$ will be called a normal function of $\mathfrak{G}$ with respect to y. A similar definition holds for a normal function with respect to x.

THEOREM IX. *Let $f(x,y)$ be a normal function of the nonrectifiable Green field $\mathfrak{G}$ with respect to y. Let $\mathfrak{B}$ be the set on $\mathfrak{P}_x$ where $f_2 = \dfrac{\partial f}{\partial y}$ exists. Let $B = \mathfrak{P}_x - \mathfrak{B}$ be discrete. Let f_2 be limited and integrable over $\mathfrak{B}$. Then*

$$\int_C f(x, y)\, dx = - \int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}. \tag{1}$$

Let I_x be the projection of the singular points of C on the x-axis. I_x is discrete by the definition of a nonrectifiable Green field. Let D be a normal division of $\mathfrak{G}$ of norm d. Let us look at the columns of cells parallel to the y-axis, which contain no points of I. In each of these columns, there is one or more partial fields of $\mathfrak{G}$. These are finite in number, as D is a normal division.

Let these fields be $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_m$, with boundaries $\lambda_1, \lambda_2, \ldots, \lambda_m$. Let $\mathfrak{B}_\iota$ be the part of $\mathfrak{B}$ in $\mathfrak{G}_\iota$.

Each $\mathfrak{G}_\iota$ satisfies the conditions of Theorem VII; therefore

$$\int_{\lambda_\iota} f(x, y)\, dx = \int_{\mathfrak{B}_\iota} \frac{\partial f}{\partial y}\, d\mathfrak{B}_\iota. \tag{2}$$

Let l_ι be the part of λ_ι belonging to C. For the rest of λ_ι, $dx = 0$, since it is parallel to the y-axis. Thus

$$\int_{l_\iota} f(x, y)\, dx = \int_{\lambda_\iota} f(x, y)\, dx. \tag{3}$$

Let $\mathfrak{B}_0 = \sum_{\iota=1}^{m} \mathfrak{B}_\iota$, $C_\delta = \sum_{\iota=1}^{m} l_\iota$. From (2), (3),

$$\int_{C_\delta} f(x, y)\, dx = -\int_{\mathfrak{B}_0} \frac{\partial f}{\partial y}\, d\mathfrak{B}_0. \tag{4}$$

As $\overline{\mathfrak{B}}_0 \doteq \overline{\mathfrak{B}}$,

$$\int_{\mathfrak{B}_0} \frac{\partial f}{\partial y}\, d\mathfrak{B}_0 \doteq \int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}.^* \tag{5}$$

Therefore $\lim\limits_{\delta=0} \int_{C_\delta} f(x, y)\, dx$ exists. Then, by definition,

$$\lim_{\delta=0} \int_{C_\delta} f(x, y)\, dx = \int_C f(x, y)\, dx. \tag{6}$$

Therefore $\int_C f(x, y)\, dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}$, from (4), (5), (6).

Remark. If x and y are interchanged as has been done before, (1) becomes

$$\int_C g(x, y)\, dy = \int_{\mathfrak{C}} \frac{\partial g}{\partial x}\, d\mathfrak{C}. \tag{7}$$

As a corollary of the preceding, we have

THEOREM X. *Let* $\Delta = \mathfrak{D}v(\mathfrak{B}, \mathfrak{C})$. *Then*

$$\int_C \{f(x, y)\, dx + g(x, y)\, dy\} = \int_\Delta \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] d\Delta. \tag{8}$$

(8) follows at once from (1) and (7).

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* Lectures, Vol. II, pp. 55, 60.

On the Asymptotic Solutions of Linear Differential Equations.

By Clyde E. Love.

Asymptotic developments for the irregular integrals of a linear differential equation have been obtained by Horn,* provided that the coefficients of the equation are themselves developable in asymptotic (or convergent) power series in the vicinity of the irregular point in question, and provided also that the roots of the so-called characteristic equation are distinct.† The important special case in which the point is a *regular* singular point ‡ of the differential equation has been studied by Bôcher § for the equation of second order, and later by Dunkel ‖ for the equation of arbitrary order. But no discussion of the general problem including the various cases of repeated roots has as yet been attempted.

By way of approach to the general solution it seems worth while to consider in some detail the cases of repeated roots for various equations of special orders. Such a study is undertaken in the following pages. We restrict ourselves, for simplicity, to equations of the second and of the third order, the method used being applicable at once to equations of any order. The irregular point is taken at infinity, and only real values of the independent variable are considered.

In point of method, we make use of two general theorems arising from Dini's ¶ researches in the theory of linear differential equations. The statement of these theorems, with an outline of their proof, is to be found in Sec. I.

The equation of second order forms the subject of Sec. II. Although the researches of Horn,** Kneser,†† Bôcher ‡‡ and others leave less to be done

* *Journal für Mathematik*, Vol. CXXXVIII (1910), pp. 159–191.

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§ Bôcher, *loc. cit.*

‖ *Proceedings of the American Academy of Arts and Sciences*, Vol. XXXVIII (1903), pp. 339-370.

¶ *Annali di Matematica*, Ser. 3, Vol. II (1898), pp. 297–324; *ibid.*, Vol. III (1899), pp. 125–183. The possibility of applying Dini's methods to the problem in hand was suggested to me by Prof. W. B. Ford.

** *Acta Mathematica*, Vol. XXIII (1900), pp. 171–202.

†† *Journal für Mathematik*, Vol. CXVI (1896), pp. 178–212; *ibid.*, Vol. CXVII (1896), pp. 72–103; *ibid.*, Vol. CXX (1899), pp. 267–275.

‡‡ Bôcher, *loc. cit.*

upon this equation than upon those of the third and higher orders, it is believed that the discussion, from the standpoint of the Dini theory, is of sufficient interest to warrant its insertion in condensed form. We begin with a brief treatment of the case of distinct roots. While the results for this case are not new, it seems best to exemplify the method by applying it first to this simple problem, so that in the subsequent discussion of more complicated cases many details may be omitted. For the equation under consideration, the only case of repeated roots that offers any difficulty is that in which the point at infinity is a "regular" singular point, and this, as noted above, has been studied by Bôcher. However, on account of his more general hypotheses he does not obtain an asymptotic solution in Poincaré's* sense, but only the dominant term of such a solution, so that the present results are a step in advance.

Questions of more interest arise in connection with the equation of third order, which is discussed at length in Sec. III. This discussion, when compared with that for the equation of second order, is of distinctly greater moment, not only because the results possess greater novelty, but because the methods used are much more suggestive in pointing the way toward a solution of the general problem. This is especially true of the case in which the characteristic equation has a simple and a double root.

I. Two General Theorems on Linear Differential Equations.†

Suppose that, in the differential equation

$$y^{(n)}+a_1(x)y^{(n-1)}+\ldots\ldots+a_n(x)y=0, \tag{1}$$

the coefficients $a_1(x), \ldots\ldots, a_n(x)$ and their first n derivatives are continuous for all positive values of x sufficiently large. Let us choose n auxiliary functions $z_1, z_2, \ldots\ldots, z_n$ of x, which, with their first n derivatives, are continuous for large positive x, and such that for the same values of x the determinant

$$Q(x)=\begin{vmatrix} z_1 & z_1' & \ldots\ldots & z_1^{(n-1)} \\ z_2 & z_2' & \ldots\ldots & z_2^{(n-1)} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ z_n & z_n' & \ldots\ldots & z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Let $A_r(x)$ denote the minor of $Q(x)$ with respect to the element $z_r^{(n-1)}$. Also define

$$Z_r(x)=z_ra_n-(z_ra_{n-1})'+\ldots\ldots+(-1)^{n-1}(z_ra_1)^{(n-1)}+(-1)^nz_r^{(n)}, \tag{3}$$
$$r=1, \ldots\ldots, n,$$

* *Acta Mathematica*, Vol. VIII (1886), p. 296.

† Cf. Dini, *loc. cit.*

and form the determinant

$$q(x, x_1) = \begin{vmatrix} z_1(x) & z_1'(x) & \dots & z_1^{(n-2)}(x) & Z_1(x_1) \\ z_2(x) & z_2'(x) & \dots & z_2^{(n-2)}(x) & Z_2(x_1) \\ \dots & \dots & \dots & \dots & \dots \\ z_n(x) & z_n'(x) & \dots & z_n^{(n-2)}(x) & Z_n(x_1) \end{vmatrix}. \tag{4}$$

Let C_r denote an arbitrary constant, and place

$$K(x, x_1) = \frac{(-1)^{n-1} q(x, x_1)}{Q(x)}, \tag{5}$$

$$\left.\begin{aligned} g_r(x) = u_{r,0}(x) = v_{r,0}(x) &= \frac{(-1)^{n-1} C_r A_r(x)}{Q(x)}, \qquad r = 1, \dots, n, \\ u_{r,\lambda}(x) = \int_a^x \int_a^{x_1} \dots \int_a^{x_{\lambda-2}} \int_a^{x_{\lambda-1}} &K(x, x_1) K(x_1, x_2) \dots \\ &K(x_{\lambda-2}, x_{\lambda-1}) K(x_{\lambda-1}, x_\lambda) g_r(x_\lambda)\, dx_\lambda dx_{\lambda-1} \dots dx_2 dx_1, \\ v_{r,\lambda}(x) = \int_x^\infty \int_{x_1}^\infty \dots \int_{x_{\lambda-2}}^\infty \int_{x_{\lambda-1}}^\infty &K(x, x_1) K(x_1, x_2) \dots \\ &K(x_{\lambda-2}, x_{\lambda-1}) K(x_{\lambda-1}, x_\lambda) g_r(x_\lambda)\, dx_\lambda dx_{\lambda-1} \dots dx_2 dx_1. \end{aligned}\right\} \tag{6}$$

Then we have

THEOREM A: *If a constant a can be found such that for all values of* $x > a$ *the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} u_{r,\lambda}(x)$$

satisfies the following conditions:

(a) *the series converges;*

(b) *the series defines a function* $y_r(x)$ *such that the series for* $y_r(x_1)$ *when multiplied by* $q(x, x_1)$ *may be integrated term by term with respect to* x_1 *from a to x;*

then for such values of x the function $y_r(x)$ *is an integral of* (1).

THEOREM B: *If for values of x greater than some constant, the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} v_{r,\lambda}(x) \tag{7}$$

satisfies the following conditions:

(a) *the series converges;*

(b) *the series for* $y_r(x_1)$ *when multiplied by* $q(x, x_1)$ *may be integrated term by term with respect to* x_1 *from x to* ∞;

(c) *the series defines a function* $y_r(x)$ *such that each of the integrals*

$$\int_x^\infty y_r(x) Z_s(x)\, dx, \qquad s = 1, \dots, n,$$

has a meaning;

then for such values of x the function $y_r(x)$ *is an integral of* (1).

We have, as a corollary of the foregoing,

THEOREM VIII. *Let* $\Delta = \mathfrak{D}\, v\,(\mathfrak{B}, \mathfrak{C})$. *Then, from* (1) *and* (12) *and Theorem* VI,

$$\int_C \{ f(x, y)\, dx + g(x, y)\, dy \} = \int_\Delta \left\{ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\} d\Delta. \tag{13}$$

The relation (13) is Green's theorem proved for a very general class of functions over a field which may be cut, by each one of a certain set of parallels, in an infinite number of points. This set may contain an infinite number of lines. The functions do not need to be continuous with respect to one variable on all lines in the field, but only on a certain set of lines $\mathfrak{P}_x$ or $\mathfrak{P}_y$. The value of the function on lines not belonging to this set is of no consequence, provided the function is integrable over the boundary of the field. The derivative need not exist for every point on the lines $\mathfrak{P}_x$ or $\mathfrak{P}_y$, but must exist for a certain set, over which it is integrable. The derivative may exist for points in the field not on $\mathfrak{P}_x$ or $\mathfrak{P}_y$. When this is true, the derivative may not be integrable over the whole field for which it exists.

§ 2. *Nonrectifiable Green Fields.*

Up to the present, line integrals have been defined only for rectifiable curves. If now we look at the relation (12) obtained in Theorem VIII, we see the right side has a meaning whether the boundary of $\mathfrak{G}$ is rectifiable or not. Let us see, then, if it is not possible to extend our definition of a line integral so that the left side of this relation has a meaning when the curve C is not rectifiable. This can be done for a class of curves defined as follows:

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$$x = X(t), \quad y = Y(t), \qquad t \text{ in } \mathfrak{A} = (\alpha < \beta).$$

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3°. C_{ab} is rectifiable except in the vicinity of points of I.

By 3° is meant the following: Let Δ be a division of $\mathfrak{A}$ of norm δ. As A is discrete, if $\delta < \delta_0$ there are intervals containing no points of A. Let their sum be $\mathfrak{A}_d$, and let C_δ be the part of C_{ab} corresponding to $\mathfrak{A}_\delta$. Then C_δ is rectifiable for each $\delta < \delta_0$, but C_{ab} is not rectifiable.

An example of such a curve is

$$x = t, \quad y = t \sin \frac{1}{t}, \qquad 0 < t \leq 1,$$
$$= 0, \qquad\qquad t = 0.$$

This curve is rectifiable except for the point $t = 0$.

The set I will be called the singular points of the nonrectifiable curve C_{ab}.

Let $f(x, y)$ be defined and limited over C_{ab}. If $f(x, y)$ is integrable over C_δ for each $\delta < \delta_0$, $f(x, y)$ is said to be regular over C_{ab}.

Let $f(x, y)$ be regular over the nonrectifiable curve C. If

$$\lim_{\delta=0} \int_{C_\delta} f(x, y)\, dx \text{ and } \lim_{\delta=0} \int_{C_\delta} f(x, y)\, dy$$

exist, these limits are denoted by

$$\int_{C_{ab}} f(x, y)\, dx \quad \text{and} \quad \int_{C_{ab}} f(x, y)\, dy.$$

If both limits exist, $f(x, y)$ is said to be integrable over C_{ab}.

A field $\mathfrak{G}$ whose boundary C satisfies the following conditions will be called a *nonrectifiable Green field.*

1°. C is a closed Jordan curve defined by

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and rectifiable except for a set I whose images on the axes are discrete.

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4°. C has only a finite number of segments parallel to the axes.

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Let $f(x,y)$ be regular over the nonrectifiable boundary C of a Green field $\mathfrak{G}$, and have limited fluctuation on $\mathfrak{P}_x$ with respect to y. Then $f(x,y)$ will be called a normal function of $\mathfrak{G}$ with respect to y. A similar definition holds for a normal function with respect to x.

THEOREM IX. *Let $f(x,y)$ be a normal function of the nonrectifiable Green field $\mathfrak{G}$ with respect to y. Let $\mathfrak{B}$ be the set on $\mathfrak{P}_x$ where $f_2 = \dfrac{\partial f}{\partial y}$ exists. Let $B = \mathfrak{P}_x - \mathfrak{B}$ be discrete. Let f_2 be limited and integrable over $\mathfrak{B}$. Then*

$$\int_C f(x, y)\, dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}. \tag{1}$$

Let I_x be the projection of the singular points of C on the x-axis. I_x is discrete by the definition of a nonrectifiable Green field. Let D be a normal division of $\mathfrak{G}$ of norm d. Let us look at the columns of cells parallel to the y-axis, which contain no points of I. In each of these columns, there is one or more partial fields of $\mathfrak{G}$. These are finite in number, as D is a normal division.

Let the fields be $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_m$, with boundaries $\lambda_1, \lambda_2, \ldots, \lambda_m$. Let $\mathfrak{B}_\iota$ be the part of $\mathfrak{B}$ in $\mathfrak{G}_\iota$.

Each $\mathfrak{G}_\iota$ satisfies the conditions of Theorem VII; therefore

$$\int_{\lambda_\iota} f(x, y)\, dx = -\int_{\mathfrak{B}_\iota} \frac{\partial f}{\partial y}\, d\mathfrak{B}_\iota. \tag{2}$$

Let l_ι be the part of λ_ι belonging to C. For the rest of λ_ι, $dx = 0$, since it is parallel to the y-axis. Thus

$$\int_{\lambda_\iota} f(x, y)\, dx = \int_{l_\iota} f(x, y)\, dx. \tag{3}$$

Let $\mathfrak{B}_0 = \sum_{\iota=1}^{m} \mathfrak{B}_\iota$, $C_\delta = \sum_{\iota=1}^{m} l_\iota$. From (2), (3),

$$\int_{C_\delta} f(x, y)\, dx = -\int_{\mathfrak{B}_0} \frac{\partial f}{\partial y}\, d\mathfrak{B}_0. \tag{4}$$

As $\overline{\mathfrak{B}}_0 \doteq \overline{\mathfrak{B}}$,

$$\int_{\mathfrak{B}_0} \frac{\partial f}{\partial y}\, d\mathfrak{B}_0 \doteq \int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}.^* \tag{5}$$

Therefore $\lim_{\delta=0} \int_{C_\delta} f(x, y)\, dx$ exists. Then, by definition,

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Therefore $\int_C f(x, y)\, dx = -\int_{\mathfrak{B}} \frac{\partial f}{\partial y}\, d\mathfrak{B}$, from (4), (5), (6).

Remark. If x and y are interchanged as has been done before, (1) becomes

$$\int_C g(x, y)\, dy = \int_{\mathfrak{C}} \frac{\partial g}{\partial x}\, d\mathfrak{C}. \tag{7}$$

As a corollary of the preceding, we have

THEOREM X. *Let $\Delta = \mathfrak{D}\, p\, (\mathfrak{B}, \mathfrak{C})$. Then*

$$\int_C \{f(x, y)\, dx + g(x, y)\, dy\} = \int_\Delta \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] d\Delta. \tag{8}$$

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On the Asymptotic Solutions of Linear Differential Equations.

By Clyde E. Love.

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By way of approach to the general solution it seems worth while to consider in some detail the cases of repeated roots for various equations of special orders. Such a study is undertaken in the following pages. We restrict ourselves, for simplicity, to equations of the second and of the third order, the method used being applicable at once to equations of any order. The irregular point is taken at infinity, and only real values of the independent variable are considered.

In point of method, we make use of two general theorems arising from Dini's¶ researches in the theory of linear differential equations. The statement of these theorems, with an outline of their proof, is to be found in Sec. I.

The equation of second order forms the subject of Sec. II. Although the researches of Horn,** Kneser,†† Bôcher‡‡ and others leave less to be done

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§ Bôcher, *loc. cit.*

‖ *Proceedings of the American Academy of Arts and Sciences*, Vol. XXXVIII (1903), pp. 339–370.

¶ *Annali di Matematica*, Ser. 3, Vol. II (1898), pp. 297–324; *ibid.*, Vol. III (1899), pp. 125–183. The possibility of applying Dini's methods to the problem in hand was suggested to me by Prof. W. B. Ford.

** *Acta Mathematica*, Vol. XXIII (1900), pp. 171–202.

†† *Journal für Mathematik*, Vol. CXVI (1896), pp. 178–212; *ibid.*, Vol. CXVII (1896), pp. 72–103; *ibid.*, Vol. CXX (1899), pp. 267–275.

‡‡ Bôcher, *loc. cit.*

upon this equation than upon those of the third and higher orders, it is believed that the discussion, from the standpoint of the Dini theory, is of sufficient interest to warrant its insertion in condensed form. We begin with a brief treatment of the case of distinct roots. While the results for this case are not new, it seems best to exemplify the method by applying it first to this simple problem, so that in the subsequent discussion of more complicated cases many details may be omitted. For the equation under consideration, the only case of repeated roots that offers any difficulty is that in which the point at infinity is a "regular" singular point, and this, as noted above, has been studied by Bôcher. However, on account of his more general hypotheses he does not obtain an asymptotic solution in Poincaré's * sense, but only the dominant term of such a solution, so that the present results are a step in advance.

Questions of more interest arise in connection with the equation of third order, which is discussed at length in Sec. III. This discussion, when compared with that for the equation of second order, is of distinctly greater moment, not only because the results possess greater novelty, but because the methods used are much more suggestive in pointing the way toward a solution of the general problem. This is especially true of the case in which the characteristic equation has a simple and a double root.

I. Two General Theorems on Linear Differential Equations.†

Suppose that, in the differential equation

$$y^{(n)}+a_1(x)y^{(n-1)}+\dots+a_n(x)y=0, \tag{1}$$

the coefficients $a_1(x), \dots, a_n(x)$ and their first n derivatives are continuous for all positive values of x sufficiently large. Let us choose n auxiliary functions $z_1, z_2, \dots, z_n$ of x, which, with their first n derivatives, are continuous for large positive x, and such that for the same values of x the determinant

$$Q(x)=\begin{vmatrix} z_1 & z_1' & \dots & z_1^{(n-1)} \\ z_2 & z_2' & \dots & z_2^{(n-1)} \\ \dots & \dots & \dots & \dots \\ z_n & z_n' & \dots & z_n^{(n-1)} \end{vmatrix} \tag{2}$$

never vanishes. Let $A_r(x)$ denote the minor of $Q(x)$ with respect to the element $z_r^{(n-1)}$. Also define

$$Z_r(x)=z_ra_n-(z_ra_{n-1})'+\dots+(-1)^{n-1}(z_ra_1)^{(n-1)}+(-1)^n z_r^{(n)}, \qquad r=1, \dots, n, \tag{3}$$

* *Acta Mathematica*, Vol. VIII (1886), p. 296.

† Cf. Dini, *loc. cit.*

and form the determinant

$$q(x, x_1) = \begin{vmatrix} z_1(x) & z_1'(x) & \dots & z_1^{(n-2)}(x) & Z_1(x_1) \\ z_2(x) & z_2'(x) & \dots & z_2^{(n-2)}(x) & Z_2(x_1) \\ \dots & \dots & \dots & \dots & \dots \\ z_n(x) & z_n'(x) & \dots & z_n^{(n-2)}(x) & Z_n(x_1) \end{vmatrix}. \tag{4}$$

Let C_r denote an arbitrary constant, and place

$$K(x, x_1) = \frac{(-1)^{n-1} q(x, x_1)}{Q(x)}, \tag{5}$$

$$\left.\begin{aligned} g_r(x) &= u_{r,0}(x) = v_{r,0}(x) = \frac{(-1)^{n-1} C_r A_r(x)}{Q(x)}, \qquad r=1, \dots, n, \\ u_{r,\lambda}(x) &= \int_a^x \int_a^{x_1} \dots \int_a^{x_{\lambda-2}} \int_a^{x_{\lambda-1}} K(x, x_1) K(x_1, x_2) \dots \\ &\qquad K(x_{\lambda-2}, x_{\lambda-1}) K(x_{\lambda-1}, x_\lambda) g_r(x_\lambda)\, dx_\lambda dx_{\lambda-1} \dots dx_2 dx_1, \\ v_{r,\lambda}(x) &= \int_x^\infty \int_{x_1}^\infty \dots \int_{x_{\lambda-2}}^\infty \int_{x_{\lambda-1}}^\infty K(x, x_1) K(x_1, x_2) \dots \\ &\qquad K(x_{\lambda-2}, x_{\lambda-1}) K(x_{\lambda-1}, x_\lambda) g_r(x_\lambda)\, dx_\lambda dx_{\lambda-1} \dots dx_2 dx_1. \end{aligned}\right\} \tag{6}$$

Then we have

THEOREM A: *If a constant a can be found such that for all values of $x > a$ the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} u_{r,\lambda}(x)$$

satisfies the following conditions:

(a) *the series converges;*

(b) *the series defines a function $y_r(x)$ such that the series for $y_r(x_1)$ when multiplied by $q(x, x_1)$ may be integrated term by term with respect to x_1 from a to x;*

then for such values of x the function $y_r(x)$ is an integral of (1).

THEOREM B: *If for values of x greater than some constant, the series*

$$y_r(x) = \sum_{\lambda=0}^{\infty} v_{r,\lambda}(x) \tag{7}$$

satisfies the following conditions:

(a) *the series converges;*

(b) *the series for $y_r(x_1)$ when multiplied by $q(x, x_1)$ may be integrated term by term with respect to x_1 from x to ∞;*

(c) *the series defines a function $y_r(x)$ such that each of the integrals*

$$\int_x^\infty y_r(x) Z_s(x)\, dx, \qquad s=1, \dots, n,$$

has a meaning;

then for such values of x the function $y_r(x)$ is an integral of (1).

To prove Theorem B, place

$$\left.\begin{aligned} &p_{s,0}=z_s, \\ &p_{s,1}=z_s a_1-z'_s=z_s a_1-p'_{s,0}, \\ &\dots\dots\dots\dots\dots\dots, \\ &p_{s,n-1}=z_s a_{n-1}-p'_{s,n-2}, \qquad s=1,2,\dots,n; \end{aligned}\right\} \tag{8}$$

$$\Phi_s(x)=\int_x^\infty y_r Z_s dx, \qquad s=1,2,\dots,r-1,r+1,\dots,n,$$

$$\Phi_r(x)=\int_x^\infty y_r Z_r dx+C_r;$$

$$\Delta_r(x)=\begin{vmatrix} p_{1,0} & p_{1,1} & \dots & p_{1,n-2} & \Phi_1(x) \\ p_{2,0} & p_{2,1} & \dots & p_{2,n-2} & \Phi_2(x) \\ \dots & \dots & \dots & \dots & \dots \\ p_{n,0} & p_{n,1} & \dots & p_{n,n-2} & \Phi_n(x) \end{vmatrix};$$

$$\Delta(x)=\begin{vmatrix} p_{1,0} & p_{1,1} & \dots & p_{1,n-1} \\ p_{2,0} & p_{2,1} & \dots & p_{2,n-1} \\ \dots & \dots & \dots & \dots \\ p_{n,0} & p_{n,1} & \dots & p_{n,n-1} \end{vmatrix}=(-1)^{\frac{n(n-1)}{2}}Q(x).$$

Now by condition (c) the series (7) may be written

$$y_r(x)=g_r(x)+\int_x^\infty y_r(x_1)K(x,x_1)dx_1. \tag{9}$$

By substituting the values of $g_r(x)$ and $K(x,x_1)$ in (9), we find

$$y_r(x)=\frac{\Delta_r(x)}{\Delta(x)},$$

so that it suffices for our proof to show that this function is an integral of (1).

To do this, consider the system of n functions $\eta_0, \eta_1, \dots, \eta_{n-1}$, each defined for all values of x sufficiently large by means of the following system of n linear equations:

$$p_{s,0}\eta_{n-1}+p_{s,1}\eta_{n-2}+\dots+p_{s,n-1}\eta_0=\Phi_s(x), \qquad s=1,2,\dots,n. \tag{10}$$

We have at once

$$\eta_0=\frac{\Delta_r(x)}{\Delta(x)}=y_r. \tag{11}$$

Upon differentiating equations (10) with respect to x, and making use of (8) and (11), we find

$$z_s\theta+p'_{s,0}\theta_1+p'_{s,1}\theta_2+\dots+p'_{s,n-2}\theta_{n-1}=0, \qquad s=1,2,\dots,n, \tag{12}$$

where

$$\theta=\eta'_{n-1}+a_1(x)\eta'_{n-2}+\dots+a_{n-1}(x)\eta'_0+a_n(x)\eta_0, \tag{13}$$

$$\left.\begin{aligned} \theta_1&=\eta_{n-1}-\eta'_{n-2}, \\ \theta_2&=\eta_{n-2}-\eta'_{n-3}, \\ &\dots\dots\dots\dots, \\ \theta_{n-1}&=\eta_1-\eta'_0. \end{aligned}\right\} \tag{14}$$

The system (12) consists of n homogeneous linear equations in the n quantities $\theta, \theta_1, \ldots, \theta_{n-1}$. By virtue of the relation

$$p'_{t,s}=z_t a_{s+1}-p_{t,s+1}, \qquad t=1, 2, \ldots, n; \; s=0, 1, \ldots, n-2,$$

the discriminant of the system reduces at once to $(-1)^{n-1}Q(x)$, and hence by our hypotheses does not vanish for any value of x under consideration, whence

$$\theta \equiv \theta_1 \equiv \ldots \equiv \theta_{n-1} \equiv 0,$$

or, by (11) and (14),

$$\eta_s=y_r^{(s)}, \qquad s=1, 2, \ldots, n.$$

Substituting in (13), we find

$$y_r^{(n)}+a_1(x)y_r^{(n-1)}+\ldots+a_n(x)y_r \equiv 0,$$

which was to be proved.

The proof of Theorem A follows similar lines, and may be omitted.

II. The Equation of Second Order.

In the differential equation

$$y''+a(x)y'+b(x)y=0, \tag{15}$$

suppose that for sufficiently large positive real values of x we may write

$$a(x)\sim x^k\left[a_0+\frac{a_1}{x}+\ldots\right],$$

$$b(x)\sim x^{2k}\left[b_0+\frac{b_1}{x}+\ldots\right],$$

k being 0 or a positive integer, and suppose also that $a'(x)$ has an asymptotic development.

1. *Distinct Roots.*

Consider first the case in which the roots m_1, m_2 of the characteristic equation

$$m^2+a_0m+b_0=0$$

are distinct.

Let us choose two auxiliary functions $z_1(x)$, $z_2(x)$ (cf. Sec. I) of the form

$$z_r=e^{-f_r(x)+\phi_r(x)}x^{-\alpha_{r,0}}, \qquad r=1, 2, \tag{16}$$

where

$$f_r(x)=\frac{m_r x^{k+1}}{k+1}+\frac{\alpha_{r,-k}x^k}{k}+\frac{\alpha_{r,-k+1}x^{k-1}}{k-1}+\ldots+\alpha_{r,-1}x,$$

and

$$\phi_r(x)=\frac{\alpha_{r\,1}}{x}+\frac{\alpha_{r\,2}}{2x^2}+\ldots+\frac{\alpha_{r,s-1}}{(s-1)x^{s-1}}.$$

The undetermined coefficients $\alpha_{r,-k}, \ldots, \alpha_{r,s-1}$ and the arbitrary integer s will be selected presently.

Upon forming $Z_r(x)$ as given by (3), we find*

$$Z_r(x) = z_r x^{2k}\left[\frac{\zeta_{r,1}}{x} + \frac{\zeta_{r,2}}{x^2} + \ldots + \frac{\zeta_{r,s+k-1}}{x^{s+k-1}} + \frac{\zeta_{r,s+k}}{x^{s+k}} P_{1,r}(x)\right], \qquad r=1, 2,$$

in which $\zeta_{r,1}, \zeta_{r,2}, \ldots, \zeta_{r,s+k}$ are certain easily determined functions of $\alpha_{r,-k}, \ldots, \alpha_{r,s-1}$.

Now let us place

$$\zeta_{r,1} = \zeta_{r,2} = \ldots = \zeta_{r,s+k-1} = 0, \quad \zeta_{r,s+k} = (-1)^r (m_2 - m_1), \qquad r=1, 2,$$

and determine the coefficients† $\alpha_{r,-k}, \ldots, \alpha_{r,s-1}$ by means of this system of equations. Under our present hypotheses the equations can always be solved, and the functions z_1, z_2 thus determined will be unique. Let our notation be such that ‡

$$R[f_1(x)] \leqq R[f_2(x)],$$

when x is sufficiently large.

Put

$$\alpha_{1,0} + \alpha_{2,0} = \alpha_0, \quad \alpha_{r,0} - k = \rho_r, \quad f_1(x) + f_2(x) = f(x).$$

Then we may write, by (4), (2), (6) and (5), respectively,

$$q(x, x_1) = \sum_{r=1}^{2} e^{f_r(x) - f(x)} x^{\alpha_{r,0} - \alpha_0} P_{2,r}(x) e^{-f_r(x_1)} x_1^{-\rho_r - s} P_{1,r}(x_1),$$

$$Q(x) = (m_1 - m_2) e^{-f(x)} x^{-\alpha_0 + k} P_1(x),$$

$$g_r(x) = \frac{C_r(-1)^r}{m_1 - m_2} e^{f_r(x)} x^{\rho_r} P_{3,r}(x), \qquad r=1, 2,$$

$$K(x, x_1) = \sum_{r=1}^{2} e^{f_r(x)} x^{\rho_r} P_{3,r}(x) e^{-f_r(x_1)} x_1^{-\rho_r - s} P_{1,r}(x_1).$$

An arbitrarily large integer p having been fixed, let us choose s so large that

$$s > p + 2 + 2|\alpha_{1,0}| + 2|\alpha_{2,0}|;$$

also take $C_r = (-1)^r (m_1 - m_2)$ and form the function $v_{1,\lambda}(x)$ occurring in Theorem B. Then the function y_1 of that theorem takes the form

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*Throughout the present work the symbol $P(x)$ (generally written with subscripts) denotes a function expressible in the form

$$P(x) = 1 + \frac{A_1}{x} + \ldots + \frac{A_p + \epsilon_p(x)}{x^p}, \quad \lim_{x=\infty} \epsilon_p(x) = 0,$$

p being an arbitrary integer.

† As soon as $\alpha_{1,0}, \alpha_{2,0}$ are determined, s is selected to fulfil a condition pointed out later.

‡ Throughout the work, $R[x]$ denotes the real part of x.

$$y_1 = e^{f_1(x)} x^{\rho_1} P_{3,r}(x) + \sum_{\lambda=1}^{\infty} v_{1,\lambda}(x).$$

Upon recalling the way in which s was chosen, together with the fact that

$$R[f_1(x) - f_2(x)] \leqq 0,$$

it appears that the function $|K(x, x_1) g_1(x_1)|$ is a monotonically decreasing function of x_1, whence

$$\int_{x_{\lambda-1}}^{\infty} |K(x_{\lambda-1}, x_\lambda) g_1(x_\lambda)| \, dx_\lambda < |K(x_{\lambda-1}, x_{\lambda-1}) g_1(x_{\lambda-1})| < 4 \, |e^{f_1(x_{\lambda-1})} x_{\lambda-1}^{\rho_1 - (p+2)}|.$$

Further,

$$\int_{x_{\lambda-2}}^{\infty} \int_{x_{\lambda-1}}^{\infty} |K(x_{\lambda-2}, x_{\lambda-1})| \, |K(x_{\lambda-1}, x_\lambda) g_1(x_\lambda)| \, dx_\lambda dx_{\lambda-1} < 4^2 \, |e^{f_1(x_{\lambda-2})} x_{\lambda-2}^{\rho_1 - 2(p+2)}|.$$

Proceeding in this way, we find that

$$|v_{1,\lambda}(x)| < \frac{4^\lambda}{(p+1)^\lambda x^{\lambda(p+1)}} \, |e^{f_1(x)} x^{\rho_1}|,$$

and therefore that

$$\left| \sum_{\lambda=1}^{\infty} [v_{1,\lambda}(x)] \right| < \frac{8}{p+1} \cdot \frac{1}{x^{p+1}} \left| e^{f_1(x)} x^{\rho_1} \right|, \tag{17}$$

when x is sufficiently large. This inequality being established, it follows very readily that all the conditions of Theorem B are satisfied, and hence the function y_1 is an actual solution of (15).

As a consequence of (17), we may write

$$y_1 = e^{f_1(x)} x^{\rho_1} P_{4,1}(x), \tag{18}$$

and therefore

$$y_1 \sim e^{f_1(x)} x^{\rho_1} \left[1 + \frac{A_{1,1}}{x} + \dots \right]. \tag{19}$$

It is not difficult to show that y_1' and y_1'' also possess asymptotic developments, which may of course be found by differentiating (19).

Now if $R[f_2(x)] = R[f_1(x)]$, we may write at once by the same argument

$$y_2 = e^{f_2(x)} x^{\rho_2} P_{4,2}(x). \tag{20}$$

If, however, $R[f_1(x)] < R[f_2(x)]$, the argument fails, since some of the improper integrals in $v_{2,\lambda}(x)$ cease to have a meaning. To obtain a solution in this case, it will be convenient to revise our system of auxiliary functions, and then apply Theorem A.

In the adjoint equation

$$z'' - (az)' + bz = 0, \tag{21}$$

corresponding to (15), the coefficients satisfy all the hypotheses of this section. Thus (21) has a solution $\bar{z}_2$ of the form (18), which is found by direct computation to be

$$\bar{z}_2 = e^{-f_2(x)} x^{-a_{2,0}} P_{5,2}(x).$$

In the system (16) let us replace z_2 by this function $\bar{z}_2$, thus making $Z_2(x) \equiv 0$. With this change we find that

$$K(x, x_1) = e^{f_1(x)} x^{\rho_1} P_{6,1}(x) e^{-f_1(x_1)} x_1^{-\rho_1 - s} P_{1,1}(x_1),$$

while $g_2(x)$ is in essentials unchanged.

Now form the function $u_{2,\lambda}(x)$ of Theorem A. Then by that theorem we find

$$y_2 = e^{f_2(x)} x^{\rho_2} P_{4,2}(x) + \sum_{\lambda=1}^{\infty} u_{2,\lambda}(x).$$

Now upon recalling that

$$R[f_1(x) - f_2(x)] < 0,$$

we see that, if a be chosen sufficiently large,

$$\left| \sum_{\lambda=1}^{\infty} [u_{2,\lambda}(x)] \right| < \frac{4}{p+1} \cdot \frac{1}{x^{p+1}} \left| e^{f_2(x)} x^{\rho_2} \right|,$$

whence both the conditions of Theorem A are satisfied, and y_2 is a solution of (15). It may evidently be expressed in the form (20).

2. *Equal Roots.*

To consider the case $m_1 = m_2$ it will be convenient to reduce the differential equation to the form

$$y'' + b(x) y = 0. \tag{22}$$

Now if $m_1 = m_2$, we must have either

$$b(x) \sim x^{2k} \left[\frac{b_1}{x} + \frac{b_2}{x^2} + \dots \right], \qquad b_1 \neq 0, \tag{23}$$

or

$$b(x) \sim \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots. \tag{24}$$

When $b(x)$ has the form (23), we may reduce the problem to the case of distinct roots by merely introducing* $t = x^{\frac{1}{2}}$ as a new independent variable.

It remains only to study the case in which (24) holds. Here the point at infinity is a regular singular point† of the differential equation, so that the problem is closely related to that solved by Bôcher.† Hence only the briefest of discussions is necessary.

*Cf. Kneser, *loc. cit.* (third paper), p. 275.

†Cf. Bôcher, *loc. cit.*

Let us choose two auxiliary functions

$$z_1 = x^{\rho_1}\left[1+\frac{\alpha_1}{x}+\dots\dots+\frac{\alpha_{s-1}}{x^{s-1}}\right],$$

$$z_2 = x^{\rho_2}\left[1+\frac{\beta_1}{x}+\dots\dots+\frac{\beta_{s-1}}{x^{s-1}}\right]+Bz_1 \log x,$$

and determine the constants by the use of $Z_1(x)$ and $Z_2(x)$, as in the case of distinct roots. We find $B \neq 0$ whenever $\rho_2=\rho_1$, and, in general, also when $\rho_2-\rho_1$ is a positive integer, but otherwise $B=0$. By use of Theorem B we may obtain the asymptotic forms of two independent solutions y_1, y_2 of (22). The developments are of course *formally* identical with those obtained when the point at infinity is a regular point in the sense of the Fuchs theory.

The results for the equation of second order may be summarized in

THEOREM I: *In the differential equation*

$$y''+b(x)y=0, \tag{22}$$

suppose that $b(x)$ *is a real or complex function developable asymptotically, for large real positive values of* x, *in the form*

$$b(x) \sim x^{2k}\left[b_0+\frac{b_1}{x}+\dots\dots\right],$$

where k *is* 0 *or a positive integer. Then, for the same values of* x, *equation* (22) *possesses two linearly independent solutions* y_1, y_2 *such that*

(a) *if* $b_0 \neq 0$, *i. e., if the roots* m_1, m_2 *of the characteristic equation* $m^2+b_0=0$ *are distinct, we may write*

$$y_r \sim e^{f_r(x)}x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\dots\dots\right], \qquad r=1,2,$$

where

$$f_r(x)=\frac{m_r x^{k+1}}{k+1}+\frac{\alpha_{r,-k}x^k}{k}+\dots\dots+\alpha_{r,-1}x;$$

(b) *if* $b_0=0$, $b_1 \neq 0$, *we may write*

$$y_r \sim e^{f_r(x)}x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\dots\dots+\frac{1}{x^{\frac{1}{2}}}\left(B_{r,0}+\frac{B_{r,1}}{x}+\dots\dots\right)\right], \quad r=1,2,$$

where

$$f_r(x)=\frac{\alpha_{r,-2k-1}x^{k+\frac{1}{2}}}{k+\frac{1}{2}}+\frac{\alpha_{r,-2k}x^k}{k}+\dots\dots+\frac{\alpha_{r,-1}x^{\frac{1}{2}}}{\frac{1}{2}};$$

(c) *if* $k=b_0=b_1=0$, *we may write, in general,*

$$y_r \sim x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\dots\dots\right], \qquad r=1,2;$$

(d) *but if* $\rho_2=\rho_1$ *or, in general, if* $\rho_2-\rho_1$ *is a positive integer, we have*

$$y_1 \sim x^{\rho_1}\left[1+\frac{A_{1,1}}{x}+\ldots.\right],$$

$$y_2 \sim y_1 \log x + x^{\rho_2}\left[A_{2,0}+\frac{A_{2,1}}{x}+\ldots.\right].$$

III. The Equation of Third Order.

Take for consideration a differential equation of the third order, which we shall suppose reduced to the form

$$y'''+b(x)y'+c(x)y=0. \tag{25}$$

Let $b(x)$ and $c(x)$ be developable, when x is large, in the form

$$b(x) \sim x^{2k}\left[b_0+\frac{b_1}{x}+\ldots.\right],$$

$$c(x) \sim x^{3k}\left[c_0+\frac{c_1}{x}+\ldots.\right],$$

and suppose that $b'(x)$ also has an asymptotic development.

If the roots m_1, m_2, m_3 of the characteristic equation

$$m^3+b_0m+c_0=0$$

are distinct, the asymptotic solutions are well-known.* We therefore pass at once to the cases of multiple roots.

1. *A Simple and a Double Root.*

Suppose first that two of the roots are equal—say $m_1 \neq m_2 = m_3$.

Let us select three auxiliary functions

$$z = e^{-f_r(x)+\phi_r(x)} x^{-\alpha_{r,0}}, \qquad r=1, 2, 3,$$

in which

$$f_r(x)=\frac{m_r x^{k+1}}{k+1}+\frac{\alpha_{r,-2k-1}x^{k+\frac{1}{2}}}{k+\frac{1}{2}}+\frac{\alpha_{r,-2k}x^k}{k}+\ldots.+\frac{\alpha_{r,-1}x^{\frac{1}{2}}}{\frac{1}{2}},$$

$$\phi_r(x)=\frac{\alpha_{r,1}}{\frac{1}{2}x^{\frac{1}{2}}}+\frac{\alpha_{r,2}}{x}+\ldots.+\frac{\alpha_{r,2s-2}}{(s-1)x^{s-1}}.$$

Upon forming $Z_r(x)$, we find that

$$Z_r(x)=z_r x^{3k}\left[\frac{\zeta_{r,1}}{x^{\frac{1}{2}}}+\frac{\zeta_{r,2}}{x}+\ldots.+\frac{\zeta_{r,2s+2k-1}}{x^{s+k-\frac{1}{2}}}+\frac{\zeta_{r,2s+2k}}{x^{s+k}}P_{1,r}(x^{\frac{1}{2}})\right], \qquad r=1, 2, 3,$$

$\zeta_{r,1}, \ldots, \zeta_{r,2s+2k}$ being certain easily formed functions of the undetermined constants in $f_r(x)$ and $\phi_r(x)$.

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* Cf. Horn, *Journal für Mathematik*, Vol. CXXXVIII (1910), pp. 159–191. The results are easily obtainable by the present theory.

Let us try to determine these constants by placing

$$\zeta_{r,1}=\ldots\ldots=\zeta_{r,2s+2k-1}=0,\quad \zeta_{r,2s+2k}=\theta_r, \tag{27}$$

the constants θ_r being for the present unspecified. Inspection of the equations thus formed shows that the function z_1 corresponding to the simple root m_1 is always determinate, and that the same is true of one of the other auxiliary functions, say z_3. In determining z_2 by (27), difficulty may arise. Suppose that the first h coefficients of z_2, as given by (27), coincide with the corresponding ones in z_3, h being 0 or some positive integer. If $h\leqq 2k$, z_2 may be determined by (27). If $h=2k+1$, there is no difficulty unless the difference $\alpha_{3,0}-\alpha_{2,0}$ is an integer, in which case the equations in general become illusory. Finally, if $h>2k+1$, *i. e.*, $\alpha_{3,0}=\alpha_{2,0}$, z_2 coincides with z_3 throughout.

We consider first the ordinary case in which equations (27) serve to determine three definite, distinct functions of the form (26).

Place $f_1(x)+f_2(x)+f_3(x)=f(x)$,

$$\delta_1=\alpha_{3,-2k-1+h}-\alpha_{2,-2k-1+h},$$
$$\delta_2=-\delta_3=m_1-m_2,$$
$$\delta=\delta_1\delta_2\delta_3,$$
$$\theta_r=\frac{\delta}{\delta_r},\qquad r=1,2,3,$$
$$\beta_1=0,\quad \beta_2=\beta_3=\frac{h+1}{2},$$
$$\rho_r=\alpha_{r,0}-2k+\beta_r.$$

Then we get

$$Q(x)=\delta e^{-f(x)}x^{-\alpha_0+3k-\frac{h+1}{2}}P_1(x^{\frac{1}{2}}),$$
$$g_r(x)=C_r\frac{\delta_r}{\delta}e^{f_r(x)}x^{\rho_r}P_{2,r}(x^{\frac{1}{2}}),\qquad r=1,2,3,$$

$$K(x,x_1)=\sum_{r=1}^{3}e^{f_r(x)}x^{\rho_r}P_{2,r}(x^{\frac{1}{2}})e^{-f_r(x_1)}x_1^{\beta_r-\rho_r-s}P_{1,r}(x_1^{\frac{1}{2}}). \tag{28}$$

Consider first the case in which the functions $f_r(x)$ have distinct real parts. Suppose for definiteness that

$$R[f_1(x)]<R[f_2(x)]<R[f_3(x)]. \tag{29}$$

Take $C_r=\theta_r$, and form $v_{1,\lambda}(x)$. Then by Theorem B we have

$$y_1=e^{f_1(x)}x^{\rho_1}P_{2,r}(x^{\frac{1}{2}})+\sum_{\lambda=1}^{\infty}v_{1,\lambda}(x). \tag{30}$$

In $v_{1,\lambda}(x)$, we have

$$x_\lambda\geqq x_{\lambda-1}\geqq\ldots\ldots\geqq x_1\geqq x.$$

Further, by (29),

$$R[f_1(x)-f_r(x)]<0, \qquad r=2, 3.$$

Whence, if we choose

$$s>p+2+h+1+\sum_{r=1}^{3}|R[\rho_r]|,$$

p being, as usual, an arbitrary integer, it appears by argument like that used in Sec. II that

$$|v_{1,\lambda}(x)|<\frac{2\cdot 3^\lambda}{(p+1)^\lambda x^{\lambda(p+1)}}|e^{f_1(x)}x^{\rho_1}|.$$

Thus

$$\left|\sum_{\lambda=1}^{\infty}[v_{1,\lambda}(x)]\right|<\frac{4\cdot 3}{p+1}\cdot\frac{1}{x^{p+1}}\left|e^{f_1(x)}x^{\rho_1}\right|.$$

This inequality once established, it is easy to show that y_1 satisfies the conditions of Theorem B, and is thus a solution of (25). Further, we may evidently write

$$y_1=e^{f_1(x)}x^{\rho_1}P_{3,1}(x^{\frac{1}{3}}). \tag{31}$$

To obtain a second solution, let us employ a device similar to that used in Sec. II. The adjoint equation

$$z'''+(bz)'-cz=0 \tag{32}$$

may be shown by (31) to have a solution $\bar{z}_3$ expressible in the form

$$\bar{z}_3=e^{-f_3(x)}x^{-a_{3,0}}P_{4,3}(x^{\frac{1}{3}}).$$

We shall replace the function z_3 by this $\bar{z}_3$, thus making $Z_3(x)\equiv 0$, so that in $K(x, x_1)$ r takes only the values 1, 2.

Form $u_{3,\lambda}(x)$. Then by Theorem A

$$y_3=e^{f_3(x)}x^{\rho_3}P_{5,3}(x^{\frac{1}{3}})+\sum_{\lambda=1}^{\infty}u_{3,\lambda}(x).$$

Now we have in the present instance

$$x_\lambda\leqq x_{\lambda-1}\leqq\ldots\leqq x_1\leqq x,$$

and

$$R[f_3(x)-f_r(x)]>0, \qquad r=1, 2,$$

so that

$$|u_{3,\lambda}(x)|<\frac{2\cdot 2^\lambda}{(p+1)^\lambda x^{\lambda(p+1)}}|e^{f_3(x)}x^{\rho_3}|,$$

and

$$\left|\sum_{\lambda=1}^{\infty}[u_{3,\lambda}(x)]\right|<\frac{2\cdot 4}{p+1}\cdot\frac{1}{x^{p+1}}\left|e^{f_3(x)}x^{\rho_3}\right|,$$

whence y_3 is a solution of (25). It may be put in the form

$$y_3=e^{f_3(x)}x^{\rho_3}P_{3,3}(x^{\frac{1}{k}}).$$

We see now that (32) has a solution

$$\bar{z}_1=e^{-f_1(x)}x^{-\alpha_{1,0}}P_{4,1}(x^{\frac{1}{k}}).$$

Take this as an auxiliary function instead of z_1, so that $Z_1(x)\equiv 0$, and $K(x, x_1)$ reduces to the form

$$K(x, x_1)=e^{f_2(x)}x^{\rho_2}P_{6,2}(x^{\frac{1}{k}})e^{-f_2(x_1)}x_1^{\beta_2-\rho_2-s}P_{7,2}(x_1)^{\frac{1}{k}}.$$

Upon writing out y_2 by Theorem B, we find by argument now familiar that

$$y_2=e^{f_2(x)}x^{\rho_2}P_{3,2}(x^{\frac{1}{k}}).$$

This evidently disposes of the case in which the functions $f_r(x)$ have distinct real parts. Only very slight changes in the argument are needed if two real parts are equal, while if all three are equal the three solutions may be written down by Theorem B at once.

By direct substitution in (25), it appears that all the terms in y_1 involving fractional powers of x disappear.

We return now to the exceptional cases noted above, in which equations (27) do not serve to determine three functions of the form (26).

Suppose equations (27) become illusory, in which case $\alpha_{3,0}-\alpha_{2,0}$ must be an integer. Take

$$\left.\begin{aligned}
z_1&=e^{-f_1(x)+\phi_1(x)}x^{-\alpha_{1,0}},\\
z_2&=e^{-f_2(x)+\phi_2(x)}x^{-\alpha_{2,0}}+\beta z_3\log x, \qquad \beta\neq 0,\\
z_3&=e^{-f_2(x)+\phi_3(x)}x^{-\alpha_{3,0}},
\end{aligned}\right\} \tag{33}$$

in which

$$f_r(x)=\frac{m_r x^{k+1}}{k+1}+\frac{\alpha_{r,-k}x^k}{k}+\ldots\ldots+\alpha_{r,-1}x, \qquad r=1, 2,$$

$$\phi_r(x)=\frac{\alpha_{r,1}}{x}+\frac{\alpha_{r,2}}{2x^2}+\ldots\ldots+\frac{\alpha_{r,s-1}}{(s-1)x^{s-1}}, \qquad r=1, 2, 3.$$

The constants can now be determined without trouble in the usual way. Placing

$$\begin{aligned}
&\delta_1=\alpha_{3,0}-\alpha_{2,0},\\
&\delta_2=-\delta_3=m_1-m_2,\\
&\delta\ =\delta_1\delta_2\delta_3,\\
&\rho_1=\alpha_{1,0}-2k, \ \rho_r=\alpha_{r,0}-k+1, \qquad r=2, 3,
\end{aligned}$$

we find

$$\left.\begin{aligned}
g_r(x)&=C_r\frac{\delta_r}{\delta}e^{f_r(x)}x^{\rho_r}P_{8,r}(x), \qquad r=1, 2,\\
g_3(x)&=C_3\frac{\delta_3}{\delta}e^{f_2(x)}x^{\rho_3}P_{8,3}(x)+\beta g_2(x)\ \log x,
\end{aligned}\right\} \tag{34}$$

$$Z_1(x)=e^{-f_1(x)}x^{-\alpha_{1,0}+2k-s}P_{9,1}(x),$$
$$Z_2(x)=e^{-f_2(x)}x^{-\alpha_{2,0}+2k-s}P_{9,2}(x)+\beta z_3\log x,$$
$$Z_3(x)=e^{-f_3(x)}x^{-\alpha_{3,0}+2k-s}P_{9,3}(x),$$
$$K(x,x_1)=\sum_{r=1}^{3}g_r(x)Z_r(x_1).$$

By familiar reasoning it may be shown that (25) has three solutions of the form

$$\left.\begin{aligned} y_r&=e^{f_r(x)}x^{\rho_r}P_{10,r}(x), \qquad r=1,2,\\ y_3&=y_2\log x+Be^{f_2(x)}x^{\rho_3}P_{10,3}(x), \end{aligned}\right\} \tag{35}$$

where B is a constant.

There remains only the case in which the functions z_2 and z_3, as given by (27), are identical.

Take the auxiliary functions as in (33). We find that now $\alpha_{3,0}=\alpha_{2,0}$, $\alpha_{3,1}\neq\alpha_{2,1}$. If we place

$$\delta_1=\alpha_{3,1}-\alpha_{2,1},$$
$$\delta_2=-\delta_3=m_1-m_2,$$
$$\delta=\delta_1\delta_2\delta_3,$$

the functions $g_r(x)$ have the same form as in (34), and the three integrals y_r take the form (35), except that now $\rho_3=\rho_2$.

2. *A Triple Root.*

If $m_1=m_2=m_3$, we have either

$$\text{(a)}\ \begin{cases} b(x)\sim x^{2k}\left[\dfrac{b_1}{x}+\dfrac{b_2}{x^2}+\dots\right],\\ c(x)\sim x^{3k}\left[\dfrac{c_1}{x}+\dfrac{c_2}{x_2}+\dots\right],\end{cases}$$

where b_1, c_1, c_2 are not all 0; or else

$$\text{(b)}\ \begin{cases} b(x)\sim \dfrac{b_2}{x^2}+\dfrac{b_3}{x^3}+\dots,\\ c(x)\sim \dfrac{c_3}{x^3}+\dfrac{c_4}{x^4}+\dots.\end{cases}$$

In (a), if $c_1\neq0$ or if $b_1=c_1=0$, $c_2\neq0$, we need only introduce the new independent variable $t=x^{\frac{1}{3}}$ in order to reduce the problem to the case of distinct roots.

If $c_1=0$, $b_1\neq0$, a similar reduction results if the new variable $t=x^{\frac{1}{2}}$ be introduced. One of the solutions is found to proceed in powers of x, the other two in powers of $x^{\frac{1}{2}}$.

There remains only (b). This case may be disposed of by Theorem B. On account of the closely related discussion by Dunkel, mentioned above,* together with the formal analogy between this problem and that in which the point at infinity is a regular point in the ordinary sense, we content ourselves with a mere statement of results, in the theorem below.

In summary, our results for the equation of the third order are as follows:†

THEOREM II: *In the differential equation*

$$y'''+b(x)y'+c(x)y=0, \tag{25}$$

suppose that $b(x)$ *and* $c(x)$ *are real or complex functions developable asymptotically, when* x *is large, real and positive, in the form*

$$b(x)\sim x^{2k}\left[b_0+\frac{b_1}{x}+\dots\right],$$

$$c(x)\sim x^{3k}\left[c_0+\frac{c_1}{x}+\dots\right],$$

where k *is* 0 *or a positive integer, and suppose that* $b'(x)$ *also has an asymptotic development. Then for the same values of* x *equation* (25) *has three linearly independent solutions* y_1, y_2, y_3 *possessing asymptotic developments as follows:*

(a) *If the roots* m_1, m_2, m_3 *of the characteristic equation*

$$m^3+b_0m+c_0=0$$

are distinct, we may write

$$y_r\sim e^{f_r(x)}x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\dots\right], \qquad r=1,2,3,$$

where

$$f_r(x)=\frac{m_rx^{k+1}}{k+1}+\frac{\alpha_{r,-k}x^k}{k}+\dots+\alpha_{r,-1}x.$$

(b) *If* $m_1\neq m_2=m_3$, *we may write in general*

$$y_1\sim e^{f_1(x)}x^{\rho_1}\left[1+\frac{A_{1,1}}{x}+\dots\right],$$

$$y_r\sim e^{f_r(x)}x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\dots+\frac{1}{x^{\frac{1}{2}}}\left(B_{r,0}+\frac{B_{r,1}}{x}+\dots\right)\right], \qquad r=2,3,$$

where $f_1(x)$ *has the same form as in* (a) *and*

$$f_r(x)=\frac{m_rx^{k+1}}{k+1}+\frac{\alpha_{r,-2k-1}x^{k+\frac{1}{2}}}{k+\frac{1}{2}}+\frac{\alpha_{r,-2k}x^k}{k}+\dots+\frac{\alpha_{r,-1}x^{\frac{1}{2}}}{\frac{1}{2}}, \quad r=2,3;$$

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*Cf. Dunkel, *loc. cit.*

†The results for the case of distinct roots are included merely for completeness.

(c) *but if* $\rho_3=\rho_2$, *or in general if* $\rho_3-\rho_2$ *is a positive integer, we have*

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\ldots\right], \qquad r=1,\,2,$$

$$y_3 \sim y_2 \log x + e^{f_2(x)} x^{\rho_3}\left[A_{3,0}+\frac{A_{3,1}}{x}+\ldots\right],$$

where $f_1(x)$ *and* $f_2(x)$ *have the same form as in* (a).

(d) *If* $m_1=m_2=m_3$ *and either* $c_1 \neq 0$ *or* $b_1=c_1=0$, $c_2 \neq 0$, *we may write*

$$y_r \sim e^{f_r(x)} x^{\rho_r}\left[1+\frac{A_{r,1}}{x}+\ldots+\frac{1}{x^{\frac{1}{3}}}\left(B_{r,0}\frac{B_{r,1}}{x}+\ldots\right)\right.$$
$$\left.+\frac{1}{x^{\frac{2}{3}}}\left(C_{r,0}+\frac{C_{r\,1}}{x}+\ldots\right)\right], \quad r=1,2,3,$$

where

$$f_r(x)=\frac{m_r x^{k+1}}{k+1}+\frac{\alpha_{r,-3k-2}x^{k+\frac{2}{3}}}{k+\frac{2}{3}}+\frac{\alpha_{r\,-3k-1}x^{k+\frac{1}{3}}}{k+\frac{1}{3}}+\ldots+\frac{\alpha_{r,-1}x^{\frac{1}{3}}}{\frac{1}{3}};$$

(e) *if* $c_1=0$, $b_1 \neq 0$, y_1, y_2, y_3 *have expansions of the same form as in* (b);

(f) *if* $k=b_1=c_1=c_2=0$, *we may write*

$$y_1 \sim x^{\rho_1}\left[1+\frac{A_{1,1}}{x}+\ldots\right],$$

$$y_2 \sim Ay_1 \log x + x^{\rho_2}\left[A_{2,0}+\frac{A_{2,1}}{x}+\ldots\right],$$

$$y_2 \sim By_1 \log^2 x + x^{\rho_2}\log x\left[B_{3,0}+\frac{B_{3,1}}{x}+\ldots\right]$$
$$+x^{\rho_3}\left[A_{3,0}+\frac{A_{3,1}}{x}+\ldots\right].$$

This evidently covers all cases that may arise in connection with the equation of the third order.

UNIVERSITY OF MICHIGAN, *April,* 1913.

Restricted Systems of Equations. *

By Arthur B. Coble.

Throughout the following account geometric ideas and language are employed exclusively, though the material is algebraic. A homogeneous equation of degree l in $n+1$ variables is called a spread of order l in S_n. A manifold of dimension r is indicated by M_r. According to Kronecker† it can always be defined as the totality of points on at most $n+1$ spreads. The simplest problem in the theory of restricted systems of equations is the following: Given n spreads in S_n of orders $\lambda_1, \ldots, \lambda_n$, all of which contain M_r, in how many points of S_n outside of M_r do they meet? A very wide range of problems in enumerative geometry and algebra can be expressed in terms of restricted systems of equations. With this in mind, Salmon‡ began to develop a theory of such equations. His method was inductive, but the general case was not touched. Furthermore, the results which he obtained were proved only for the case where the manifold M_r is the complete intersection of $n-r$ spreads in S_n, a manifold of a type referred to hereafter as "regular." The examples given at the end of § 2 show that such results may or may not be true for the general case. That no systematic development of the subject of restricted systems has been attempted is due, no doubt, to the success of Schubert § and others in the field of enumerative geometry with the aid of the principle of the conservation of number and a certain symbolic calculus.

The object of this paper—the first of a series under the same title—is to give a general account of the theory of restricted systems of equations. The simple problem formulated above is considered in § 1. A solution is obtained in terms of the orders λ, and of $r+1$ so-called "index numbers of M_r in S_n" which depend only on the M_r in S_n. Certain theorems relating to the determination of these index numbers are derived, and applications to specific problems are made.

* Written under the auspices of the Carnegie Institution of Washington, D. C.

† *Crelle*, 92.

‡ "Lessons on Modern Higher Algebra." Cf. E. Lasker, "Zur Theorie der Moduln und Ideale," *Matematische Annalen*, Vol. LX (1905), in particular p. 44 and p. 112.

§ "Kalkül der abzählenden Geometrie," Leipzig (1879); cf. Pascal-Schepp, *Repertorium*, Vol. II, Chap. XV (1902).

In § 2 there is treated the more general problem in which S_n is replaced by a manifold M_n in a linear space of any dimension greater than n. For this case the $r+1$ "relative index numbers of M_r as to M_n" are introduced. It is shown that they have properties entirely analogous to those of the ordinary index numbers. Some relations connecting the two kinds of index numbers are also given.

Further problems are readily suggested, and these will be discussed in later papers. Possibly the most important is that of the "incomplete restricted system," namely: Given $n-r+k$ spreads of orders $\lambda_1, \ldots, \lambda_{n-r+k}$ on M_r in S_n which meet in a residual M_{r-k} which has an M_{r-k-1} in common with M_r; what are the index numbers of M_{r-k} and of M_{r-k-1} in S_n, and what are the relative index numbers of M_{r-k-1} as to M_r and as to M_{r-k}? For $k=r$ we have again the problem of § 1. An obvious generalization is the problem of the "incomplete relative restricted system." It appears that if $r \geqq 2+k$, the ordinary index numbers of M_r are not sufficient for the solution and further index numbers must be introduced. Closely related to the above inquiry is that as to the index numbers of a composite manifold in terms of those of its constituent manifolds.

Beginning with the fact that k manifolds in S_n of orders $\lambda_1, \ldots, \lambda_k$ and dimensions $r_1, \ldots, r_k$, where $\sum_i r_i = n(k-1)$, ordinarily meet in $\prod_{i=1}^{k} \lambda_i$ points, we may ask what is the reduction in this number due to the fact that the manifolds all contain M_r, $r < r_i$. This question can be extended in the same directions as the one originally put, and doubtless leads to the most general one in the subject.

Some attention will be devoted to the determination of the index numbers of certain spreads, such as those defined by matrices and those which occur in mapping.

§ 1. *The Index Numbers of a Spread in a Linear Space.*

1. Let M_r be a manifold of dimension r in a space S_n whose section by an arbitrary $S_{n-(r-k)}$, $0 \leqq k \leqq r$, is M_k. In $S_{n-(r-k)}$ let M_k be on $n-(r-k)$ spreads of dimension $n-(r-k)-1$ and of orders $\lambda_1, \lambda_2, \ldots, \lambda_{n-(r-k)}$ which meet further in a finite number, O_k, of points not on M_k. If σ_j be the sum of the products of the λ's, j at a time, then *the* $(k+1)$*-th index number of* M_r *in* S_n is defined by the formula

$$(1) \quad \alpha_k = \sigma_{n-(r-k)} - \alpha_0 \sigma_k - \alpha_1 \sigma_{k-1} - \alpha_2 \sigma_{k-2} - \ldots - \alpha_{k-1} \sigma_1 - O_k,$$

in terms of the orders λ, the number O_k, and the earlier index numbers α_0, α_1,

$\ldots, \alpha_{k-1}$, which are similarly defined by means of sections of M_r in S_{n-r}, $S_{n-r+1}, \ldots, S_{n-(r-k+1)}$. In particular, α_0 is the *order* of M_r. The *last* index number, α_r, is defined by n spreads of orders λ_i in S_n and this particular form of (1):

$$(2) \qquad \alpha_r = \sigma_n - \alpha_0\sigma_r - \alpha_1\sigma_{r-1} - \alpha_2\sigma_{r-2} - \ldots - \alpha_{r-1}\sigma_1 - O_r;$$

and it is convenient to say that

(3) *The $r+1$ index numbers of M_r in S_n are the last index numbers of M_r and its successive linear sections.*

For a given set of index numbers, $\alpha_0, \alpha_1, \ldots$, and a given set of orders, $\lambda_1, \ldots, \lambda_h$, we shall often use a symbol A_j defined by

$$(4) \qquad A_j = \alpha_0\sigma_j + \alpha_1\sigma_{j-1} + \alpha_2\sigma_{j-2} + \ldots + \alpha_{j-1}\sigma_1 + \alpha_j.$$

Then, if σ' and A' refer only to the orders $\lambda_2, \ldots, \lambda_h$, we have

$$(5) \qquad \sigma_j = \sigma'_j + \lambda_1\sigma'_{j-1}, \quad A_j = A'_j + \lambda_1 A'_{j-1}.$$

2. The index numbers defined above are independent of the orders of the spreads employed in the definition. This is certainly true of the first index number α_0, the order of M_r, and according to (3) needs to be proved only for the last index number α_r. Beginning with (2), which we write $O_r = \sigma_n - A_r$, let the order λ_1 of the spread f_1 be increased by unity by adding to f_1 an arbitrary linear spread L which cuts M_r in M_{r-1}. Then $O''_r = \sigma''_n - A''_r$, where the superscript refers to the new order λ_1+1 and the possible new index number α''_r. The points O''_r outside of M_r are made up of the points O_r and the points O'_{r-1} outside of M_{r-1} in L. But $O'_{r-1} = \sigma'_{n-1} - A'_{r-1}$, where the superscript refers to the orders $\lambda_2, \ldots, \lambda_n$ and the index numbers $\alpha_0, \ldots, \alpha_{r-1}$. Thus $O''_r = O_r + O'_{r-1} = \sigma_n - A_r + \sigma'_{n-1} - A'_{r-1} = \sigma''_n - A_r - A'_{r-1}$. Hence $A''_r = A_r + A'_{r-1}$, and we see from (5) that $\alpha''_r = \alpha_r$. By induction the index numbers are independent of any increase in any of the orders due to the addition of linear spreads. According to Schubert's principle of the conservation of number, they are independent of any possible increase. Since the original orders might have been the lowest possible ones, the index numbers are entirely independent of the orders. The application of Schubert's principle here is eminently proper. We may regard the spread M_r as defined by $f_1, \ldots, f_n$ and the exclusion of the O_r outside points. Then $f_1 L, \ldots, f_n$, with the exclusion of O''_r, define M_r in precisely the same way for our present purpose. To be sure, $f_1 L$ has M_{r-1} as a locus of double points at least while f_1 itself may have only simple points on M_{r-1}. But this does not affect the number of outside intersections. For in both cases $f_2, \ldots, f_n$ meet in M_r and a residual curve which cuts M_r in γ points.

To obtain the legitimate number of outside intersections, we have only to choose L so that it contains none of these γ points.

(6) *The index numbers of M_r in S_n do not depend upon the orders of the spreads used to define them. They depend only on M_r itself and the dimension in which M_r lies.*

The dependence upon the dimension is given below in (14).

3. From (6) and (2) we have at once that

(7) *The order of a restricted system of equations in S_n with the common solution M_r is given by the formula*

$$O_r = \sigma_n - \alpha_0 \sigma_r - \alpha_1 \sigma_{r-1} - \alpha_2 \sigma_{r-2} - \ldots\ldots - \alpha_{r-1} \sigma_1 - \alpha_r = \sigma_n - A_r,$$

in which $\alpha_0, \ldots, \alpha_r$ are the index numbers of M_r in S_n and the σ's refer to the orders of the given equations.

Thus if the order of one restricted system for M_r itself and for each of its successive sections is known and thereby the α's are determined, then the order of *any* restricted system involving M or any of its linear sections can be obtained. The first problem in the theory of restricted systems is the determination of the index numbers of given spreads. Some theorems relating to this will now be derived.

Let us call M_r in S_n a *complete manifold of excess* k if it is the complete intersection of $n-r+k$ $(0 \overline{<} k \overline{<} r)$ spreads in S_n. If, in particular, $k=0$, M_r will be called a *regular manifold*; an *ordinary manifold*, if it lies in an S_{r+1} in S_n.

4. Let then M_r be a complete manifold of excess k defined in S_n by spreads of orders $\lambda_1, \lambda_2, \ldots, \lambda_{n-r+k}$, of which λ_1 is the maximum. In (1), $O_k=0$, and α_k is determined in terms of the given orders and the earlier index numbers by the formula $0=\sigma_{n-r+k}-A_k$. In $S_{n-r+k+1}$ an additional spread is required to determine α_{k+1}. This can always be taken equal to λ_1 in such a way that $O_{k+1}=0$, and for sections of greater dimension further spreads of order λ_1 can be used. By repeated application of (4) we can write the equations defining the index numbers beyond α_{k-1} as follows, the σ's referring of course to the $n-r+k$ given orders:

$$\begin{aligned}
0 &= \sigma_{n-r+k} - [A_k], \\
0 &= \lambda_1 \sigma_{n-r+k} - [\lambda_1 A_k + A_{k+1}], \\
0 &= \lambda_1^2 \sigma_{n-r+k} - [\lambda_1^2 A_k + 2\lambda_1 A_{k+1} + A_{k+2}], \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
0 &= \lambda_1^j \sigma_{n-r+k} - [\lambda_1^j A_k + \binom{j}{1} \lambda_1^{j-1} A_{k+1} + \binom{j}{2} \lambda_1^{j-2} A_{k+2} + \ldots\ldots \\
&\qquad + \binom{j}{j-1} \lambda_1 A_{k+j-1} + \binom{j}{j} A_{k+j}].
\end{aligned}$$

Multiplying these equations in order, beginning with the last, by the terms in the development of $(1-\lambda_1)^j$ and using the formula

$$(8)\quad \binom{a}{0}\binom{a+b}{c}-\binom{a}{1}\binom{a+b-1}{c}+\binom{a}{2}\binom{a+b-2}{c}-\dots\\ +(-1)^{a+b-c}\binom{a}{a+b-c}\binom{c}{c}=\begin{cases}0 \text{ if } c<a,\\ 1 \text{ if } c=a,\end{cases}$$

we get $A_{k+j}=0$, if $j>0$. Hence the index numbers, beginning with α_k, are determined by the equations

$$(9)\qquad A_k=\sigma_{n-r+k},\quad A_{k+j}=0,\qquad j=1,2,\dots,r-k.$$

In the case of a regular manifold, $k=0$ and $A_0=\alpha_0=\sigma_{n-r}$ is the order d of the manifold. Then equations (9) read thus:

$$\begin{aligned}
d&=\alpha_0,\\
0&=\alpha_0\sigma_1+\alpha_1,\\
0&=\alpha_0\sigma_2+\alpha_1\sigma_1+\alpha_2,\\
&\dots\dots\dots\dots,\\
0&=\alpha_0\sigma_j+\alpha_1\sigma_{j-1}+\alpha_2\sigma_{j-2}+\dots+\alpha_{j-1}\sigma_1+\alpha_j.
\end{aligned}$$

Multiplying these in order, beginning with the last, by $1, -P_1, P_2, -P_3, \dots, (-1)^j P_j$, where the P's are the complete symmetric polynomials formed from the given orders, and recalling that

$$(10)\qquad \sigma_l-P_1\sigma_{l-1}+P_2\sigma_{l-2}-P_3\sigma_{l-3}+\dots+(-1)^l P_l=0,$$

we find that $\alpha_j=(-1)^j d\,P_j$.

(11) *The index numbers $\alpha_k, \dots, \alpha_r$ of the complete manifold M_r in S_n of excess k are determined in terms of the given orders and the earlier index numbers by the equations* (9); *those of a regular manifold are $\alpha_j=(-1)^j d\,P_j$, $j=0,\dots,r$, where d is the product, P_j the complete symmetric polynomial of order j, formed from the orders of the $n-r$ spreads in S_n which cut out M_r; those of an ordinary manifold M_{n-1} in S_n of order d are $\alpha_j=(-1)^j d^{j+1}$, $j=0,\dots,n-1$.*

In particular, if M_r is a linear manifold S in S_n, it is regular, being cut out by $n-r$ ordinary linear spreads, and P_j reduces to the number of terms in the complete symmetric polynomial of order j in $n-r$ variables, which is known to be $\binom{n-r+j-1}{j}$, whence

(12) *The index numbers of an S_r in S_n are*

$$\alpha_j=(-1)^j\binom{n-r+j-1}{j},\qquad j=0,1,\dots,r.$$

5. If M_r in S_n has index numbers α, then M_r in an S_{n+1} containing S_n has index numbers α', where

$$(13) \qquad \alpha'_j = \alpha_j - \alpha_{j-1} + \alpha_{j-2} - \alpha_{j-3} + \dots + (-1)^j \alpha_0 .$$

This is true of α'_0, the order of M_r; let us assume it for $\alpha'_1, \dots, \alpha'_{r-1}$ and prove it for α'_r. If in S_n α_r is determined from $O_r = \sigma_n - A_r$, in S_{n+1} α'_r is determined by an additional linear spread, the S_n itself, O_r being the same as before. Hence $O_r = 1 \cdot \sigma_n - A'_r$, where in A'_r $\sigma'_k = \sigma_k + \sigma_{k-1}$. Thus

$$A_r = A'_r = \alpha'_0(\sigma_r + \sigma_{r-1}) + \alpha'_1(\sigma_{r-1} + \sigma_{r-2}) + \dots + \alpha'_{r-1}(\sigma_1 + 1) + \alpha'_r ,$$

or

$$A_r = \alpha'_0 \sigma_r + (\alpha'_0 + \alpha'_1)\sigma_{r-1} + (\alpha'_1 + \alpha'_2)\sigma_{r-2} + \dots + (\alpha'_{r-2} + \alpha'_{r-1})\sigma_1 + (\alpha'_{r-1} + \alpha'_r).$$

Assuming (13) up to $j = r$, this determines α'_r, which is of the same form since $\alpha'_k + \alpha'_{k-1} = \alpha_k$. By repeated use of (13) we find that

(14) *If the index numbers of M_r in S_n are $\alpha_0, \dots, \alpha_r$, the index numbers of the same M_r in S_{n+p} are*

$$\alpha_j^{(p)} = \alpha_j - \binom{p}{1}\alpha_{j-1} + \binom{p+1}{2}\alpha_{j-2} - \binom{p+2}{3}\alpha_{j-3} + \dots + (-1)^j \binom{p+j-1}{j}\alpha_0 ,$$
$$j = 0, 1, \dots, r.$$

This, applied to an S_r in S_n, yields through (12) the relation

$$(15) \quad \binom{b}{0}\binom{a+c}{a} + \binom{b+1}{1}\binom{a+c-1}{a} + \binom{b+2}{2}\binom{a+c-2}{a} + \dots$$
$$+ \binom{b+c}{c}\binom{a}{a} = \binom{a+b+c+1}{c}.$$

6. Let $M_r(\alpha)$ in S_n be cut by the spread F of order q in $M_{r-1}(\alpha')$. Then

$$(16) \quad \begin{cases} \alpha'_0 = q\alpha_0; \text{ let us assume} \\ \alpha'_1 = q\alpha_1 - q^2\alpha_0 , \\ \alpha'_2 = q\alpha_2 - q^2\alpha_1 + q^3\alpha_0 , \\ \dots\dots\dots\dots\dots\dots , \\ \alpha'_{r-2} = q\alpha_{r-2} - q^2\alpha_{r-3} + q^3\alpha_{r-4} - \dots + (-1)^{r-2}q^{r-1}\alpha_0 ; \text{ and prove that} \\ \alpha'_{r-1} = q\alpha_{r-1} - q^2\alpha_{r-2} + q^3\alpha_{r-3} - \dots + (-1)^{r-1}q^r\alpha_0 . \end{cases}$$

Let spreads of orders $\lambda_1, \dots, \lambda_{n-1}$ on M_r meet outside M_r in a curve C whose order (from a linear section) is $O_{r-1} = \sigma_{n-1} - A_{r-1}$, the σ's being formed from $\lambda_1, \dots, \lambda_{n-1}$. These $n-1$ spreads together with F make up n spreads on M_{r-1} which meet again in O'_{r-1} points outside M_{r-1}, the points where F meets C. Hence $O'_{r-1} = q\, O_{r-1} = \sigma'_n - A'_{r-1}$, A'_{r-1} being formed from the α''s and σ''s, which in turn are formed from $\lambda_1, \dots, \lambda_{n-1}, q$. Hence $A'_{r-1} = q\, A_{r-1}$ and α'_{r-1} is determined by

$$\alpha'_0(\sigma_r+q\sigma_{r-1})+\alpha'_1(\sigma_{r-1}+q\sigma_{r-2})+\ldots.+\alpha'_{r-2}(\sigma_1+q)+\alpha'_{r-1}$$
$$=q(\alpha_0\sigma_r+\ldots.+\alpha_{r-1}),$$

or

$$\alpha'_0\sigma_r+(\alpha'_1+q\alpha'_0)\sigma_{r-1}+\ldots.+(\alpha'_{r-1}+q\alpha'_{r-2})=q(\alpha_0\sigma_r+\ldots.+\alpha_{r-1}).$$

Assuming (16) true throughout, this equation is satisfied since then $\alpha'_j+q\alpha'_{j-1}=q\alpha_j$, $j=0,1,\ldots.,r-1$. Hence

(17) *If $M_r(\alpha)$ in S_n is cut by a spread of order q in an M_{r-1}, the index numbers α' of M_{r-1} in S_n are given by* (16) *in terms of q and the α's.*

For $q=1$ we have again the formula (13). If M_r is a regular manifold, so also is M_{r-1}, and (16) merely expresses that $P'_k=q^k+q^{k-1}P_1+\ldots.+P_k$, where P'_i is formed from $\lambda_1,\ldots.,\lambda_{n-1}$, q and P_i from $\lambda_1,\ldots.,\lambda_{n-1}$ only.

7. The theorem (17) can be generalized as follows:

(18) *If $M_r(\alpha)$ and $M_s(\beta)$ in S_n meet in an $M_{r+s-n}(\gamma)$*, then $\gamma_k=\sum_{i=0}^{k}\alpha_i\beta_{k-i}$.

Let us prove this provisionally for the case where one of the manifolds, say $M_s(\beta)$, is regular, being determined by $n-s$ spreads of orders $p, q, \ldots.$. Then M_{r+s-n} is the meet of $M_r(\alpha)$ and these spreads in order, and the index numbers of the successive sections according to (17) are

$$\begin{array}{lll}
p\alpha_0, & pq\alpha_0, & \ldots., \\
p\alpha_1-p^2\alpha_0, & pq\alpha_1-pq(p+q)\alpha_0, & \ldots., \\
p\alpha_2-p^2\alpha_1+p^3\alpha_0, & pq\alpha_2-pq(p+q)\alpha_1+pq(p^2+q^2+pq)\alpha_0, & \ldots., \\
\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots.
\end{array}$$

Substituting the index numbers of $M_s(\beta)$ as given by (11), we have the desired formula. A more general statement is:

(19) *If in S_n b spreads $M_{r_1}(\alpha^{(1)})$, $M_{r_2}(\alpha^{(2)})$, $\ldots.$, $M_{r_b}(\alpha^{(b)})$ meet in an $M_{\Sigma r_i-n(b-1)}$ with index numbers β, then $\beta_k=\Sigma\alpha^{(1)}_{i_1}\alpha^{(2)}_{i_2}\ldots.\alpha^{(b)}_{i_b}$, where $i_1+i_2+\ldots.+i_b=k$.*

In (17), (18), (19) we have generalizations of (11) relating to regular manifolds. If in (19) the dimensions $r_1,\ldots.,r_b$ are all equal to $n-1$, $M(\beta)$ is a regular M_{n-b}.

8. Given $M_r(\alpha)$ in S_n; let α' be the index numbers of M_r doubled, *i. e.* $[M_r]^2$. A section of M_r by an S_{n-r} is α_0 points, which are doubled if M_r is doubled, and account for $2^{n-r}\alpha_0$ intersections, whence $\alpha'_0=2^{n-r}\alpha_0$. Let us assume that $\alpha'_k=2^{n-r+k}\alpha_k$ for $k=0,1,\ldots.,r-1$ and prove that $\alpha'_r=2^n\alpha_r$. Let M_r be determined to within certain outside points by $f_1,\ldots.,f_n$, of orders $\lambda_1,\ldots.,\lambda_n$, and also by $g_1,\ldots.,g_n$, of orders $\mu_1,\ldots.,\mu_n$. Then $[M_r]^2$ is determined by $f_1\cdot g_1,\ldots.,f_n\cdot g_n$ to within O'_r outside points, or

$$\alpha'_r=\prod_1^n(\lambda_i+\mu_i)-\alpha'_0\sigma_r(\lambda_i+\mu_i)-\alpha'_1\sigma_{r-1}(\lambda_i+\mu_i)-\ldots.-\alpha'_{r-1}\sigma_1(\lambda_i+\mu_i)-O'_r.$$

The points O'_r arise from any k spreads f and $n-k$ complementary spreads g, whence

$$O'_r = \Sigma\,[\lambda_1\,\lambda_2\ldots\ldots\lambda_k\,\mu_{k+1}\ldots\ldots\mu_n - \alpha_0\,\sigma_r - \alpha_1\,\sigma_{r-1} - \ldots\ldots - \alpha_{r-1}\,\sigma_1 - \alpha_r],$$

where Σ refers to the 2^n choices of $\lambda_1, \ldots\ldots, \lambda_k, \mu_{k+1}, \ldots\ldots, \mu_n$. Collecting the terms in O'_r, we find that

$$O'_r = \prod_1^n (\lambda_i + \mu_i) - 2^{n-r}\alpha_0\,\sigma_r\,(\lambda_i + \mu_i) - 2^{n-r+1}\alpha_1\,\sigma_{r-1}\,(\lambda_i + \mu_i) - \ldots\ldots$$
$$- 2^{n-1}\alpha_{r-1}\sigma_1\,(\lambda_i + \mu_i) - 2^n\alpha_r.$$

Substituting this value in α'_r and using the assumed values of $\alpha'_0, \ldots\ldots, \alpha'_{r-1}$, we find $\alpha'_r = 2^n\alpha_r$, whence

(20) *If the index numbers of M_r in S_n are α, the index numbers of $[M_r]^2$ are $\alpha'_j = 2^{n-r+j}\alpha_j$, $j = 0, 1, \ldots\ldots, r$.*

Using an additional set of spreads $h_1, \ldots\ldots, h_n$ on M_r of orders $\nu_1, \ldots\ldots, \nu_n$, and a similar argument, we find the index numbers of $[M_r]^3$, etc. The result is as follows:

(21) *The index numbers of an l-fold M_r in S_n in terms of those of the simple M_r are $\alpha'_j = l^{n-r+j}\alpha_j$.*

9. Let $M_r(\alpha)$ in S_n be contained simply on spreads of orders $\lambda_1, \ldots\ldots, \lambda_n$. We ask for the number O_r of points outside of M_r common to n spreads $f_1, \ldots\ldots, f_n$ of orders $l_1, \ldots\ldots, l_n$, which contain M_r $k_1, \ldots\ldots, k_n$ times respectively. Consider the n degenerate spreads,

$$u_1^{(1)} \cdot u_2^{(1)} \cdot \ldots\ldots \cdot u_{k_1}^{(1)} \cdot P^{(l_1 - k_1\lambda_1)}, \ldots\ldots, u_1^{(n)} \cdot u_2^{(n)} \cdot \ldots\ldots \cdot u_{k_n}^{(n)}\,P^{(l_n - k_n\lambda_n)},$$

where $u_i^{(h)}$ is a spread of order λ_h containing M_r simply, and $P^{(l_i - k_i\lambda_i)}$ is a spread of order $l_i - k_i\lambda_i$ in general position with regard to M_r. These degenerate spreads satisfy the conditions of the problem, and from them O_r can be calculated. We find that

$$O_r = \prod_1^n (l_i - k_i\lambda_i) + \Sigma\prod_1^{n-1} (l_i - k_i\lambda_i)\,k_n\lambda_n + \Sigma\prod_1^{n-2} (l_i - k_i\lambda_i)\,\lambda_{n-1}\lambda_n k_{n-1}k_n$$
$$+ \ldots\ldots + \Sigma\prod_1^{r} (l_i - k_i\lambda_i)\,(\lambda_{r+1}\lambda_{r+2}\ldots\ldots\lambda_n - \alpha_0)\,k_{r+1}k_{r+2}\ldots\ldots k_n$$
$$+ \Sigma\prod_1^{r-1} (l_i - k_i\lambda_i)\,(\lambda_r\lambda_{r+1}\ldots\ldots\lambda_n - \alpha_0\sigma_1 - \alpha_1)\,k_r k_{r+1}\ldots\ldots k_n$$
$$+ \Sigma\prod_1^{r-2} (l_i - k_i\lambda_i)\,(\lambda_{r-1}\lambda_r\ldots\ldots\lambda_n - \alpha_0\sigma_2 - \alpha_1\sigma_1 - \alpha_2)\,k_{r-1}k_r\ldots\ldots k_n$$
$$+ \ldots\ldots + (\lambda_1\lambda_2\ldots\ldots\lambda_n - \alpha_0\sigma_r - \alpha_1\sigma_{r-1}\ldots\ldots - \alpha_r)\,k_1k_2\ldots\ldots k_n.$$

To divide this by $k_1 \cdot k_2 \cdot \ldots\ldots \cdot k_n$ amounts to replacing l_i by $\dfrac{l_i}{k_i}$ and dropping the other k's. Then O_r takes the form $\overline{O}_r$, which is obtained from n spreads

$v^{(i)} \cdot P\left(\frac{l_i}{k_i} - \lambda_i\right)$, $v^{(i)}$ being of order λ_i, *i. e.*, the number $\overline{O}_r$ obtained from n spreads of orders $\frac{l_i}{k_i}$ containing M_r simply. Hence

(22) *The number of points not on $M_r(\alpha)$ in S_n and common to spreads $f_1, \ldots, f_n$ of orders $l_1, \ldots, l_n$, which contain M_r $k_1, \ldots, k_n$ times respectively, is given by the formula*

$$O_r = [\sigma_n - \alpha_0 \sigma_r - \alpha_1 \sigma_{r-1} - \alpha_2 \sigma_{-2} - \ldots - \alpha_r]\, k_1 k_2 \ldots k_n,$$

where the σ's refer to $\frac{l_1}{k_1}, \ldots, \frac{l_n}{k_n}$.

If the k's are all equal, this furnishes the same result as (21).

10. In S_n let M'_r and M''_s be two *non-incident* manifolds with index numbers α' and α'' respectively, where $s \overline{<} r$ and $r + s < n$. Let the index numbers of the two manifolds considered as a whole be α. From their sections we see that $\alpha_j = \alpha'_j$ for $j = 0, 1, \ldots, r - s - 1$. In determining α_{r-s}, we find that O_{r-s} is O'_{r-s} diminished by α''_0, the order of M''_s, whence $\alpha_{r-s} = \alpha'_{r-s} + \alpha''_0$. Let us assume, then, that $\alpha_j = \alpha'_j + \alpha''_{j-(r-s)}$ for $j = 0, 1, \ldots, r-1$, and prove that the formula holds when $j = r$ also. If to within certain outside points M'_r is determined by spreads $f_1, \ldots, f_n$ of orders $\lambda_1, \ldots, \lambda_n$, and M''_s by spreads $g_1, \ldots, g_n$ of orders $\mu_1, \ldots, \mu_n$, then $M'_r \cdot M''_s$ is determined by spreads $f_1 \cdot g_1, \ldots, f_n \cdot g_n$ of orders $\lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n$ to within O_r outside points. These points arise from the intersection of the meet of $f_1, \ldots, f_k$ (residual to M'_r if $k \overline{>} n - r$) and the meet of $g_{k+1}, \ldots, g_n$ (residual to M''_s if $n - k \overline{>} n - s$). The orders of these residual intersections are determined by a proper section, and we find that

$$O_r = \Sigma \{\lambda_1 \lambda_2 \ldots \lambda_k - \alpha'_0 \sigma_{k-(n-r)} - \alpha'_1 \sigma_{k-1-(n-r)} - \ldots - \alpha'_{k-(n-r)}\} \cdot \{\mu_{k+1} \mu_{k+2} \ldots \mu_n - \alpha''_0 \sigma_{s-k} - \alpha''_1 \sigma_{s-1-k} - \ldots - \alpha''_{s-k}\},$$

the Σ referring to all possible complementary choices of n λ's and μ's, and the σ's in a brace referring to the quantities appearing in the first product of the brace. From this value of O_r we can find α_r. If we suppose that $\alpha_r = \alpha'_r + \alpha''_s$, we have only to verify that O_r above and

$$\begin{aligned} O = {} & \sigma_n(\lambda_i + \mu_i) - \alpha'_0 \sigma_r (\lambda_i + \mu_i) - \ldots - \alpha'_{r-s-1} \sigma_{s+1} (\lambda_i + \mu_i) \\ & - (\alpha'_{r-s} + \alpha''_0)\, \sigma_s (\lambda_i + \mu_i) - \ldots - (\alpha'_{r-1} + \alpha''_{s-1})\, \sigma_1 (\lambda_i + \mu_i) - (\alpha'_r + \alpha''_s) \end{aligned}$$

are the same. In the first value the terms free of α' and α'' are $\sigma_n(\lambda_i + \mu_i)$, as also in the second. There are no terms in the first expression containing both an α' and an α'', since $r + s < n$. In each expression an α' or α'' is multiplied by all possible terms of a definite simultaneous degree in λ and μ, whence

their coefficients must be the same. This identifies the two expressions, and the assumed formula for α_j is generally true.

(23) *If two non-incident spreads M'_r and M''_s in S_n ($s \leqq r$, $r+s<n$) have index numbers α' and α'' respectively, the two together constitute a manifold which has the index numbers $\alpha_j = \alpha'_j + \alpha''_{j-(r-s)}$, $j=0,1,\ldots,r$.*

The generalization of (23) to the case of any number of manifolds, no two of which have common points, is obvious. The argument used does not apply (nor is the result true) in the case where M'_r and M''_s have common points, say a common M'''_t, *unless* M'''_t be doubled. This is due to the nature of the spreads used above to determine O_r.

11. Given a curve $M_1(\alpha)$ in S_n, $n-1$ spreads of orders $\lambda_1,\ldots,\lambda_n$ on it meet in a residual curve $M_1(\beta)$ which cuts $M_1(\alpha)$ in $M_0(\vartheta)$ points. From a section $\alpha_0+\beta_0=\sigma_{n-1}$. A further spread of order l on $M_1(\alpha)$ cuts $M_1(\beta)$ in $\beta_0 l - \vartheta_0$ points outside $M_1(\alpha)$. This number is given by $O_1 = l\sigma_{n-1} - \alpha_0(\sigma_1+l) - \alpha_1$. Equating the two numbers, we find from the coefficients of l that $\alpha_0+\beta_0=\sigma_{n-1}$ and $\vartheta_0 = \alpha_0\sigma_1+\alpha_1$. The two curves are mutually related, since they constitute a regular curve, whence also $\vartheta_0=\beta_0\sigma_1+\beta_1$. The index numbers of the two according to (11) are σ_{n-1} and $-\sigma_1\sigma_{n-1}$ or $\alpha_0+\beta_0$ and $-(\alpha_0+\beta_0)\sigma_1=\alpha_1+\beta_1-2\vartheta_0$. That these are the index numbers of any two curves $M_1(\alpha)$ and $M_1(\beta)$ with ϑ_0 common points can be proved directly. Let $n-1$ spreads on the two meet again in $M_1(\gamma)$, which cuts $M_1(\alpha)$ in η_0 points and $M_1(\beta)$ in ζ_0 points. If the index numbers of the pair $M_1(\alpha)$, $M_1(\beta)$ are $\alpha_0+\beta_0$, $\alpha_1+\beta_1+x_{\alpha\beta}$, and similarly for the other pairs, then we have from the above, regarding each curve in turn as residual, the following relations:

$$
\begin{aligned}
(\alpha_0+\beta_0)\sigma_1+\alpha_1+\beta_1+x_{\alpha\beta} &= \eta_0+\zeta_0=\gamma_0\sigma_1+\gamma_1,\\
(\beta_0+\gamma_0)\sigma_1+\beta_1+\gamma_1+x_{\beta\gamma} &= \zeta_0+\vartheta_0=\alpha_0\sigma_1+\alpha_1,\\
(\gamma_0+\alpha_0)\sigma_1+\gamma_1+\alpha_1+x_{\alpha\gamma} &= \vartheta_0+\eta_0=\beta_0\sigma_1+\beta_1.
\end{aligned}
$$

From the first and the sum of the last two we find that $x_{\alpha\beta}=-2\vartheta_0$. Hence

(24) *If $n-1$ spreads of orders $\lambda_1,\ldots,\lambda_n$ on $M_1(\alpha)$ in S_n meet in a residual $M_1(\beta)$ which has $M_0(\vartheta)$ points in common with $M_1(\alpha)$, then the index numbers β_0, β_1, ϑ_0 are determined from the equations*

$$\alpha_0+\beta_0=\sigma_{n-1},\quad \alpha_0\sigma_1+\alpha_1=\vartheta_0=\beta_0\sigma_1+\beta_1.$$

The index numbers of an $M_1(\alpha)$ and $M_1(\beta)$ with common $M_0(\vartheta)$ are $\alpha_0+\beta_0$, $\alpha_1+\beta_1-2\vartheta_0$.

Given a curve $M_1(\alpha)$ in S_3 of order ν, genus π, and rank γ (class of the projection) with d apparent double points, let the cones on $M_1(\alpha)$ with vertices at P_1 and P_2 such that $\overline{P_1P_2}$ is skew to $M_1(\alpha)$ meet in a residual curve $M_1(\beta)$

of order $\nu^2 - \nu$ which meets $M_1(\alpha)$ in ϑ_0 points. These ϑ_0 points arise (a) from the points where a double generator on P_1 or P_2 meets $M_1(\alpha)$, *i. e.*, $4d$ points, and (b) from the γ points on $M_1(\alpha)$ of contact of planes on $\overline{P_1 P_2}$. Thus $\vartheta_0 = 4d + \gamma$, where $\gamma = 2(\nu - 1 + \pi)$ and $d = \frac{1}{2}(\nu - 1)(\nu - 2) - \pi$. But $\vartheta_0 = \alpha_0 \sigma_1 + \alpha_1 = \nu(2\nu) + \alpha_1$, whence $\alpha_1 = -2\nu - 2(\nu - 1 + \pi)$. If also $M_1(\alpha)$ has δ actual double points, $M_1(\beta)$ has the same double points, each of which counts for two points in ϑ_0, whence $\vartheta_0 = 4(d + \delta) + \gamma - 2\delta$, and $\alpha_1 = -2\nu - 2(\nu - 1 + \pi + \delta)$. Hence

(25) *The index numbers of a curve in S_3 of order ν, genus π, and rank γ, with δ actual double points, are*

$$\alpha_0 = \nu, \quad \alpha_1 = -2\nu - 2(\nu - 1 + \pi + \delta) = -2\nu - \gamma - 2\delta;$$

i. e., they are the same as those of ν lines with $\nu - 1 + \pi + \delta$ intersections.

Using these index numbers and (22), we find that

(26) *Three surfaces of orders n_1, n_2, n_3 which contain k_1, k_2, k_3 times respectively a space curve of order ν and rank γ, with δ actual double points, meet again in*

$$n_1 n_2 n_3 - \nu(n_1 k_2 k_3 + n_2 k_3 k_1 + n_3 k_1 k_2) + k_1 k_2 k_3 (2\nu + \gamma + 2\delta)$$

*points.**

The index numbers of a curve with a higher singularity do not depend on the order of the singularity alone. For example, if the tangents at a triple point lie in a plane, the curve behaves at that point like three lines in a plane with three intersections, otherwise like two lines meeting a third with two intersections.

(27) *The index numbers in S_n of a curve of order ν and genus π, with δ actual double points, are ν, $-\nu(n-1) - 2(\nu - 1 + \pi + \delta)$.*

This is certainly true according to (24), if the curve in S_n, as in S_3, can be replaced by ν lines with $(\nu - 1 + \pi + \delta)$ intersections. It merely requires that the second index number shall increase by ν, if the curve is projected into the space of next lower dimension. This requirement is satisfied on projection from S_3 to S_2, since the index numbers in S_2 are ν, $-\nu^2$ [see (11)]. It is satisfied also on projection from an $S_{k+\pi}$ to an $S_{k+\pi-1}$, when the curve itself lies in an S_k [see (13)]. We shall assume provisionally that the requirement is satisfied for intermediate projections.

12. It is clear that the formula (7) for the order of a restricted system will in most applications be a rather complicated expression. In order to

* Cf. Pascal-Schepp, *Repertorium*, Vol. II (1902), p. 212.

simplify it, certain numerical relations among the binomial coefficients are given here. Most and possibly all of these are known; but for lack of definite references proofs of the more complicated relations are indicated.

$$1^\circ.\quad \binom{a}{b}=\binom{a}{a-b}.$$

$$2^\circ.\quad \binom{a}{b}\binom{a-b}{c}=\binom{a}{b+c}\binom{b+c}{c}=\binom{a}{c}\binom{a-c}{b}.$$

$$3^\circ.\quad \binom{a}{b}=\binom{a-1}{b}+\binom{a-1}{b-1}.$$

$$4^\circ.\quad \binom{a+1}{b}=\binom{a}{b}+\binom{a-1}{b-1}+\binom{a-2}{b-2}+\ldots\ldots+\binom{a-b}{0}.$$

This is proved by repeated use of 3°.

$$5^\circ.\quad \binom{a+c}{b}=\binom{c-1}{0}\binom{a}{b}+\binom{c}{1}\binom{a-1}{b-1}+\binom{c+1}{2}\binom{a-2}{b-2}+\ldots\ldots$$
$$+\binom{c+b-1}{b}\binom{a-b}{0}.$$

This is proved by repeated use of 4° [see (15)].

$$6^\circ.\quad \binom{a+c}{b}=\binom{c}{0}\binom{a}{b}+\binom{c}{1}\binom{a}{b-1}+\binom{c}{2}\binom{a}{b-2}+\ldots\ldots$$
$$+\binom{c}{c}\binom{a}{b-c}.$$

This is proved by repeated use of 3°.

$$7^\circ.\quad \binom{a}{b}=\binom{a+1}{b}-\binom{a+1}{b-1}+\binom{a+1}{b-2}-\ldots\ldots+(-1)^b\binom{a+1}{0}.$$

This follows immediately from 3°.

$$8^\circ.\quad \binom{a}{b}=\binom{c-1}{0}\binom{a+c}{b}-\binom{c}{1}\binom{a+c}{b-1}+\binom{c+1}{2}\binom{a+c}{b-2}-\ldots\ldots$$
$$+(-1)^b\binom{c+b-1}{b}\binom{a+c}{0}.$$

This is proved by using 7° repeatedly.

$$9^\circ.\quad \binom{a}{0}\binom{a}{b}s^{a-b}-\binom{a}{1}\binom{a-1}{b}s^{a-b-1}t+\binom{a}{2}\binom{a-2}{b}s^{a-b-2}t^2-\ldots\ldots$$
$$+(-1)^{a-b}\binom{a}{a-b}\binom{b}{b}t^b=\binom{a}{b}(s-t)^{a-b}.$$

The factor $\binom{a}{b}$ can be removed from each term by using 2°.

$$10^\circ.\quad \binom{a}{0}\binom{a+c}{b}-\binom{a}{1}\binom{a+c-1}{b}+\binom{a}{2}\binom{a+c-2}{b}-\ldots\ldots$$
$$+(-1)^{a+c-b}\binom{a}{a+c-b}\binom{b}{b}=0,\ \text{if}\ b<a.$$

Reduce the upper numbers in $\binom{a+c}{b}$, $\binom{a+c-1}{b}$, ... by means of 6° and apply 9° for $s=t=1$.

11°. $$\binom{a-1}{0}\binom{a+c}{b}s^b-\binom{a}{1}\binom{a+c}{b-1}s^{b-1}t+\binom{a+1}{2}\binom{a+c}{b-2}s^{b-2}t^2 - \ldots + (-1)^b\binom{a+b-1}{b}\binom{a+c}{0}t^b$$
$$=\binom{c-b}{0}\binom{a+c}{b}(s-t)^b+\binom{c-b+1}{1}\binom{a+c}{b-1}(s-t)^{b-1}t + \binom{c-b+2}{2}\binom{a+c}{b-2}(s-t)^{b-2}t^2+\ldots+\binom{c}{b}\binom{a+c}{0}t^b$$
$$=\binom{a-1}{0}\binom{c}{b}s^b+\binom{a}{1}\binom{c-1}{b-1}s^{b-1}(s-t) + \binom{a+1}{2}\binom{c+2}{b+2}s^{b-2}(s-t)^2+\ldots+\binom{a+b-1}{b}\binom{c-b}{0}(s-t)^b.$$

In the first expression raise the upper number of the first factor of each term by $c-b+1$, according to 8°. Each term of the result can then be modified by 2°. Collecting the coefficients of $\binom{c-b}{0}$, $\binom{c-b+1}{1}$, $\binom{c-b+2}{2}$, ..., and using the expansion of $(s-t)^j$, the second expression is obtained. If in this $a+c$ be reduced by a or more, according to 5°, and the coefficients of $\binom{c}{b}$, $\binom{c-1}{b-1}$, $\binom{c-2}{b-2}$, ... be collected, the third expression is obtained.

13. If point conics in a plane be mapped on the points of an S_5, the ∞^2 repeated lines of the plane are mapped on the points of the Veronese surface, F_2^4. The point conics apolar to a given line conic are mapped on an S_4 in S_5, those apolar to two given line conics on an S_3 in S_5. The latter system contains the squares of the common lines of the two line conics, whence α_0 of F_2^4 is 4. A section of F_2^4 by an S_4 is a quartic curve necessarily rational, since it lies in S_4 or since it is the map of the lines of a line conic. Using (27) for $\nu=n=4$, $\pi=\delta=0$, we find that $\alpha_1=-18$. The point conics which, in line form, are apolar to a given point conic map on a quadric which contains F_2^4. Since there is only one proper conic apolar to five given point conics, five such quadrics on F_2^4 meet in one outside point, whence to determine α_2 we have the equation $1=2^5-\alpha_0\binom{5}{2}2^2-\alpha_1\binom{5}{1}2-\alpha_2$ or $\alpha_2=51$.

(28) *The index numbers of the Veronese surface F_2^4 in S_5 are* 4, —18, 51.

If $(ax)^2=(a'x)^2=\ldots$. be a variable point conic, and $(bx)^2=(b'x)^2=$ .... be a fixed point conic, then in the discriminant equation of the pencil,

$$(a\,a'\,a'')^2+3\,\lambda\,(a\,a'\,b)^2+3\,\lambda^2\,(a\,b\,b')^2+\lambda^3\,(b\,b'\,b'')^2=0,$$

$(a\,a'\,a'')^2=0$ is a cubic spread in S_5 containing F_2^4 doubly, $(a\,a'\,b)^2=0$ is a quadric spread containing F_2^4 simply, $(a\,b\,b')^2=0$ is an S_4, while $(b\,b'\,b'')^2$ is a constant. Hence the discriminant of this cubic equation—the tact-invariant of the two conics—is in S_5 a sextic spread containing F_2^4 doubly. According to (20) the index numbers of F_2^4 doubled, are $2^3\cdot 4$, $-2^4\cdot 18$, $2^5\cdot 51$, and five such sextic spreads will meet outside F_2^4 in

$$O_2=6^5-\binom{5}{2}6^2\cdot 2^3\cdot 4+\binom{5}{1}6\cdot 2^4\cdot 18-2^5\cdot 51=2^5\cdot 102=3264$$

points, whence we have the well-known theorem: *

(29) *There are* 3264 *proper conics which touch five given proper conics.*

14. Let us ask how many proper conics touch a general rational plane quintic five times. Conics cut the quintic in sets of ten points determined by binary ten-ics apolar to five binary ten-ics. We want, then, the number of squared quintics apolar to the five ten-ics exclusive of the ∞^2 quintics determined by line sections whose squares obviously satisfy the apolarity conditions. Map binary quintics upon the points of an S_5, and the line sections will map on points of an S_2. The apolarity condition of the squared quintic and one of the five ten-ics represents in S_5 a quadric which contains the S_2. The index numbers of the S_2 in S_5 are 1, -3, 6, whence five quadrics on S_2 meet outside of S_2 in $O_2=2^5-\binom{5}{2}2^2\cdot 1+\binom{5}{1}2\cdot 3-6=16$ points, or

(30) *There are* 16 *proper conics which touch a general rational plane quintic curve five times.*†

15. A plane curve of order r, f^r, is determined by $\frac{1}{2}r(r+3)$ constants. If it degenerates into an f^s and an f^{r-s}, $s<r$, the two contain only $\lambda+\mu$ constants, where $\lambda=\frac{1}{2}s(s+3)$ and $\mu=\frac{1}{2}(r-s)(r-s+3)$, whence to degenerate thus is $s(r-s)$ conditions. Given then a linear system of $\infty^{s(r-s)}$ f^r's, how many members of the system degenerate into an f^s and an f^{r-s}. The system is apolar to $\lambda+\mu$ linearly independent curves of class r; *i. e.*, the coefficients of its members satisfy $\lambda+\mu$ apolarity conditions. Map the curves f^s upon the points of an S_λ, the curves f^{r-s} upon the points of an S_μ which lies skew to S_λ in an $S_{\lambda+\mu+1}$. Then a product $f^s\cdot f^{r-s}$ can be mapped by any point of the line joining the maps of f^s and of f^{r-s}. Taking a section by an $S_{\lambda+\mu}$, such a line or such a product is represented by a point. Conversely, a point in $S_{\lambda+\mu}$

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* Pascal-Schepp, *Repertorium*, Vol. II (1902), p. 433.

† Cf. F. Morley, "The Contact Conics of the Plane Quintic Curve," *Johns Hopkins University Circular* (1912), No. 2.

represents such a product unless it lies in the section $S_{\lambda-1}$ or the section $S_{\mu-1}$, in which case f^{r-s} or f^s respectively vanishes identically. The apolarity relations furnish $\lambda+\mu$ quadrics in $S_{\lambda+\mu}$ on $S_{\lambda-1}$ and $S_{\mu-1}$. The index numbers of $S_{\lambda-1}$ in $S_{\lambda+\mu}$ are $1, -\binom{\mu+1}{1}, \binom{\mu+2}{2}, \ldots, (-1)^{\lambda-1}\binom{\lambda+\mu-1}{\lambda-1}$; those of $S_{\mu-1}$ are $1, -\binom{\lambda+1}{1}, \binom{\lambda+2}{2}, \ldots, (-1)^{\mu-1}\binom{\lambda+\mu-1}{\mu-1}$. Since the $S_{\lambda-1}$ and $S_{\mu-1}$ have no common points, according to (23) their index numbers are additive and the $\lambda+\mu$ quadrics meet outside of $S_{\lambda-1}$ and $S_{\mu-1}$ in

$$
\begin{aligned}
O = 2^{\lambda+\mu} &- \Big[\binom{\lambda+\mu}{\lambda-1}2^{\lambda-1} - \binom{\lambda+\mu}{\lambda-2}2^{\lambda-2}\cdot\binom{\mu+1}{1} + \binom{\lambda+\mu}{\lambda-3}2^{\lambda-3}\cdot\binom{\mu+2}{2} \\
&\qquad - \ldots + (-1)^{\lambda}\binom{\lambda+\mu}{0}2^{0}\cdot\binom{\mu+\lambda-1}{\lambda-1}\Big] \\
&- \Big[\binom{\lambda+\mu}{\mu-1}2^{\mu-1} - \binom{\lambda+\mu}{\mu-2}2^{\mu-2}\cdot\binom{\lambda+1}{1} + \binom{\lambda+\mu}{\mu-3}2^{\mu-3}\cdot\binom{\lambda+2}{2} \\
&\qquad - \ldots + (-1)^{\mu}\binom{\lambda+\mu}{0}2^{0}\cdot\binom{\lambda+\mu-1}{\mu-1}\Big].
\end{aligned}
$$

Apply 11° to the brackets and they become respectively

$$\Big[\binom{\lambda+\mu}{\mu-1}+\binom{\lambda+\mu}{\lambda-2}+\ldots+\binom{\lambda+\mu}{0}\Big] \text{ and } \Big[\binom{\lambda+\mu}{\mu-1}+\binom{\lambda+\mu}{\mu-2}+\ldots+\binom{\lambda+\mu}{0}\Big].$$

The sum of the two is $(1+1)^{\lambda+\mu}-\binom{\lambda+\mu}{\lambda}$, whence $O=\binom{\lambda+\mu}{\lambda}$.

(31) *In a linear system of* $\infty^{s(r-s)}$ *plane curves of order* r *there are* $\binom{\lambda+\mu}{\lambda}$ *which break up into a curve of order* s *and a curve of order* $r-s$, $\lambda=\frac{1}{2}s(s+3)$, $\mu=\frac{1}{2}(r-s)(r-s+3)$.

The same result is obtained at once by the methods of Schubert, if the $\lambda+\mu$ curves of class r be taken as r-fold points.

16. Given a rational curve of order ν in S_k, how many S_{k-1}'s meet it λ times in μ coincident points, $\lambda(\mu-1)=k$? The S_{k-1}'s determine on S_k $k+1$ linearly independent binary forms f^ν which are apolar to $\nu-k$ forms ϕ^ν. We want the number of forms in the system f^ν of the type $[g^\lambda]^\mu\cdot h^{\nu-k-\lambda}$. The forms g^λ map on an S_λ, the forms $h^{\nu-k-\lambda}$ on an $S_{\nu-k-\lambda}$ skew to S_λ in an $S_{\nu-k+1}$. Taking a section by an $S_{\nu+k}$, g^λ and $h^{\nu-k-\lambda}$ are represented by points outside the exceptional sections $S_{\lambda-1}$ and $S_{\nu-k-\lambda-1}$. Then the $\nu-k$ apolarity conditions give rise in $S_{\nu-k}$ to $\nu-k$ spreads of order $\mu+1$ which contain $S_{\nu-k-\lambda-1}$ μ times and $S_{\lambda-1}$ once. In $S_{\nu-k}$ the index numbers of $S_{\lambda-1}$ are

$$1, -\binom{\nu-k-\lambda+1}{1}, \binom{\nu-k-\lambda+2}{2}, \ldots, (-1)^{\lambda-1}\binom{\nu-k-1}{\lambda-1};$$

those of the μ-fold $S_{\nu-k-\lambda-1}$ are

$$\mu^{\lambda+1},\ -\mu^{\lambda+2}\binom{\lambda+1}{1},\ \mu^{\lambda+3}\binom{\lambda+2}{2},\ \ldots,\ (-1)^{\nu-k-\lambda-1}\mu^{\nu-k}\binom{\nu-k-1}{\nu-k-\lambda-1}.$$

Since the two manifolds have no point in common, we have

$$\begin{aligned}O = (\mu+1)^{\nu-k} &- \Big[\binom{\nu-k}{\lambda-1}(\mu+1)^{\lambda-1} - \binom{\nu-k}{\lambda-2}(\mu+1)^{\lambda-2}\binom{\nu-k-\lambda+1}{1} \\ &\qquad + \ldots + (-1)^{\lambda-1}\binom{\nu-k}{0}(\mu+1)^{0}\binom{\nu-k-1}{\lambda-1}\Big] \\ &- \Big[\binom{\nu-k}{\nu-k-\lambda-1}(\mu+1)^{\nu-k-\lambda-1}\mu^{\lambda+1} \\ &\qquad - \binom{\nu-k}{\nu-k-\lambda-2}(\mu+1)^{\nu-k-\lambda-2}\mu^{\lambda+2}\binom{\lambda+1}{1} + \ldots \\ &\qquad + (-1)^{\nu-k-\lambda-1}\binom{\nu-k}{0}(\mu+1)^{0}\mu^{\nu-k}\binom{\nu-k-1}{\nu-k-\lambda-1}\Big].\end{aligned}$$

Applying 11° to these brackets, they become respectively

$$\Big[\binom{\nu-k}{\lambda-1}\mu^{\lambda-1} + \binom{\nu-k}{\lambda-2}\mu^{\lambda-2} + \ldots + \binom{\nu-k}{0}\mu^{0}\Big]$$

and

$$\Big[\binom{\nu-k}{\lambda+1} + \binom{\nu-k}{\lambda+2}\mu + \ldots + \binom{\nu-k}{0}\mu^{\nu-k-\lambda-1}\Big]\mu^{\lambda+1}.$$

Hence the sum of the two is $(\mu+1)^{\nu-k} - \mu^{\lambda}\binom{\nu-k}{\lambda}$ and $O = \mu^{\lambda}\binom{\nu-k}{\lambda}$.

(32) *The number of S_{k-1}'s which meet a rational curve of order ν in S_k λ times in μ coincident points, where $\lambda(\mu-1) = k$, is* $\mu^{\lambda}\binom{\nu-k}{\lambda}$.

The formula furnishes for a rational curve in S_2 the number of flexes and double tangents, for a rational curve in S_3 the number of hyperosculating and triple tangent planes, for a rational curve in S_4 the number of hyperosculating, doubly osculating, and quadruply tangent S_3's, etc.

The examples given above are drawn from a rather restricted field, since we have thus far developed explicitly the index numbers of regular manifolds only, or of combinations of them which have no common points. The extension of (32) to the case where the points of a section come together in any prescribed fashion can not yet be derived,* since, if more than two sets appear, the corresponding linear spreads have common points.

§ 2. *Relative Index Numbers.*

17. Given in S_n a manifold $M_r(\alpha)$ and a manifold $M_{r-k}(\gamma)$ upon it, $0<k\leqq r$, then r spreads $f_1, \ldots, f_r$ of orders $\lambda_1, \ldots, \lambda_r$ will meet $M_r(\alpha)$ in general in $\alpha_0\sigma$ points. But if the spreads f_i all contain $M_{r-k}(\gamma)$, we define

* I am indebted to Professor Morley for the formula which applies to this general case.

the relative index number, $(\alpha\gamma)_{r-k}$, *of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$ in terms of the orders λ_i and the earlier relative index numbers [which are similarly defined for successive sections of $M_r(\alpha)$ and $M_{r-k}(\gamma)$] by the equation

(33) $O_{r-k}=\alpha_0\sigma_r-(\alpha\gamma)_0\sigma_{r-k}-(\alpha\gamma)_1\sigma_{r-k-1}-(\alpha\gamma)_2\sigma_{r-k-2}-\ldots-(\alpha\gamma)_{r-k}$,

where O_{r-k} is the number of points of $M_r(\alpha)$ outside of $M_{r-k}(\gamma)$ cut out by the spreads f_i, and the σ's refer to the orders λ. In particular, $(\alpha\gamma)_0$ is the order γ_0 of $M_{r-k}(\gamma)$. We shall now prove that

(34) *The relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$ *are independent of the spreads used to define them, and also of the dimension in which the base spread* $M_r(\alpha)$ *lies. The order of* A RELATIVE RESTRICTED SYSTEM OF EQUATIONS *is given by* (33) *in terms of the orders of the equations and the relative index numbers.*

For if λ_1 increase to λ_1+1 by using the spread $f_1\cdot L$, where L is a general S_{n-1}, then, from a section by L, O_{r-k} is increased by

$$O'_{r-k-1}=\alpha_0\sigma'_{r-1}-(\alpha\gamma)_0\sigma'_{r-k-1}-(\alpha\gamma)_1\sigma'_{r-k-2}-\ldots-(\alpha\gamma)_{r-k-1},$$

where the σ''s refer to $\lambda_2,\ldots,\lambda_n$. But this is precisely the increase on the right of (33) due to the change in λ_1; *i. e.*, $(\alpha\gamma)_{r-k}$ is unaltered and is independent of the orders. The same is true of the earlier index numbers. Again, if $M_r(\alpha)$ is supposed to lie in an S_{n+1} containing S_n, then, in S_{n+1}, spreads of the same orders λ determine the same number O_{r-k} of points of $M_r(\alpha)$ outside of $M_{r-k}(\gamma)$.

Evidently the index numbers α of $M_r(\alpha)$ in S_n, as defined in § 1, are merely the relative index numbers of $M_r(\alpha)$ as to S_n. We might define the *absolute index numbers* of M_r to be the relative index numbers of M_r as to the linear space of lowest dimension in which M_r lies, a linear space S_r being supposed to lie in an S_{r+1}. Thus the absolute index numbers of an S_r are

$$1,\ -1,\ \ldots,\ (-1)^k,\ \ldots,\qquad k=0,\ \ldots,\ r.$$

18. The theorems of § 1 can be generalized to apply to relative index numbers. The extension of (11) is:

(35) *If* $M_{r-k}(\gamma)$ *is the regular meet of* $M_r(\alpha)$ *with spreads of orders* $l_1,\ldots,l_k$, *the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$ *are*

$$(\alpha\gamma)_j=(-1)^j\alpha_0\tau_kT_j,\qquad j=0,\ \ldots,\ r-k,$$

where τ *and* T *are the symmetric and complete symmetric polynomials in* $l_1,\ldots,l_k$.

For further spreads of orders $\lambda_1,\ldots,\lambda_{r-k}$ on $M_{r-k}(\gamma)$, the numbers $O_1,\ldots,O_{r-k}$ are all zero. First $(\alpha\gamma)_0=\gamma_0=\alpha_0\tau_k$; also $O_1=0=\alpha_0\tau_k\lambda_1-(\alpha\gamma)_0(\tau_1+\lambda_1)-(\alpha\gamma)_1$, whence $(\alpha\gamma)_1=-\alpha_0\tau_kT_1$. Assume $(\alpha\gamma)_j=(-1)^j\alpha_0\tau_kT_j$ for $j=0,1,\ldots,r-k-1$, and let the σ's refer to $l_1,\ldots,l_{r-k}$.

Then

$$O_{r-k}=0=\alpha_0\tau_k\sigma_{r-k}-(\alpha\gamma)_0[\sigma_{r-k}+\sigma_{r-k-1}\tau_1+\ldots\ldots+\sigma_{r-2k}\tau_k]$$
$$-(\alpha\gamma)_1[\sigma_{r-k-1}+\sigma_{r-k-2}\tau_1+\ldots\ldots+\sigma_{r-2k-1}\tau_k]-\ldots\ldots$$
$$-(\alpha\gamma)_{r-k-1}(\sigma_1+\tau_1)-(\alpha\gamma)_{r-k}.$$

This equation determines $(\alpha\gamma)_{r-k}$. It is satisfied when also $(\alpha\gamma)_{r-k}=(-1)^{r-k}\tau_k T_{r-k}$; for then we find that

$$0=\alpha_0\tau_k[\sigma_{r-k}(1-1)+\sigma_{r-k-1}(-\tau_1+T_1)+\sigma_{r-k-2}(-\tau_2+\tau_1T_1-T_2)+\ldots\ldots$$
$$+(-\tau_{r-k}+\tau_{r-k-1}T_1-\tau_{r-k-2}T_2+\ldots\ldots+(-1)^{r-k+1}T_{r-k})],$$

which according to (10) is true.

The following theorem is the analogue of (17) and is similarly proved:

(36) *If* $(\alpha\gamma)_j$ *are the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$, *the relative index numbers* $(\alpha\gamma)'_j$, *of* M_{r-k-1}, *the complete meet of* $M_{r-k}(\gamma)$ *and a spread of order* q, *as to* $M_r(\alpha)$, *are*

$$(\alpha\gamma)'_j=q(\alpha\gamma)_j-q^2(\alpha\gamma)_{j-1}+q^2(\alpha\gamma)_{j-2}-\ldots\ldots+(-1)^j(\alpha\gamma)_0,$$
$$j=0,1,\ldots\ldots,r-k-1.$$

Let us state without proof that

(37) *The theorems* (18), (19), (20), (21), (22), (23), *when generalized as above, remain true for the relative index numbers of spreads on an* M_r.

19. The following three theorems are of different type from the foregoing:

(38) *The relative index numbers of* $M_{r-k}(\gamma)$ *as to the regular meet* $M_r(\alpha)$ *of* $n-r$ *spreads of orders* $\lambda_1,\ldots\ldots,\lambda_{n-r}$ *in* S_n, *in terms of* λ *and the index numbers* γ *of* $M_{r-k}(\gamma)$ *in* S_n, *are*

$$(\alpha\gamma)_j=\gamma_j+\gamma_{j-1}\sigma_1+\gamma_{j-2}\sigma_2+\ldots\ldots+\gamma_0\sigma_j,\quad j=0,1,\ldots\ldots,r-k,$$

where the σ*'s refer to the* λ*'s.*

Let $M_{r-k}(\gamma)$ and its sections be on as many spreads of orders $l_1, l_2,\ldots\ldots,$ in addition to those of order λ, as are necessary to determine them, and let the τ's refer to symmetric combinations of the l's. First $(\alpha\gamma)_0=\gamma_0$; then $O_1=\sigma_{n-r}\tau_{k+1}-\gamma_0(\sigma_1+\tau_1)-\gamma_1=\sigma_{n-r}\tau_{k+1}-(\alpha\gamma)_0\tau_1-(\alpha\gamma)_1$, whence $(\alpha\gamma)_1=\gamma_1+\gamma_0\sigma_1$. Assuming the theorem true for $j=0,1,\ldots\ldots,r-k-1$, we have

$$O_{r-k}=\sigma_{n-r}\tau_r-\gamma_0(\sigma_{r-k}+\sigma_{r-k-1}\tau_1+\ldots\ldots+\tau_{r-k})$$
$$-\gamma_1(\sigma_{r-k-1}+\sigma_{r-k-2}\tau_1+\ldots\ldots+\tau_{r-k-1})-\ldots\ldots-\gamma_{r-k}$$
$$=\sigma_{n-r}\tau_r-\gamma_0\tau_{r-k}-(\gamma_1+\sigma_1\gamma_0)\tau_{r-k-1}-\ldots\ldots$$
$$-(\gamma_{r-k}+\gamma_{r-k-1}\sigma_1+\ldots\ldots+\gamma_0\sigma_{r-k})$$
$$=\sigma_{n-r}\tau_r-(\alpha\gamma)_0\tau_{r-k}-(\alpha\gamma)_1\tau_{r-k-1}-\ldots\ldots-(\alpha\gamma)_{r-k}.$$

Hence $(\alpha\gamma)_{r-k}=\gamma_{r-k}+\gamma_{r-k-1}\sigma_1+\ldots\ldots+\sigma_{r-k}$.

The generalization of this to the case where the γ's are relative index numbers is:

(39) *If* $(\alpha\gamma)_j$ *are the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$, *the relative index numbers* $(\alpha_i\gamma)_j$ *of* $M_{r-k}(\gamma)$ *as to* $M_{r-i}\,(i<k)$, *the regular meet of* $M_r(\alpha)$ *and i spreads of orders* $\lambda_1, \ldots, \lambda_i$, *all on* $M_{r-k}(\gamma)$, *are*

$(\alpha_i\gamma)_j = (\alpha\gamma)_j + (\alpha\gamma)_{j-1}\sigma_1 + (\alpha\gamma)_{j-2}\sigma_2 + \ldots + (\alpha\gamma)_{j-i}\sigma_i$, $j = 0, 1, \ldots, r-k$,

where the σ*'s refer to* $\lambda_1, \ldots, \lambda_i$.

The following theorem generalizes the notion of a linear section:

(40) *If* $(\alpha\gamma)_i$ *are the relative index numbers of* $M_{r-k}(\gamma)$ *as to* $M_r(\alpha)$, *and if* $M_r(\alpha)$ *and* $M_{r-k}(\gamma)$ *be cut by a spread of order q in* M_{r-1} *and* M_{r-k-1} *respectively, then the relative index numbers of* M_{r-k-1} *as to* M_{r-1} *are*

$$(\alpha\gamma)'_j = q\,(\alpha\gamma)_j, \quad j = 0, 1, \ldots, r-k-1.$$

This is proved by finding from (36) the relative index numbers of M_{r-k-1} as to $M_r(\alpha)$ and then using (39) to get the required relative index numbers.

The theorem (14) of § 1 can be generalized as follows:

(41) *Given an* $M_r(\alpha)$ *which contains an* $M_{r-j}(\beta)$ *which contains an* $M_{r-k}(\gamma)$, $k > j$; *then*

$\beta_0(\alpha\gamma)_i = (\alpha\beta)_0(\beta\gamma)_i + (\alpha\beta)_1(\beta\gamma)_{i-1} + \ldots + (\alpha\beta)_i(\beta\gamma)_0$, $i = 0, 1, \ldots, r-k$,

provided either $M_{r-j}(\beta)$ *is a regular meet of* $M_r(\alpha)$, *or* $M_{r-k}(\gamma)$ *is a regular meet of* $M_{r-j}(\beta)$.

For in the first case we can solve the formulæ given in (39) for $(\alpha\gamma)_\nu$ in terms of $(\alpha_i\gamma)_\nu = (\beta\gamma)_\nu$ and substitute the known relative index numbers $(\alpha\beta)_\mu$. In the second case (36) can be repeatedly applied, and the known relative index numbers $(\beta\gamma)_\nu$ can be substituted.

That the provisos in (41) are essential is shown by the following example: On $M_r(\alpha) = S_5$ let us define the $M_{r-j} = M_3(\beta)$ by the matrix $\overline{123} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$, where the variables represent general linear forms in S_5. On $M_{r-j}(\beta)$ let us define $M_{r-k}(\gamma) = M_2(\gamma)$ by $x_1 = y_1 = \overline{23} = 0$. Neither $M_3(\beta)$ nor $M_2(\gamma)$ is regular in the sense of (41), and the formulæ of (41) do *not* hold. In fact the index numbers $(\alpha\beta)_i$ are the index numbers β_i in S_5. For β_0 we have in S_2 two conics, $\overline{12} = 0$, $\overline{13} = 0$, on the outside point $x_1 = y_1 = 0$, whence $\beta_0 = 3$. In S_3, S_4 and S_5, $M_3(\beta)$ is a complete spread determined by quadrics, whence $\beta_1 = -10$, $\beta_2 = 24$, $\beta_3 = -48$. Evidently $\gamma_0 = 2$. In S_4, $x_1 = y_1 = 0$ are two S_3's on $M_1(\gamma)$ which meet $M_2(\beta)$ only where $\overline{23} = 0$, *i. e.*, in no points outside $M_1(\gamma)$, whence $0 = 3 \cdot 1 \cdot 1 - (1+1)2 - (\beta\gamma)_1$, or $(\beta\gamma)_1 = -1$. In S_4, $M_1(\gamma)$ is a conic, so that

its index numbers are $(\alpha\gamma)_0=2$, $(\alpha\gamma)_1=-8$. Thus we have $\gamma_0=(\alpha\gamma)_0=2$, $(\alpha\gamma)_1=-8$, $\beta_0=(\alpha\beta)_0=3$, $(\alpha\beta)_1=-10$, $(\beta\gamma)_1=-1$. The equation $\beta_0(\alpha\gamma)_0=(\alpha\beta)_0(\beta\gamma)_0$ of course is true, but $\beta_0(\alpha\gamma)_1=(\alpha\beta)_0(\beta\gamma)_1+(\alpha\beta)_1(\beta\gamma)_0$ is not true.

On the other hand, theorem (18), which has thus far been proved only for the case where at least one manifold is regular, holds in the following example, where neither is regular. In S_5 let $M_3(\alpha)$ be defined by $\overline{123}=\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}=0$ and $M_3(\beta)$ by $\overline{123'}=\begin{vmatrix} z_1 & z_2 & z_3 \\ t_1 & t_2 & t_3 \end{vmatrix}=0$, the variables representing S_4's in general position. They meet in a curve $M_1(\gamma)$ whose index numbers, according to (18), are $\gamma_0=\alpha_0\beta_0=9$ and $\gamma_1=\alpha_0\beta_1+\alpha_1\beta_0=-60$. To verify this, note that the two quadrics $\overline{12}=0$ and $\overline{13}=0$ meet in $M_3(\alpha)$ and the S_3, $x_1=y_1=0$; also that $\overline{12'}=0$ and $\overline{13'}=0$ meet in $M_3(\beta)$ and the S_3, $z_1=t_1=0$. The four quadrics meet in a composite curve whose index numbers are 16, $-16\cdot 8$. This composite curve has the four parts:

(a) $\overline{123}=0$, $\overline{123'}=0$, with index numbers $\gamma_0=9$, γ_1;
(b) $\overline{123}=0$, $z_1=t_1=0$, with index numbers 3, -16;
(c) $x_1=y_1=0$, $\overline{123'}=0$, with index numbers 3, -16;
(d) $x_1=y_1=0$, $z_1=t_1=0$, with index numbers 1, -4.

But the pairs (a), (b) and (a), (c) each have six common points; the pairs (b), (d) and (c), (d) each have two common points; while the pairs (a), (d) and (b), (c) have no common points. Hence the index numbers of the four, according to (24), are 16, $(\gamma_1-16-16-4-2\cdot 16)$. Comparing these with 16, $-16\cdot 8$, we find that $\gamma_1=-60$, which verifies the theorem.

These examples emphasize the fact that considerable caution must be used in accepting as general theorems which have been proved only for the cases where some or all of the manifolds concerned are regular.

BALTIMORE, *October* 1, 1913.

The Canonical Types of Nets of Modular Conics.

By Albert Harris Wilson.

Introduction.

1. In this paper is treated the following problem. Given three ternary quadratic forms,

$$C_i = a_i t_1^2 + 2h_i t_1 t_2 + b_i t_2^2 + 2g_i t_1 t_3 + 2f_i t_2 t_3 + c_i t_3^2, \qquad (i=1, 2, 3), \tag{1}$$

belonging to the $GF(p^n)$; it is proposed to reduce the net of forms

$$R = xC_1 + yC_2 + zC_3 \tag{2}$$

(x, y, z likewise in the $GF(p^n)$), to canonical types by means of linear transformations operated simultaneously on the t_i, on the one hand, and on the x, y, z, on the other. The t_i will be referred to as the variables, and the x, y, z as the parameters; and a transformation of these latter (which replaces any C by a linear combination of the C_i), as a parameter change. By canonical types is to be understood what is usually implied by that term in algebra; namely, types equivalent in the aggregate, under the transformations mentioned, to the nets (2), and which contain the minimum number of arbitrary constants, such constants as do appear being invariants of the net.

The analogous problem for the ordinary complex-number field has been completely solved by Jordan.* In this field the vanishing points of the C_i are curves of the second order, and the discriminant of the net is a cubic curve, the locus of the points (x, y, z) for which the quadratic $R(t_1, t_2, t_3)=0$ is degenerate. The treatment is based upon the invariant theory and the geometric properties of the cubic, and the canonical forms are derived in an elegant manner. They are sixteen in number, classified by the mutual relations of the three curves $C_i=0$ and the form of the discriminant.

In the finite field the analogy of geometry may still be useful. The vanishing points of the quadratics are conics in the Veblen-Bussey finite geometry,†

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* C. Jordan, "Réduction d'un réseau de formes quadratiques ou bilineares," *Journal de Mathématiques* (1906), 7.

† Veblen-Bussey, " Finite Projective Geometries," *Transactions of the American Mathematical Society*, Vol. VII, 1906.

and those of the discriminant of the net a cubic curve. Unquestionably the most advantageous method of attacking the problem would be the usual one of a classification based upon the discriminant; but the theory of the cubic curve and its invariants in finite geometry is at present in so undeveloped a state that little progress could be made with it. Under the circumstances it has seemed best to make use of the following purely algebraical method, though aid is at times derived from geometrical intuition, and frequently geometrical nomenclature is employed.

2. *Analysis of the Problem.* The net (2), $R=xC_1+yC_2+zC_3$, regarded as a quadratic in t_1, t_2, t_3, $R=a_{11}t_1^2+2a_{12}t_1t_2+a_{22}t_2^2+2a_{13}t_1t_3+2a_{23}t_2t_3+a_{33}t_3^2$, has for its discriminant

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

a ternary cubic form in x, y and z. The discussion of the net may be separated into parts according to the rank of the determinant D.

In the first place, the case of the identical vanishing of D (for all values of x, y and z) may be excluded, as this would mean that the ternary quadratic $R(t_1, t_2, t_3)$ was reducible to a binary form. Also the case where the quadratics C_i are not linearly independent may be set aside, for then the net degenerates into a family of two forms, or into a single form. The determinant may then be of rank 1, 2 or 3, meaning by this of minimum rank 1, 2 or 3 for any values of x, y, z not all zero; and it is on this basis of division that the problem is discussed in what follows.

D is of rank 1. Values of x, y, z exist (always excluding $x=y=z=0$) for which the first minors of D vanish simultaneously. In this case the net contains a unary form; and conversely, values of x, y, z which make R a unary form will cause all the first minors to vanish.

D is of rank 2. Values of x, y, z exist which will make D vanish, but no values exist which will cause all the first minors to vanish. In this case the net contains a binary form, but no unary form; the converse of this statement is likewise true.

D is of rank 3. No values of x, y, z exist for which D vanishes. The net in this case contains neither a binary nor a unary form. The existence of this net is in itself remarkable and occurs only when the coefficients of R are subjected to narrow conditions.*

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* The discriminant cubic is the locus of the points (x, y, z) for which the corresponding conic degenerates into two lines or into a double line. The above classification of the discriminant is then a classification of the nets into those which contain (i), degenerate conics which are double lines; (ii), degenerate

PART I. THE DISCRIMINANT OF THE NET IS OF RANK 1.

3. *Separation of the Cases.* There is by hypothesis a unary form in the net, and we may set $C_1=t_1^2$. Then by a parameter change, subtracting C_1 from C_2 and C_3,

$$C_2=at_2^2+bt_2t_3+ct_3^2+dt_1t_2+et_1t_3,\quad C_3=a't_2^2+b't_2t_3+c't_3^2+d't_1t_2+e't_1t_3.^*$$

One of the following three cases must exist:

(a) No one of the terms t_2^2, t_2t_3, t_3^2 appears either in C_2 or C_3.

(b) One or more of the terms t_2^2, t_2t_3, t_3^2 appears in one form of the net, but not in the others.

(c) The terms t_2^2, t_2t_3, t_3^2 appear in both forms, C_2 and C_3, in such a way that this case is not reducible to (b).

4. *In the first of the hypotheses,* (a), writing the forms C_1, C_2, C_3, in order, to indicate the net, we have t_1^2, $at_1t_2+bt_1t_3$, $a't_1t_2+b't_1t_3$, which is obviously readily reduced in all cases to

$$\text{I.}\dagger\qquad t_1^2,\quad 2t_1t_2,\quad 2t_1t_3. \tag{4}$$

5. *In the second of the hypotheses,* (b), *of* § 3, the net is

$$t_1^2,\quad at_1t_2+bt_1t_3,\quad f+d't_1t_3+e't_1t_2, \tag{5}$$

where f represents the binary quadratic $a't_2^2+b't_2t_3+c't_3^2$. f may be transformed, without affecting the forms of C_1 or C_2, into one of the following: t_2^2, t_3^2, $t_2^2+t_3^2$, or $t_2^2+vt_3^2$, where v represents a particular not-square.‡ The last two forms are usually treated together and written $t_2^2+mt_3^2$, where $m=1$ or v.

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conics not double lines; and (iii), no degenerate conics. Imaginary points are not considered in this paper.

The number of nets of conics can be gotten as follows: Let $P_k=\dfrac{p^{(k+1)n}-1}{p^n-1}$ be the number of points in a linear space, S_k. The linear system of conics are in one-to-one correspondence with the points of an S_5. The number of nets of conics is the number of planes (S_2's) in S_5. A plane is determined by a *proper* triangle. Such a triangle can be chosen in $\dfrac{P_5(P_5-P_0)(P_5-P_1)}{1\cdot 2\cdot 3}$ ways in S_5; the first point in P_5 ways, the second point (any point except the first) in P_5-P_0 ways, and the third point (any point except those on the join of the first two) in P_5-P_1 ways. But the same plane is determined by any triangle in it. In a plane there are $\dfrac{P_2(P_2-P_0)(P_2-P_1)}{1\cdot 2\cdot 3}$ proper triangles. Hence there are $\dfrac{P_5(P_5-P_0)(P_5-P_1)}{P_2(P_2-P_0)(P_2-P_1)}$ planes in S_5 or nets of conics in S_2.

*In the processes of reduction the letters representing the coefficients are usually repeated, even though they are in the course of the transformation replaced by combinations of the coefficients; for only the types are sought. Only when special values (such as 0) may arise, which affect the validity of the results, is it necessary to attend to the actual values of the coefficients.

† With these same numerals the nets are listed in a final summary.

‡ Dickson, "Linear Groups," §§ 168, 169. The process of transformation of $at_2^2+bt_2t_3+ct_3^2$ by which the term t_2t_3 is eliminated [*i. e.*, by $t_2=t_2'-bt_3'/(2a)$, $t_1=t_1'$, $t_3=t_3'$] we may call, for brevity, completing the square on the terms $at_2^2+bt_2t_3$.

Let $f=t_2^2$ in (5). Completing the square in C_3 and making an easy parameter change, we have t_1^2, $at_1t_2+bt_1t_3$, $t_2^2+b't_1t_3$. If here $b'=0$, there results by $t_3'=at_2+bt_3$ (as $b\neq 0$) the single net

$$\text{IV.} \qquad t_1^2, \quad t_2^2, \quad 2t_1t_3. \tag{6}$$

If $b'\neq 0$ and $b=0$, by $t_2=b't_2'$, $t_1=b't_1'$,

$$\text{II.} \qquad t_1^2, \quad 2t_1t_2, \quad t_2^2+2t_1t_3. \tag{7}$$

If $b'\neq 0$ and $b\neq 0$, we have, on multiplying the variables t_1 and t_2 by b', t_1^2, $at_1t_2+t_1t_3$, $t_2^2+t_1t_3$; and by the transformation $t_1=t_1'$, $t_2=t_2'+at_1'/2$, $t_3=t_3'-at_2'-a^2t_1'/2$, followed by parameter changes, this net becomes t_1^2, t_2^2, $2t_1t_3$, which is IV.

Let $f=t_2t_3$ in (5). If $b=0$, the second form C_2 becomes t_1t_2; if $b\neq 0$, then, by the transformation $t_1=t_1'$, $t_2=t_3'$, $bt_3=t_2'-at_3'$, C_2 becomes t_1t_2. In any case t_1t_2 may be canceled from C_3, and we have t_1^2, t_1t_2, $t_2t_3+c't_1t_3$. If $c'=0$, directly, or, if $c'\neq 0$, by $t_2=t_2'+c't_1'$, this becomes

$$\text{V.} \qquad t_1^2, \quad 2t_1t_2, \quad 2t_2t_3. \tag{8}$$

Let $f=t_2^2+mt_3^2$ in (5). Completing the square on $t_2^2+e't_1t_2$ and $mt_3^2+d't_1t_3$ in C_3, and making parameter changes, there results t_1^2, $at_1t_2+bt_1t_3$, $t_2^2+mt_3^2$. We may here assume that $a\neq 0$; for if $a=0$, then by interchanging t_2 and t_3 the term t_1t_2 is restored in C_2. Set, then, $a=1$. By the transformation $t_1=t_1'$, $t_2=t_2'-bt_3'$, $t_3=t_3'+bt_2'/m$, provided the determinant $1+b^2/m$ does not vanish, the net becomes

$$\text{VII.} \qquad t_1^2, \quad 2t_1t_2, \quad t_2^2+mt_3^2. \tag{9}$$

If, on the other hand, the determinant $1+b^2/m$ of the transformation just used does vanish, multiply t_3 by $-b$: t_1^2, $t_1t_2+mt_1t_3$, $t_2^2-m^2t_3^2$, which by $t_2=(t_2'+t_3')/2$, $t_3=(t_2'-t_3')/2m$ becomes t_1^2, $2t_1t_2$, $2t_2t_3$, a net already enumerated as V.

6. *In the third of the hypotheses*, (c), *of* § 3, the net is written

$$t_1^2, \quad at_2^2+bt_2t_3+ct_3^2+dt_1t_2+et_1t_3, \quad a't_2^2+b't_2t_3+c't_3^2+d't_1t_2+e't_1t_3.$$

Obviously one square term must be present in C_2 or C_3, and this may, without essential restriction, be taken to be t_2^2 in C_2. If the square be completed on the terms $at_2^2+bt_2t_3$ of C_2, and t_2^2 be canceled from C_3, there results

$$t_1^2, \quad t_2^2+ct_3^2+dt_1t_2+et_1t_3, \quad b't_2t_3+c't_3^2+d't_1t_2+e't_1t_3. \tag{10}$$

If $b'=0$, (10) becomes

$$t_1^2, \quad t_2^2+dt_1t_2+et_1t_3, \quad t_3^2+d't_1t_2+e't_1t_3 \tag{11}$$

(as t_3^2 must be present in C_3 by hypothesis). If $b' \neq 0$ in (10), then, by $b't_2+c't_3=t_2'$ afterwards canceling the resulting t_2t_3 from C_2, the net is

$$t_1^2, \quad t_2^2+ct_3^2+dt_1t_2+et_1t_3, \quad t_2t_3+d't_1t_2+e't_1t_3. \tag{12}$$

If $c=0$ in (12), by $t_1=t_1'$, $t_2=t_2'-dt_1'/2$, $t_3=t_3'-d't_1'$, the net reduces readily to

$$t_1^2, \quad t_2^2+et_1t_3, \quad t_2t_3+e't_1t_3. \tag{13}$$

If in (13) $e=0$,

$$t_1^2, \quad t_2^2, \quad t_2t_3+e't_1t_3. \tag{14}$$

If in (13) $e \neq 0 = 2$ (say),

$$t_1^2, \quad t_2^2+2t_1t_3, \quad t_2t_3+e't_1t_3. \tag{15}$$

If in (14) $e'=0$, the net is t_1^2, t_2^2, t_2t_3, which is seen to be equivalent to IV by interchanging t_1 and t_2. If in (14) $e' \neq 0 = 1$ (say), there results

$$\text{VI.} \qquad t_1^2, \quad t_2^2, \quad 2(t_2t_3+t_1t_3). \tag{16}$$

The net (15) for $e' \neq 0 = 1$ (say) is equivalent to the same net for $e'=0$, *i. e.*, t_1^2, $t_2^2+2t_1t_3$, $2(t_2t_3+t_1t_3)$ to $t_1'^2$, $t_2'^2+2t_1't_3'$, $2t_2't_3'$, by the transformation $t_1=-9t_1'$, $t_2=3(t_1'+t_2')$, $t_3=2t_1'+t_2'-t_3'$, with non-vanishing determinant,

$$\text{IX.} \qquad t_1^2, \quad 2t_2t_3, \quad t_2^2+2t_1t_3. \tag{17}$$

In this reduction it is assumed that $p \neq 3$.

If in (12) $c=$ a square (not 0), then, on multiplying t_2 by the square root of c, the form C_2 becomes $t_2^2+t_3^2+dt_1t_2+et_1t_3$. By $C_2'=C_2+2C_3$, followed by $t_2=t_2'+t_3'$, $t_3=t_2'-t_3'$, and a parameter change, the net (11) is again obtained.

If finally, in (12), c is a not-square, set it equal to a particular not-square v; there remain for further reduction the nets (11) and

$$t_1^2, \quad t_2^2+vt_3^2+dt_1t_2+et_1t_3, \quad t_2t_3+d't_1t_2+e't_1t_3. \tag{18}$$

In (11) complete the square on $t_2^2+dt_1t_2$ in C_2 and on $t_3^2+e't_1t_3$ in C_3, and get

$$t_1^2, \quad t_2^2+et_1t_3, \quad t_3^2+d't_1t_2. \tag{19}$$

If in (19) $e=d'=0$, we have

$$\text{III.} \qquad t_1^2, \quad t_2^2, \quad t_3^2. \tag{20}$$

If in (19) $e=0$, $d' \neq 0$, or $e \neq 0$, $d'=0$, by obvious changes we have

$$t_1^2, \quad t_2^2, \quad t_3^2+2t_1t_2. \tag{21}$$

If in (19) $e \neq 0$, $d' \neq 0$, multiply t_1 by $2/e$ and get

$$t_1^2, \quad t_2^2+2t_1t_3, \quad t_3^2+3\rho t_1t_2, \tag{22}$$

where ρ is a parameter, arbitrary, except not 0.

In (18) the squares may be completed to eliminate the terms t_1t_2 and t_1t_3 in C_2, giving

$$t_1^2, \quad t_2^2+vt_3^2, \quad t_2t_3+d't_1t_2+e't_1t_3. \tag{23}$$

If in (23) $d'=e'=0$, there results

$$\text{VIII.} \qquad t_1^2, \quad 2t_2t_3, \quad t_2^2+vt_3^2. \tag{24}$$

If in (23) $d'=0$, $e'\neq 0$, or $d'\neq 0$, $e'=0$,

$$\text{X.} \qquad t_1^2, \quad 2(t_1t_3+t_2t_3), \quad t_2^2+vt_3^2. \tag{25}$$

If in (23) $d'\neq 0$, $e'\neq 0$, then, multiplying t_3 by d' and t_2 by e', we have finally

$$\text{XI.} \qquad t_1^2, \quad 2(t_1t_2+t_1t_3+t_2t_3), \quad t_2^2+\alpha t_3^2, \tag{26}$$

where α is an arbitrary parameter not equal to zero.

7. Two of the nets just obtained and not listed by the Roman numerals, namely, (21) and (22), may now be included under the net XI (26).

In the first place, for $\alpha=1$ in (26), by $C_2'=C_1+C_2+C_3$, the net may be written t_1^2, $(t_1+t_2+t_3)^2$, $t_2^2+t_3^2$; and this, by $t_1=t_1'$, $t_2=-t_1'+t_2'-t_3'$, $t_3=t_3'$, reduces to t_1^2, t_2^2, $t_1^2+t_2^2+2t_3^2-2t_1t_2+2t_1t_3-2t_2t_3$. Cancel t_1^2, t_2^2, and eliminate t_1t_3 and t_2t_3 by completing the square, and there results t_1^2, t_2^2, $t_3^2+2t_1t_2$, which is (21).

Again, the net (22) may be included under XI (26) for α = a square = k^2, $\neq 1$. In fact, if h is chosen so that $h^3=\rho\dfrac{1+k}{1-k}$ (which for every ρ is possible unless $p^n=3$), the substitution

$$t_1=-2t_1', \quad t_2=t_1'+\frac{1+k}{2h}t_2'+\frac{1+k}{2h^2}t_3', \quad t_3=t_1'-\frac{1+k}{2hk}t_2'+\frac{1+k}{2h^2k}t_3'$$

will transform t_1^2, $2(t_1t_2+t_1t_3+t_2t_3)$, and $t_2^2+k^2t_3^2$ into members of the net $xt_1^2+y(t_2^2+2t_1t_3)+z(t_3^2+2\rho t_1t_2)$.

8. *Independence of the Nets.* It remains to determine whether the nets just found are independent of each other; that is, incapable of being transformed, one into the other, by linear transformations of the variables or the parameters. Many of the questions of equivalence are answered by an examination of the invariants of the nets. The rank of the determinant shows the independence, as classes, of the three classes of nets which are separately examined in this paper; no net which contains a unary form can be equivalent to a net which contains no unary form. The numerical invariants, the number

of the unary forms and the number of the binary forms, will in many cases serve to distinguish the nets. Further, the form of the discriminant $D(x, y, z)$, which is multiplied in the transformation on the t_i by the square of the determinant of transformation, will aid in answering the question.

9. As an illustration of the method of reckoning these invariants, consider the case of the net XI, $xt_1^2+2y(t_1t_2+t_1t_3+t_2t_3)+z(t_2^2+\alpha t_3^2)$.

$$D=\begin{vmatrix} x & y & y \\ y & z & y \\ y & y & \alpha z \end{vmatrix}=2y^3-(\alpha+1)y^2z-xy^2+\alpha xz^2.$$

For $D=0$, $(y^2-\alpha z^2)x=2y^3-(\alpha+1)y^2z$. In the case $\alpha=1$, the discriminant is factorable, and this case should be treated separately; however, here, by § 7, the net is equivalent to the simpler one t_1^2, t_2^2, $t_3^2+2t_1t_2$, of which it is readily determined that the number of binary forms, $B,=2(p^n-1)^2$, and the number of unary forms, $U,=2(p^n-1)$. Excluding the case $\alpha=1$, distinguish further the cases α = a square = k^2 (not 0), and α = a not-square.

$\alpha=k^2$. If $y=z=0$, the determinant vanishes for x arbitrary, only not 0; *i. e.*, for p^n-1 values. If y and z are arbitrary, only not $y=z=0$, there are $p^{2n}-1$ sets of values; but from these must be deducted the number of sets for which the multiplier of x also vanishes (except $y=z=0$); for these will not cause D to vanish unless $\alpha=1$. These latter sets given by $y=\pm kz$ are $2(p^n-1)$ in number. The total number of vanishing sets is then $p^n-1+p^{2n}-1-2(p^n-1)=p^n(p^n-1)$. To find how many of these are unary forms, consider the first minors of the discriminant: $xz-y^2$, $y(x-y)$, $y(y-z)$, $\alpha xz-y^2$, $y(\alpha z-y)$, αz^2-y^2. As $\alpha\neq 1$, these vanish together only for $y=z=0$; *i. e.*, for p^n-1 sets of values of x, y, and z. Hence, finally, $B=p^n(p^n-1)-(p^n-1)=(p^n-1)^2$, $U=p^n-1$.

α = a not-square. The multiplier of x can not be made to vanish except for $y=z=0$, for which, for x arbitrary (not 0), the discriminant vanishes. Further, as in the case of $\alpha=k^2$. The number of vanishing sets for D is then $p^{2n}+p^n-2$, or $B=(p^n+1)(p^n-1)$, $U=p^n-1$.

10. By an examination of the character of the discriminant and the numerical invariants, all questions of equivalence are answered except the following: IV may be equivalent to VI, and X may be equivalent to XI for $\alpha=v$. (See pp. 208, 209 for table of nets.)

If IV is equivalent to VI, then each of the forms t_1^2, t_2^2, t_1t_2 must go into one of the type $xt_1^2+2y(t_1t_3+t_2t_3)+zt_2^2$ by a transformation $t_1=at_1'$, $t_2=bt_2'$, $t_3=c_1t_1'+c_2t_2'+c_3t_3'$, or else $t_1=at_2'$, $t_2=bt_1'$, $t_3=c_1t_1'+c_2t_2'+c_3t_3'$. By these t_1t_3

becomes $a(c_1t_1^2+c_2t_1t_2+c_3t_1t_3)$ or $a(c_1t_1t_2+c_2t_2^2+c_3t_2t_3)$; but neither of these is of the required type.

11. *Equivalence of the Nets* X *and* XI *for* α = *a not-square* v. If $p^n=4k-1$, *i. e.*, if -1 is a not-square, set $v=-1$. Then the transformation

$$t_1=-2t_1', \quad t_2=2t_1'+t_2'+t_3', \quad t_3=t_2'-t_3'$$

will take each form of X into XI; *i. e.*, t_1^2, $t_1t_3+t_2t_3$, and $t_2^2+t_3^2$ into

$$Xt_1^2+2Y(t_1't_2'+t_1't_3'+t_2't_3')+Z(t_2'^2-t_3'^2).$$

If $p^n=4k+1$ (-1 a square), the equivalence of the two nets is conditional. As an example of discussions of this kind, this case is treated in detail.

The net $xC_1+yC_2+zC_3$ is equivalent to $XC_1'+YC_2'+ZC_3'$ if C_1, C_2, and C_3 are severally capable of being transformed into forms of the type $XC_1'+YC_2'+ZC_3'$. The form t_1^2 must go into the form $at_1'^2$, as there is but one (essential) unary form in each of the nets. Hence the reduction is effected, if at all, by a transformation of the form

$$t_1=at_1', \quad t_2=b_1t_1'+b_2t_2'+b_3t_3', \quad t_3=c_1t_1'+c_2t_2'+c_3t_3', \tag{27}$$

of determinant $a(b_2c_3-b_3c_2)$. By this the three forms of XI must be transformed into forms of the type X. Substituting in the last two forms of XI, and equating coefficients with those of X, we have the two sets of equations:

$$\begin{aligned} 2(ab_1+ac_1+b_1c_1)&=X, & b_1^2+vc_1^2&=X',\\ 2b_2c_2&=Z, & b_2^2+vc_2^2&=Z',\\ 2b_3c_3&=vZ, & b_3^2+vc_3^2&=vZ',\\ a(b_2+c_2)+b_1c_2+b_2c_1&=0, & b_1b_2+vc_1c_2&=0,\\ a(b_3+c_3)+b_1c_3+b_3c_1&=Y, & b_1b_3+vc_1c_3&=Y',\\ b_2c_3+b_3c_2&=Y, & b_2b_3+vc_2c_3&=Y'. \end{aligned}$$

From the second and third of each set Z may be eliminated, and from the fifth and sixth Y, giving,

(A₁) $b_3c_3-vb_2c_2=0$, (B₁) $vb_2^2-b_3^2+v^2c_2^2-vc_3^2=0$,

(A₂) $a(b_2+c_2)+b_1c_2+b_2c_1=0$, (B₂) $b_1b_2+vc_1c_2=0$,

(A₃) $a(b_3+c_3)+b_3(c_1-c_2)+c_3(b_1-b_2)=0$, (B₃) $b_3(b_1-b_2)+vc_3(c_1-c_2)=0$.

It is readily seen that none of the letters can be 0. Setting from (A₁): $b_2=lb_3$, $vc_2=c_3/l$, $(l\neq 0)$, there results from (B₁) $lb_3=c_3$, or $lb_3=-c_3$. Supposing first $lb_3=c_3$, we have $lb_3=c_3=b_2=lvc_2$; and substituting successively in the remaining equations, (A₂), (A₃), (B₂), (B₃), we find all the coefficients expressed in terms of a, as follows:

$$\left.\begin{aligned} 2lb_3&=-(1+vl)a, & b_2&=lb_3, & b_1(1-l^2v)&=2lb_3, \\ c_3&=lb_3, & vc_2&=b_3, & c_1(1-l^2v)&=-2l^2b_3, \end{aligned}\right\} \tag{28}$$

which will effect the transformation in question, provided

$$l^3v^2+3l^2v+3lv+1=0. \tag{29}$$

As $p^n=4k+1$, $v\neq-1$, $1+l^2v\neq0$, $1-l^2v\neq0$, the last expression is, to within a non-vanishing factor, the discriminant of the transformation.

Similarly, from the second hypothesis, $lb_3=-c_3$, there are derived:

$$\left.\begin{aligned} 2lb_3&=(1-vl)a, & b_2&=lb_3, & (1-l^2v)b_1&=2lb_3, \\ c_3&=-lb_3, & vc_2&=-b_3, & (1-l^2v)c_1&=2l^2b_3, \end{aligned}\right\} \tag{30}$$

provided

$$l^3v^2-3l^2v+3lv-1=0. \tag{31}$$

If either of the equations (29), (31) is reducible, the other is also. Now the conditions for the irreducibility of equations in Galois fields has been fully discussed by Professor Dickson,* and, in particular, the following result obtained: The necessary and sufficient conditions that the cubic $x^3+bx+c=0$ be irreducible, are

(1) $\quad -4b^3-27c^2=$ a square $\neq0$, say $81m^2$, (32)

(2) $\quad \frac{1}{2}(-c+m\sqrt{-3})=$ a not-cube for field $[GF(p^n), \sqrt{-3}]$. (33)

Multiply the roots of (29) by v and write $l-1$ for l, and it takes the following form: $l^3+3(v-1)l-2(v-1)=0$. The conditions for irreducibility are now

$$\left.\begin{aligned} &(1) \quad -108(v-1)^3-108(v-1)^2=81m^2, \\ &(2) \quad \tfrac{1}{2}[2(v-1)+m\sqrt{-3}]=\text{ a not-cube in } [GF(p^n), \sqrt{-3}]. \end{aligned}\right\} \tag{34}$$

The first condition requires that 3 be a not-square (as $p^n=4k+1$); hence $p^n=12k+5$. Take $v=-3$, then $m=8$, and the second condition requires that $4(-1+\sqrt{-3})$, or equally, that $\frac{1}{2}(-1+\sqrt{-3})$, shall be a not-cube. This latter number is ω, a cube root of 1. If now $\omega=r^3$, then $\omega^3=r^9=1$, where $r^3\neq1$, r in $GF(p^{2n})$. Hence 9 is a factor of $p^{2n}-1$; but as $p^n-1\equiv1$ (mod. 3), it follows that $p^n+1\equiv0$ (mod. 9). From $p^n=12k+5$ follows then $k\equiv1$ (mod. 3).

* *Bulletin of the American Mathematical Society*, October, 1906.

Hence the equation $l^3v^2+3l^2v+3lv+1=0$ is reducible, and the net X equivalent to XI, except for $p^n=36k+17$; and it results from equations $(A_1), \ldots, (B_3)$ that in this latter case the nets are not equivalent.

PART II. THE DISCRIMINANT OF THE NET IS OF RANK 2.

12. *Separation of the Cases.* There is by hypothesis a binary form in the net, but no unary form; hence we may set

$$\left.\begin{aligned} C_1&=t_1^2+mt_2^2, \quad (m=1 \text{ or } v, \text{ a particular not-square}), \\ C_2&=at_1^2+bt_1t_2+ct_1t_3+dt_2t_3+et_3^2, \\ C_3&=a't_1^2+b't_1t_2+c't_1t_3+d't_2t_3+e't_3^2, \end{aligned}\right\} \tag{35}$$

canceling t_2^2 from C_2 and C_3. Under any circumstances we may take $e=0$. If also $e'=0$, cancel t_2t_3 from C_2 and write the net:

$$t_1^2+mt_2^2, \quad at_1^2+bt_1t_2+ct_1t_3, \quad a't_1^2+b't_1t_2+c't_1t_3+d't_2t_3. \tag{36}$$

If $e'\neq 0$ in (35) the square may be completed to eliminate from C_3 the terms t_1t_3 and t_2t_3, and the net be written:

$$t_1^2+mt_2^2, \quad at_1^2+bt_1t_2+ct_1t_3+dt_2t_3, \quad a't_1^2+b't_1t_2+t_3^2. \tag{37}$$

13. In the net (36) distinguish the cases $c=0$, $c\neq 0$. If $c=0$, then $b\neq 0$; otherwise a unary is present. Setting $b=1$, and canceling t_1t_2 from C_3:

$$t_1^2+mt_2^2, \quad at_1^2+t_1t_2, \quad a't_1^2+c't_1t_3+d't_2t_3. \tag{38}$$

If $c'\neq 0$, then, by $t_3'=at_1+bt_2+ct_3$, and canceling t_1t_3 from C_3':

$$t_1^2+mt_2^2, \quad t_1t_3, \quad a't_1^2+b't_1t_2+d't_2t_3. \tag{39}$$

The net (38), by $t_2'=at_1+t_2$ and parameter change, becomes

$$t_1^2+mt_2^2, \quad t_1t_2, \quad a't_1^2+c't_1t_3+d't_2t_3. \tag{40}$$

Since c' and d' are not both 0 in (40), say $c'\neq 0=1$, set $t_3'=a't_1+t_3$, and the net becomes $t_1^2+mt_2^2$, t_1t_2, $t_1t_3+d't_2t_3$; or, if $d'\neq 0$, $t_1^2+ft_2^2$, t_1t_2, $t_1t_3+t_2t_3$, on multiplying t_1 by d'. Write $f=r^2$ or $f=r^2v$ according as f is a square or a not-square; the net is brought, by the substitution $t_1=t_1'r+t_2'$, $t_2=-\frac{r}{f}t_1'-t_2'$, $t_3=t_3'$, into the form

$$\text{XII.} \qquad t_1^2+vt_2^2, \quad 2t_1t_2, \quad 2t_1t_3. \tag{41}$$

The transformation is inadmissible if $f=1$; but in this case, by adding $2C_2$ to C_1, a unary results. If $d'=0$ in (40), net XII is reached directly.

In the net (39) $d'=0$ readily reduces to (41); set $d'=1$. For $a'=b'=0$ there results

$$\text{XIII.} \qquad t_1^2+mt_2^2, \quad 2t_1t_3, \quad 2t_2t_3. \tag{42}$$

If $b'\neq 0$, multiplying t_3 by b':

$$\text{XIV.}\quad t_1^2+mt_2^2,\quad 2t_1t_3,\quad \beta t_1^2+2t_1t_2+2t_2t_3,\qquad (\beta \text{ arbitrary}). \tag{43}$$

If $a'\neq 0$, $b'=0$, the net is $t_1^2+mt_2^2$, t_1t_3, $at_1^2+t_2t_3$. By $C_3-aC_1=C_3'$, followed by $t_3=-amt_3'$: $t_1^2+mt_2^2$, t_1t_3, $t_2(t_2+t_3)$. By $t_3'=t_2+t_3$: $t_1^2+mt_2^2$, $t_1t_2+t_1t_3$, t_2t_3. Interchanging t_1 and t_2 and multiplying C_1, there results: $t_1^2+mt_2^2$, t_1t_3, $t_1t_2+t_2t_3$, which is XIV for $\beta=0$.

14. If in net (37) $c=d=0$, then $b\neq 0$, and setting $t_2'=at_1+bt_2$, we have, on canceling t_1t_2 from the resulting C_2 and C_3, $t_1^2+ft_2^2$, t_1t_2, $a't_1^2+t_3^2$. As above, f may be reduced to v, and multiplying t_3 by $\sqrt{a'}$ if a' is a square, and by b if $a'=vb^2$, the net becomes

$$\text{XV.}\quad t_1^2+vt_2^2,\quad t_1t_2,\quad t_1^2+mt_3^2. \tag{44}$$

If c and d are not both 0 in the net (37), we may take $c\neq 0$ and set $c=1$. Then, by the transformation

$$t_1'=t_1+dt_2,\quad t_2'=-\frac{d}{m}t_1+t_2,\quad t_3'=t_3, \tag{45}$$

if the determinant $\dfrac{d^2}{m}+1\neq 0$, the net takes the form

$$t_1^2+mt_2^2,\quad at_1^2+bt_1t_2+t_1t_3,\quad a't_1^2+b't_1t_2+t_3^2.$$

By $t_3'=at_1+t_3$, followed by a parameter change, we have

$$t_1^2+mt_2^2,\quad bt_1t_2+t_1t_3,\quad a't_1^2+b't_1t_2+t_3^2. \tag{46}$$

To reduce this, certain values of the coefficients must be excepted. For $b=b'=0$, the net is easily seen to reduce to XV. For $b=0$, $b'\neq 0$, we may write $t_3=b''t_3'$, where $b'=m'b''^2$, $(m'=1$ or $v)$, and we obtain

$$t_1^2+mt_2^2,\quad t_1t_3,\quad a't_1^2+2m't_1t_2+t_3^2;$$

and finally, setting $m't_1=t_1'$, the net becomes

$$\text{XVI.}\quad t_1^2+\lambda t_2^2,\quad 2t_1t_3,\quad \gamma t_1^2+2t_1t_2+t_3^2, \tag{47}$$
$$(\lambda=1,\ v,\ v^2, \text{ or } v^3;\ \gamma \text{ arbitrary}).$$

In (46), for $b\neq 0$ set $t_3=bt_3'$ and cancel t_1t_2 from C_3: $t_1^2+mt_2^2$, $t_1t_3+t_1t_2$, $a't_1^2+b't_1t_3+t_3^2$. In this complete the square to eliminate t_1t_3 from C_3, and obtain

$$\text{XVII.}\quad t_1^2+mt_2^2,\quad \delta t_1^2+2t_1t_2+2t_1t_3,\quad \varepsilon t_1^2+t_3^2, \tag{48}$$
$$(\delta,\ \varepsilon \text{ arbitrary, except } \varepsilon\neq 0).$$

15. There remains to be considered, from the net (37), the case where the determinant of the transformation (46), $\frac{d^2}{m}+1$, vanishes. For this case $t_1=\frac{1}{2}(t'_1+t'_2)$, $t'_2=\frac{1}{2d}(t'_1-t'_2)$, and easy parameter changes, give t_1t_2, $at_1^2+bt_2^2+t_1t_3$, $a't_1^2+b't_2^2+t_3^2$; and this, by $t'_3=at_1+t_3$, becomes

$$t_1t_2,\quad bt_2^2+t_1t_3,\quad a't_1^2+b't_2^2+t_3^2. \tag{49}$$

If in (49) $b\neq 0$, we obtain, on multiplying t_1 and t_3 by suitably chosen constants,

$$t_1t_2,\quad t_2^2+t_1t_3,\quad at_1^2+\lambda t_2^2+t_3^2,\qquad (\lambda=0,\ 1,\ \text{or } v).$$

If $\lambda=0$, permute t_2 and t_3, and multiply these letters to reduce the net to

$$t_1^2+mt_2^2,\quad t_1t_3,\quad t_3^2+m't_1t_2,$$

which is XVI for $\gamma=0$. If $\lambda\neq 0$, by $C'_3=C_3-\lambda C_2$, and then completing the square on $t_3^2-\lambda t_1t_3$ in C'_3, we have $t_3^2+at_1^2$, t_1t_2, $\frac{\lambda}{2}t_1^2+t_1t_3+t_2^2$. Permute here t_3 and t_2, multiply these letters by suitably chosen constants, and get

$$t_1^2+mt_2^2,\quad t_1t_3,\quad \rho t_1^2+2m't_1t_2+t_3^2,$$

which is XVI.

If in (49) $b=0$, by obvious changes the net becomes

$$t_1t_2,\quad t_1t_3,\quad \lambda t_1^2+\lambda' t_2^2+t_3^2,\qquad (\lambda,\ \lambda'=0,\ 1,\ \text{or } v). \tag{50}$$

$\lambda=\lambda'=0$ is excluded; otherwise C_3 is a unary form. $\lambda=0$ gives XIII. $\lambda'=0$ gives XII. $\lambda=\lambda'=1$ or v, gives, by $C'_3=2C_1+C_3$, t_1t_2, t_1t_3, $(t_1+t_2)^2+mt_3^2$; this, by $t'_2=t_1+t_2$, permuting t_1 and t_3 in the result, and eliminating t_2t_3, becomes

$$t_1^2+mt_2^2,\quad t_1t_2+t_1t_3,\quad mt_1^2+t_3^2,$$

which is included in XVII. $\lambda=1$, $\lambda'=v$ is equivalent to $t_1^2+t_2^2+vt_3^2$, t_1t_2, t_1t_3. By $C'_1=2C_2+C_1$, $t'_2=t_1+t_2$, and an interchange of t_1 and t_3 in the result:

$$t_1^2+vt_2^2,\quad t_1t_3,\quad t_3^2+t_2t_3,$$

which is XVI for $\gamma=0$. $\lambda=v$, $\lambda'=1$, gives

$$vt_1^2+t_2^2+t_3^2,\quad t_1t_2,\quad t_1t_3;$$

this net is equivalent to XVII for $\delta=0$, $\epsilon=m=1$ by the transformation

$$t_1=b_3t'_2-b_2t'_3,\quad t_2=t'_1+b_2t'_2+b_3t'_3,\quad t_3=t'_1-b_2t'_2-b_3t'_3,$$

where $v(b_2^2+b_3^2)=1$, applied to XVII.

16. *Independence of the Nets of Part* II. To use the discriminant to its full value in separating the nets, we examine for what cases the discriminants

of XIV, XVI, and XVII are factorable. To illustrate the method, the details are given for the case of XIV.

$D=2yz^2+mxy^2-z^2(x+\beta z)$. Set $y=rx+sz$ in this and simplify. D becomes

$$mr^2x^3+2mrsx^2z+(2r-1+ms^2)xz^2+(2s-\beta)z^3.$$

In order that this vanish identically, $r=0$, $s^2=\dfrac{1}{m}$, $\beta=2s$; then $m=1$, $s=\pm 1$, $\beta=\pm 2$; and for this value of β, $y\pm z$ is a factor of $D=(y\pm z)(2z^2+xy\mp xz)$. There is evidently no factor free of y.

Examining similarly the discriminant of XVI, we find that it is never factorable.

The discriminant of XVII, $D=mx^2z-mxy^2+m\varepsilon xz^2+m\delta xyz-y^2z$, is found by the above method to have a factor in case $\delta=0$, $\varepsilon=\dfrac{1}{m}$; in fact, for these values $D=(xz-y^2)(mx+z)$. The number of binary forms in this case is $2p^n(p^n-1)$ or $2(p^n+1)(p^n-1)$, according as $-m$ is a square or a not-square. This number is the same as that for XV, but the two nets may be shown to be distinct by the method of § 11.

The number of binary forms is calculated as in § 9. For nets XVI and XVII this number appears to be so difficult to calculate, that it seems best to take up the question of their independence from another point of view. There are, of course, no unary forms in these nets.

It is seen at once that XII, XIII, XIV, and XV are independent of each other even for the factorable case of the discriminant of XIV. Moreover, XII, XIII, and XIV are independent of XVI and XVII. This leaves for examination the relations between the nets XIV, XVI, and XVII.

17. *Relations between* XIV *and* XVI. To examine this we proceed to attempt the transformation of the discriminant of one into that of the other multiplied by a square. To this end a device is used which depends on the following lemma:*

LEMMA. *If a ternary cubic form $f(x, y, z)$ becomes $F(X, Y, Z)$ under the linear transformation*

$$x=AX+BY+CZ,\quad y=A_1X+B_1Y+C_1Z,\quad z=A_2X+B_2Y+C_2Z, \tag{51}$$

and u denotes $f(A, A_1, A_2)$, then

$$\frac{1}{2}\frac{\partial^2 F}{\partial X^2}=x\frac{\partial u}{\partial A}+y\frac{\partial u}{\partial A_1}+z\frac{\partial u}{\partial A_2}. \tag{52}$$

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* This Lemma and its application are due to Professor Dickson.

Proof: If we set

$$u_B = B\frac{\partial u}{\partial A} + B_1\frac{\partial u}{\partial A_1} + B_2\frac{\partial u}{\partial A_2}, \quad u_C = C\frac{\partial u}{\partial A} + C_1\frac{\partial u}{\partial A_1} + C_2\frac{\partial u}{\partial A_2},$$

we have

$$F = uX^3 + u_B X^2 Y + u_C X^2 Z + \ldots,$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial X^2} = 3uX + u_B Y + u_C Z = (AX+BY+CZ)\frac{\partial u}{\partial A}$$

$$+ (A_1X+B_1Y+C_1Z)\frac{\partial u}{\partial A_1} + (A_2X+B_2Y+C_2Z)\frac{\partial u}{\partial A_2},$$

since by Euler's theorem of homogeneous functions

$$3u = A\frac{\partial u}{\partial A} + A_1\frac{\partial u}{\partial A_1} + A_2\frac{\partial u}{\partial A_2}. \quad \text{Q. E. D.}$$

Replace, in the discriminant of XVI, γ by $2\sigma/\lambda$; it then becomes

$$f = -z^3 + 2\sigma x z^2 + \lambda x^2 z - \lambda x y^2, \qquad (\lambda \neq 0). \tag{53}$$

Suppose that under the transformation (51) f becomes

$$F = d^2(2YZ^2 + mXY^2 - XZ^2 - \beta Z^3), \tag{54}$$

which differs from the discriminant of XIV only by a square factor. Since F is linear in X, (52) must vanish, so that $\frac{\partial u}{\partial A} = \frac{\partial u}{\partial A_1} = \frac{\partial u}{\partial A_2} = 0$; we have then $2\sigma A_2^2 + 2\lambda A A_2 - \lambda A_1^2 = 0$, $AA_1 = 0$, $-3A_2^2 + 4\sigma A A_2^2 + \lambda A^2 = 0$. Let the modulus p^n exceed 3. In view of the determinant of (51), A, A_1, and A_2 can not all be 0; hence, taking $A_1=0, A_2 \neq 0, A = -\sigma\lambda^{-1}A_2$, whence $\lambda = -\sigma^2, A_1 = 0, A = \sigma^{-1}A_2 \neq 0$. In view of the last relation, $z_1 = z - \sigma x$ is free of X. Hence we set

$$x = x_1, \quad y = y_1, \quad z = z_1 + \sigma x_1. \tag{55}$$

Then $f = -z_1^3 - \sigma x_1 z_1^2 + \sigma^2 x_1 y_1^2$, and

$$x_1 = AX+BY+CZ, \quad y_1 = bY+cZ, \quad z_1 = rY+sZ. \tag{56}$$

Now $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial X} + \ldots = A(\sigma^2 y_1^2 - \sigma z_1^2), \frac{\partial F}{\partial X} = d^2(mY^2 - Z^2)$. Since these binary forms must be equivalent, the ratio of their discriminants must be a square. Thus σ/m must be a square; but as in XIV m may be multiplied by any square, we may set $m = \sigma$. By the theory of binary forms the only transformations replacing $\sigma A(\sigma y_1^2 - z_1^2)$ by $d^2(\sigma Y^2 - Z^2)$ are

$$y_1 = bY + cZ, \quad z_1 = \pm\sigma c Y \pm bZ, \tag{57}$$

where $\sigma A(b^2 - \sigma c^2) = d^2$.

Set in (56) $r = \pm\sigma c$, $s = \pm b$, and noting the relation (57), transform f and equate to F. On reducing the conditions we find that the transformation

will replace f by F if, and only if,

$$d^2B=\pm A\sigma^2c^3,\quad d^2C=\pm 3A\sigma bc^2,\quad d^2\beta=\pm b^3\pm 3\sigma bc^2,\quad d^2=\mp\frac{1}{2}\sigma c(3b^2+\sigma c^2). \qquad (58)$$

The simplest values of b and c which make the final expression a square, are $b=0$, $c=-2$, with the upper signs holding. For this choice (57) and (58) give $d^2=4\sigma^2$, $A=-1$, $B=2$, $C=0$, $\beta=0$. Hence by (55) and (56)

$$x=-X+2Y,\quad y=-2Z,\quad z=-\sigma X. \qquad (59)$$

Now the net XIV, with $m=\sigma$, $\beta=0$, is $X(t_1^2-\sigma t_2^2)+2Yt_1t_3+2Z(t_1t_2+t_2t_3)$, while the net XVI, with $\lambda=-\sigma^2$ and γ replaced by $2\sigma/\lambda$ (as above), is

$$x(t_1'^2-\sigma^2t_2'^2)+2yt_1't_2'+z(-2\sigma^{-1}t_1'^2+2t_1't_2'+t_3'^2).$$

If these are equivalent under the correspondence (59), the coefficients of X, $2Y$, and $2Z$ must be equivalent:

$$t_1^2-\sigma t_2^2=t_1'^2-2\sigma t_1't_2'+\sigma^2t_2'^2-\sigma t_3'^2,\quad t_1t_3=t_1'^2-\sigma^2t_2'^2,\quad t_1t_2+t_2t_3=-2t_1't_3'.$$

Hence, apart from constant factors, $t_1=t_1'-\sigma t_2'$, $t_2=-t_3'$, $t_3=t_1'+\sigma t_2'$. Thus a net of XVI is equivalent to some net of XIV if, and only if, $\lambda\gamma^2=-4$ in XVI.

18. *Relation between* XIV *and* XVII. By examining these nets in exactly the same way as XIV and XVI were examined in § 17, we arrive at the result: A net of XVII is equivalent to some net of XIV if, and only if, in XVII $m=-1$, and $4\epsilon+(\delta\pm 2)^2=0$.*

19. *Relation between* XVI *and* XVII. The invariants, S and T, of a cubic form furnish the absolute invariant, S^3/T^2. For brevity write $\sigma=\lambda\gamma^2$ in XVI and $Q=\delta\epsilon-4\delta-4\epsilon$ in XVII. The value of this invariant for the two discriminants is as follows:

$$D_{\text{XVI}}:\quad \frac{S^3}{T^2}=-\frac{(3+\sigma)^3}{16(9+2\sigma)^2},$$

$$D_{\text{XVII}}:\quad \frac{S^3}{T^2}=-\frac{(Q^2-48\delta\epsilon)^3}{64Q^2(Q^2-72\delta\epsilon)^2}$$

(first multiplying t_2 and t_3 in XVII by δ). A necessary condition for the equivalence of the nets is the equivalence of these two invariants. If we put $a=\sigma+4$, $b=Q^2-64\delta\epsilon$, $c=-16\delta\epsilon$, and then $a'=b/c$, this condition becomes

$$\frac{(a'-1)^3}{(a'-4)(a'+1/2)}=\frac{(a-1)^3}{(a-4)(a+1/2)},$$

* The cases of equivalence between the nets XIV, XVI, and XVII just found receive a new interpretation when the discriminants of the nets are regarded as cubic curves in finite geometry. The discriminant of XIV put equal to 0 is a cubic with a double point; that of XVI has a double point if $\lambda\gamma^2+4=0$, and that of XVII if $m=-1$, $4\epsilon+(\delta\pm 2)^2=0$. The conditions derived by use of the lemma, namely,

$$\frac{\partial u}{\partial A}=\frac{\partial u}{\partial A_1}=\frac{\partial u}{\partial A_2}=0,$$

are obviously equivalent to the conditions for a double point of the cubic.

where $a=0$, $a'=0$ are, respectively, the conditions for a double point of the cubic curves $D_{XVI}=0$, $D_{XVII}=0$.

If we attack the problem of direct transformation of the net XVI into XVII, it is possible to express the conditions to which the parameters are subjected for equivalence, but these conditions are in an exceedingly long and awkward form.

PART III. THE DISCRIMINANT OF THE NET IS OF RANK 3.

20. That nets exist which contain neither a unary nor a binary form may be proved from the results of a paper by Professor Dickson.* Setting out from the theorem that any field which contains an irreducible cubic $f(r)=0$ has for the norm of the function $x+ry+r^2z$ a ternary cubic form which vanishes in the field only for $x=y=z=0$, the conditions are determined that the general cubic shall have this property. We shall apply these conditions as they are derived from a simplified form of the cubic, when, namely, by an obvious transformation, the terms in x^2y, x^2z, and xyz have been made to disappear. Then, for the cubic

$$a'x^3+d'xy^2+e'xz^2+g'y^3+h'y^2z+k'yz^2+l'z^3, \tag{60}$$

the *necessary and sufficient conditions that this vanish in the field only for* $x=y=z=0$ *are:*

$$\left.\begin{array}{ll} \text{(A)} & a'q^3+d'q+g'=0 \text{ shall be reducible in the } GF(p^n), \\ \text{(B)} & a'd'h'\neq 0, \\ \text{(C)} & d'k'+3e'g'=0, \\ \text{(D)} & e'h'+3d'l'=0, \\ \text{(E)} & 4d'^2e'+3a'h'^2-9a'g'k'=0. \end{array}\right\} \tag{61}$$

21. *Reduction of the Net.* The family of two ternary quadratics which contains no binary form may be readily reduced to $C_1=t_1^2-t_2t_3$, $C_2=2t_1t_2+t_2^2+at_3^2$, where a has a fixed value not 0. Adjoin now a third form which by hypothesis can not be reduced to a binary form; by parameter change it may be written $C_3=2bt_1t_3+ct_1^2+t_2^2+dt_3^2$, the coefficient of t_2^2 being put equal to 1, since it can not vanish, as C_3 is not a binary.

22. The discriminant of the net

$$x(t_1^2-2t_2t_3)+y(2t_1t_2+t_2^2+at_3^2)+z(2bt_1t_3+ct_1^2+t_2^2+dt_3^2) \tag{62}$$

is

$$D=\begin{vmatrix} x+cz & y & bz \\ y & y+z & -x \\ bz & -x & ay+dz \end{vmatrix} =x^3+cx^2z-axy^2-dxz^2-(a+d-2b)xyz+ay^3 \\ +(d-ac)y^2z-(ac+cd-b^2)yz^2+(b^2-cd)z^3. \tag{63}$$

In order to apply the conditions (61) cited above on this cubic, it must be so

* "Triple Algebras and Cubic Forms," *Bulletin of the American Mathematical Society*, January, 1908.

transformed that the terms in x^2z and xyz shall disappear without introducing the term x^2y. Write in D $x=x'-cz'/3$, $y=y'$, $z=z'$, and the term x^2z will disappear. The terms in $x'y'^2$ and $x'y'z'$ will become $-ax'y'^2-(a+d-2b)x'y'z'$; then by $x'=x''$, $y'=y''-(a+d-2b)z''/2a$, $z'=z''$, the term in $x'y'z'$ will disappear. The result of the two transformations is the cubic (60), where now

$$\left.\begin{aligned}
a'&=1,\\
d'&=-a,\\
e'&=(3a^2-12ab-4ac^2-6ad+12b^2-12bd+3d^2)/12a,\\
g'&=a,\\
h'&=(-9a-4ac+18b-3d)/6,\\
k'&=(3a^2-12ab+4ab^2-8abc+2ad+12b^2-4bd-d^2)/4a,\\
l'&=(-27a^3+36a^3c+162a^2b+108a^2b^2+16a^2c^3\\
&\qquad+72a^2bc-27a^2d-72a^2cd-324ab^2+216ab^3-288ab^2c\\
&\qquad+72abcd+108abd-108ab^2d+36acd^2+27ad^2\\
&\qquad+216b^3-108b^2d-54bd^2+27d^3)/216a^2.
\end{aligned}\right\}\quad(64)$$

For these values of $a', \ldots, l'$, the conditions (61) become

$$\left.\begin{aligned}
&\text{(A)}\ \ q^3-aq+a=0 \text{ shall be irreducible,}\\
&\text{(B)}\ \ a(9a+4ac-18b+3d)\neq 0,\\
&\text{(C)}\ \ ab^2-2abc+ac^2+2ad+2bd-d^2=0,\\
&\text{(D)}\ \ 4a^3c+9a^2b^2+2a^2bc-3a^2c^2-8a^2cd-6a^2d\\
&\qquad\quad+18ab^3-20ab^2c-9ab^2d+6abc^2+2abcd\\
&\qquad\quad+6abd-ac^2d+4acd^2+3ad^2+12b^2d-12bd^2+3d^3=0,\\
&\text{(E)}\ \ a^3-4a^2b+6a^2c-2a^2d-5ab^2+6abc-4abd+2acd+ad^2+3d^2=0.
\end{aligned}\right\}\quad(65)$$

23. *Reduction of the Conditions* (65). With regard to condition (A), it is easily seen that an irreducible cubic exists for every field. In fact, when $q^3-aq+a=0$ is reducible, then a $(\neq 0)$ may be expressed as $q^3/(q-1)$, and conversely. But there can not be more than p^n-2 such values (since $q\neq 0, 1$); that is, there is at least one value of a which will make the equation irreducible. (See § 25.)

The key to the simplification of the conditions (65) (C), (D), and (E) lies in (C), which may be written:

$$a(b-c)^2=d(d-2a-2b). \tag{66}$$

Suppose, first, $d=0$. Then from (C), since $a\neq 0$, $b=c$, while (D) and (E) alike reduce to $(a+b)^2=0$. That is

$$b=c=-a, \tag{67}$$

while (B) becomes $4a-27\neq 0$. Here, then, in the case $d=0$, there is a single net whose fundamental forms are

$$t_1^2-2t_2t_3,\quad 2t_1t_2+t_2^2+at_3^2,\quad -2at_1t_3-at_1^2+t_2^2, \tag{68}$$

where a is so chosen that $a\neq 0$, $4a-27\neq 0$, and $q^3-aq+a=0$ is irreducible.

Suppose, next, that $d\neq 0$ in (66). From (66) we may set $b-c=df$, whence

$$\left.\begin{aligned} 2b&=d-adf^2-2a,\\ 2c&=d-adf^2-2df-2a.\end{aligned}\right\} \tag{69}$$

For these values of b and c, (D) and (E) become, respectively,

$$\left.\begin{aligned} &a^3f^4+12a^2f^3+2a^2f^2-20af+a+12=0,\\ &(2a^3f^3-64a^2f^3-20a^2f^2+112af-5a-64)d-a^2(a^2f^3-af+1)=0;\end{aligned}\right\} \tag{70}$$

or, putting

$$af=z, \tag{71}$$

equations (70) take the form

$$\left.\begin{aligned} &z^4+12z^3+2az^2-20az+a^2+12a=0,\\ &[(2a-64)z^3-20az^2+112az-5a^2-64a]d-a^2(z^3-az+a)=0.\end{aligned}\right\} \tag{72}$$

24. *Discussion of the Quartic* (72). The quartic

$$z^4+12z^3+2az^2-20az+a^2+12a=0$$

has one, and but one, root in the field. In the paper to which reference has been made in § 11, the conditions for irreducibility of the cubic and quartic are explicitly set forth; in particular, it is there proved that a quartic has one, and but one, root in the field if the resolvent cubic is irreducible. Now the resolvent cubic of the quartic in question is

$$y^3-2ay^2+(-4a^2-288a)y+8a^3-448a^2-(12)^3a=0.$$

In order to apply the conditions for the irreducibility of the cubic, the term in y^2 is first eliminated. Set $y=2x$, and divide by 8 to simplify:

$$x^3-ax^2+(-a^2-72a)x+a^3-56a^2-216a=0.$$

In this set $x=X+a/3$:

$$X^3-(72a+4a^2/3)X+16a^3/27-80a^2-216a=0. \tag{73}$$

The discriminant of this cubic is

$$a^24^3(64a^3-9^2\cdot 16a^2-6\cdot 27\cdot 54a-(27)^3=4^3a^2(4a-27)^3. \tag{74}$$

Now the cubic $x^3-ax+a=0$ is known to be irreducible, and the two conditions for this (see § 11, end) are

$$\left.\begin{aligned} 4a-27&=\text{ a square }=81e^2\text{ (say)},\\ \frac{1}{2}(-a+ae\sqrt{-3})&=\text{ a not-cube in }[GF(p^n),\ \sqrt{-3}].\end{aligned}\right\} \tag{75}$$

Applying the conditions of irreducibility to (73):

$$\left.\begin{aligned} 4^3a^2(4a-27)^3 &= \text{a square } = 81(8\cdot 81\cdot ae^3)^2, \\ \frac{1}{2}(-b+8\cdot 81\cdot ae^3\cdot\sqrt{-3}) &= \text{a not-cube in } [GF(p^n),\ \sqrt{-3}], \end{aligned}\right\} \quad (76)$$

where $b=16a^3/27-80a^2-216a$. Using the second of (75), the second of (76) is equivalent to the statement

$$(-b+8\cdot 81\cdot ae^3\cdot\sqrt{-3})/(-a+ae\cdot\sqrt{-3}) = \text{a cube in } [GF(p^n),\ \sqrt{-3}].$$

Multiplying numerator and denominator by $-a-ae\sqrt{-3}$ and reducing, this fraction becomes $36a-36\cdot 27+\sqrt{-3}\cdot e(4a-28\cdot 27)$; but this is equal to

$$9\cdot 81e^2-(27)^2+\sqrt{-3}\cdot e[81e^2-(27)^2]=3^6(-1-e\sqrt{-3}/3)^3.$$

It is thus seen that the cubic resolvent is irreducible, and hence the quartic (72) has one, and but one, root in the field.

25. *Determination of the Coefficients from the Remaining Conditions.* Set the single root of the quartic (72) in the cubic (72), and the value of d is uniquely determined, provided the coefficient of d in that equation does not vanish. By hypothesis the term free of d, $(z^3-az+a)a^2$, is different from 0. Having determined d, (71) and (69) give f, b, and c uniquely. Condition (B), (65), becomes, as $a\neq 0$, $a^2(27-4a)+d(7z^2-4az+2a^2-6a)\neq 0$, so that in this case also, $4a-27\neq 0$.

Denote the coefficient of d in the cubic of (72) by C, and the quartic of (72) by Q, and examine

$$Q=z^4+12z^3+2az^2-20az+a^2+12a=0,$$
$$C=(2a-64)z^3-20az^2+112az-5a^2-64a=0,$$

to determine whether or not they have a common root. The first step in the highest-common-factor process leads to the identity

$$2(a-32)^2Q-[(a-32)z+(22a-384)]C=a^2T,$$

where

$$T=(4a+72)z^2-35az+2a(a+3).$$

If Q and C have a common factor, then C and T will have this same factor, and conversely. We seek, then, the eliminant of C and T which will vanish for a common root. By Sylvester's method of elimination this invariant has the following determinant form:

$$\begin{vmatrix} 4a+72 & -35a & 2a(a+3) & 0 & 0 \\ 0 & 4a+72 & -35a & 2a(a+3) & 0 \\ 0 & 0 & 4a+72 & 35a & 2a(a+3) \\ 2a-64 & -20a & 112a & -5a^2-64a & 0 \\ 0 & 2a-64 & -20a & 112a & -5a^2-64a \end{vmatrix}.$$

From the last two columns a may be removed as a factor. It is then easy to see that the highest power of a that can occur is the sixth.

To complete the proof, we shall show that except for certain low values of p^n, a may be so chosen that the discriminant does not vanish. The choice of a was such as to make the cubic $q^3-aq+a=0$ irreducible.

26. *For how many values of a is the cubic $q^3-aq+a=0$ irreducible?* If this equation be reducible, $a=q^3/(q-1)$, q a mark of the $GF(p^n)$ not 0 nor 1. Let r be also one of these marks and determine for what values of q in terms of r, other than $q=r$, the following equation is satisfied:

$$q^3/(q-1)=r^3/(r-1).$$

Removing the factor $q-r$, we have $q^2(r-1)+q(r^2-r)=r^2$, or if $r\neq 1$, $(2q+r)^2=r^2(r+3)/(r-1)$. Hence, if $(r+3)/(r-1)$ is a square, not 0, there are exactly three of the marks which give the same a; namely, the value of r making $(r+3)/(r-1)$ a square, and the two values of q from the quadratic just written. If $(r+3)/(r-1)$ is a not-square, each corresponding r gives a distinct a. Finally, if $(r+3)/(r-1)=0$, that is $r=-3$, $q=3/2$, and there is an a different from all the rest; that is to say, the two marks -3, $3/2$ always give an a, and it is different from every other a. Let now r run through the sequence of all the marks of the field except 0, 1, 3/2, and -3. Then $(r+3)/(r-1)$ will take all these values except -3, 1, 9, and 0. If, first, -3 is a square, there are $(p^n-1)/2$ not-squares among the values of $(r+3)/(r-1)$, and hence as many a's. There are $(p^n-1)/2-3=(p^n-7)/2$ squares, and therefore $(p^n-7)/6$ a's, and in addition the a that results from the two values $3/2$, -3, of r; in all, $2(p^n-1)/3$ values of a which make the cubic reducible. If -3 is a not-square, there are $(p^n-3)/2$ not-squares $(r+3)/(r-1)$, and hence as many a's; $(p^n-5)/2$ squares, and hence $(p^n-5)/6$ a's, and the additional a; in all, $2(p^n-2)/3$ values of a for which the cubic will be reducible.

The cubic $q^3-aq+a=0$ will be irreducible for $(p^n=1)/3$ or for $(p^n-2)/3+1$ values according as -3 is a square or a not-square in the field. This number will exceed 6, and hence the discriminant in § 24 be not zero if p^n is greater than 19.

27. It thus appears that there are always two nets in which no binary forms occur: one for $d=0$,

$$\text{XVIII.}\qquad t_1^2-2t_2t_3,\quad 2t_1t_2+t_2^2+at_3^2,\quad -2at_1t_3-at_1^2+t_2^2\,; \tag{77}$$

and one for which $d\neq 0$,

$$\text{XIX.}\qquad t_1^2-2t_2t_3,\quad 2t_1t_2+t_2^2+at_3^2,\quad 2bt_1t_2+ct_1^2+t_2^2+dt_3^2, \tag{78}$$

where a is fixed, not 0 nor 27/4, $q^3-aq+a=0$ irreducible, and b, c, and d determined uniquely by (69), ..., (72).

28. *The case of* $p^n=3k+1$ deserves especial notice on account of the simple form which the conditions take. In fact, for this field the family of two forms may be reduced to $x(t_1^2-2t_2t_3)+y(2ut_1t_2-t_3^2)$, u being a particular not-cube.* The net of three forms may readily be put in the form:

$$t_1^2-2t_2t_3,\quad 2ut_1t_2+t_3^2,\quad 2rt_1t_3+st_1^2+t_2^2+tt_3^2. \tag{79}$$

The discriminant of this net is

$$x^3+sx^2z-txz^2+(2ur-1)xyz+u^2y^3+u^2ty^2z-syz^2+(r^2-st)z^3.$$

The conditions that this cubic vanish only for $x=y=z=0$, are best expressed directly as obtained on page 163 of the *Bulletin of the American Mathematical Society*, January, 1908, by Professor Dickson:

$$\left.\begin{array}{ll}(\mathrm{A}') & 2utr+2t+s^2=0,\\ (\mathrm{B}') & 2s+u^2s^2t-4u^2r^2s-2urs+3u^2t^2=0,\\ (\mathrm{C}') & 3s^2+8u^2st^2-5u^2r^2t-4urt+t=0,\\ (\mathrm{D}') & 2ur-1\neq 0.\end{array}\right\} \tag{80}$$

By $(\mathrm{C}')-3(\mathrm{A}')$:

$$t(8u^2st-5u^2r^2-10ur-5)=0. \tag{81}$$

Hence, either $t=0$ or $8u^2st-5u^2r^2-10ur-5=0$. If $t=0$, by (A′) $s=0$; this brings us to an alternative condition (page 164 of above citation),

$$27u^2r^2=-(2ur-1)^3;$$

or, setting $ur=w$:

$$8w^3+15w^2+6w-1=(w+1)^2(8w-1)=0, \tag{82}$$

whence $r=-1/u$ or $r=1/8u$. Eliminating s^2 from (B′) and (A′):

$$2s-4u^2r^2s-2urs+u^2t^2-2u^3t^2r=(1-2ur)(2s+2urs+u^2t^2)=0.$$

Therefore, $2s(1+ur)=-u^2t^2$. Squaring this and eliminating s^2 by (A′), $-8(1+ur)^3t=u^4t^4$; and if $t\neq 0$, $[-2(1+ur)/ut]^3=u$, which is impossible. Hence $t=0$, and we have the nets:

$$\left.\begin{array}{lll} t_1^2-2t_2t_3, & t_3^2+2ut_1t_2, & t_2^2-(2/u)\cdot t_1t_3,\\ t_1^2-2t_2t_3, & t_3^2+2ut_1t_2, & t_2^2+(1/4u)\cdot t_1t_3.\end{array}\right\} \tag{83}$$

In each case the condition (D′) requires that the modulus be not 3.

29. *Summary.* The net $R=xC_1+yC_2+zC_3$ of ternary forms

$$C_i=a_it_1^2+2h_it_1t_2+b_it_2^2+2g_it_1t_3+2f_it_2t_3+c_it_3^2,$$

in the $GF(p^n)$, has been reduced to nineteen canonical types: namely, I, ..., XI, which contain a unary form; XII, ..., XVII, which contain a binary form but no unary; XVIII, XIX, which contain neither unary nor binary forms. All questions of inter-relations between these types have been considered and answered, except with respect to the two cases, nets XVI and XVII, and nets XVIII and XIX.

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*Dickson, *Quarterly Journal*, § 8, No. 156, 1908. The examination of the net for $p^n=3k+1$ was also made by Professor Dickson.

TABLE

	C_1	C_2	C_3
I	t_1^2	$2t_1t_2$	$2t_1t_3$
II	t_1^2	$2t_1t_2$	$t_2^2+2t_1t_3$
III	t_1^2	t_2^2	t_3^2
IV	t_1^2	t_2^2	$2t_1t_3$
V	t_1^2	$2t_1t_2$	$2t_2t_3$
VI	t_1^2	t_2^2	$2(t_1t_3+t_2t_3)$
VII	t_1^2	$2t_1t_2$	$t_2^2+mt_3^2$
VIII	t_1^2	$2t_2t_3$	$t_2^2+vt_3^2$
IX	t_1^2	$2t_2t_3$	$t_2^2+2t_1t_3$
X	t_1^2	$2(t_1t_3+t_2t_3)$	$t_2^2+vt_3^2$
XI	t_1^2	$2(t_1t_2+t_1t_3+t_2t_3)$	$t_2^2+\alpha t_3^2$
XII	$t_1^2+vt_2^2$	$2t_1t_2$	$2t_1t_3$
XIII	$t_1^2-mt_2^2$	$2t_1t_3$	$2t_2t_3$
XIV	$t_1^2-mt_2^2$	$2t_1t_3$	$\beta t_1^2+2t_1t_2+2t_2t_3$
XV	$t_1^2+vt_2^2$	$2t_1t_2$	$t_1^2+mt_3^2$
XVI	$t_1^2+\lambda t_2^2$	$2t_1t_3$	$\gamma t_1^2+2t_1t_2+t_3^2$
XVII	$t_1^2+mt_2^2$	$\delta t_1^2+2t_1t_2+2t_1t_3$	$\epsilon t_1^2+t_3^2$
XVIII	$t_1^2-2t_2t_3$	$2t_1t_2+t_2^2+at_3^2$	$-2at_1t_3-at_1^2+t_2^2$
XIX	$t_1^2-2t_2t_3$	$2t_1t_2+t_2^2+at_3^2$	$2bt_1t_2+ct_1^2+t_2^2+dt_3^2$

OF NETS.

D	B	U
0	$p^n(p^n+1)(p^n-1)$	p^n-1
z^3	$p^n(p^n-1)$	p^n-1
xyz	$3(p^n-1)^2$	$3(p^n-1)$
yz^2	$(2p^n-1)(p^n-1)$	$2(p^n-1)$
xz^2	$2p^n(p^n-1)$	p^n-1
$z^2(y-x)$	$(2p^n-1)(p^n-1)$	$2(p^n-1)$
$mz(xz-y^2)$	$2p^n(p^n-1)$	p^n-1
$x(vz^2-y^2)$	$(p^n+1)(p^n-1)$	p^n-1
z^3-xy^2	$p^n(p^n-1)$	p^n-1
$x(vz^2-y^2)-y^2z$	$(p^n+1)(p^n-1)$	p^n-1
$2y^3-(\alpha+1)y^2z-xy^2+\alpha xz^2$	for $\alpha=1$ $2(p^n-1)^2$	$2(p^n-1)$
	for $\alpha=\sigma^2\neq 1$ $(p^n-1)^2$	p^n-1
	for $\alpha=v$ $(p^n+1)(p^n-1)$	p^n-1
vxz^2	$(2p^n+1)(p^n-1)$	0
$x(my^2-z^2)$	for $m=1$ $3p^n(p^n-1)$	0
	for $m=v$ $(p^n+2)(p^n-1)$	
$2yz^2+mxy^2-z^2(x+\beta z)$	for $m=1$, $\beta^2=4$ $2p^n(p^n-1)$	0
	for $m=1$, $\beta^2\neq 4$ $p^n(p^n-1)$	
	for $m=v$ $(p^n+2)(p^n-1)$	
$mz(vx^2-vxy-y^2)$	$2(p^n+1)(p^n-1)$	0
$-z^3+\lambda\gamma xz^2+\lambda x^2z-\lambda xy^2$	?	0
$m\epsilon xz^2+z(mx^2+m\delta xy-y^2)-mxy^2$	?	0
....	0	0
....	0	0

Remarks. In the above table:

$m=1$ or v, a particular not-square.

$\lambda=1$, v, v^2, or v^3.

α, β, γ, δ, ε are arbitrary parameters, except:

(i) α, ε are not 0;

(ii) net XVI is equivalent to XIV if $\lambda\gamma^2+4=0$;

(iii) net XVII is equivalent to XIV if $m=-1$, $4\varepsilon+(\delta\pm 2)^2=0$.

a, in nets XVIII and XIX, is fixed, not 0 nor 27/4, and $q^3-aq+a=0$ is irreducible.

b, c, and d, in XIX, are determined uniquely by (69), ..., (72).

For convenience in tabulating the number of binary forms, B, in XIII and XIV, we write in these nets $-m$ for m.

On Long Waves.

By J. H. M. Wedderburn.

The present paper deals with the theory of long waves in a canal of variable section. As it is easily shown* that any case of variable section can be replaced by one in which the depth alone varies, it is assumed throughout that the breadth is constant.

The main object of the paper is to obtain special cases in which the hydrodynamical equations admit of simple solution for the case of *progressive* waves; for most of the known soluble cases, apart from the case of constant depth, refer either to motions of an oscillatory character or to a reflected train of waves. It is also shown that the problem of waves in a basin which is symmetrical about a vertical axis reduces in a simple manner to the one-dimensional case.

The ordinary approximate dynamical equations are assumed without any attempt to discuss their validity, although, so far as I know, this has never been done in a completely satisfactory manner. From the mathematical standpoint, then, our problem is merely the discussion of the solution of a certain partial differential equation. The treatment of the initial conditions which is given involves the discussion of a type of expansion in series which is in itself of considerable interest and seems capable of further extension.

1. *The Equations of Motion.* Let the x-axis be taken along the canal and the y-axis vertically upwards, the origin being at the undisturbed level of the liquid. As usual the displacements, parallel to the axes, of a point on the surface are denoted by ξ and η, while $h=h(x)$ represents the depth reckoned positive for points below the undisturbed level of the surface.

For long waves the dynamical equation then is†

$$\frac{\partial^2 \xi}{\partial t^2}=-g\frac{\partial \eta}{\partial x}, \tag{1}$$

and the equation of continuity is

$$\eta=-\frac{\partial(h\xi)}{\partial x}=-\frac{\partial w}{\partial x}, \tag{2}$$

where

$$w=h\xi. \tag{3}$$

* See for instance Lamb, "Hydrodynamics," 3d ed., p. 257; and Chrystal, "On the Hydrodynamical Theory of Seiches," *Transactions of the Royal Society of Edinburgh*, Vol. XLI, p. 614.

† Cf. Lamb, *loc. cit.*, p. 240.

If η is eliminated between (1) and (2), we get

$$\frac{\partial^2 w}{\partial t^2}=gh\frac{\partial^2 w}{\partial x^2}; \tag{4}$$

and if ξ is eliminated,

$$\frac{\partial^2 \eta}{\partial t^2}=g\frac{\partial}{\partial x}\left(h\frac{\partial \eta}{\partial x}\right), \tag{5}$$

which, if we set $\lambda=\int dx/h$, becomes

$$\frac{\partial^2 \eta}{\partial t^2}=\frac{g}{h}\frac{\partial^2 \eta}{\partial \lambda^2}, \tag{6}$$

which is of the same form as (4).

Although equations (4) and (5) have both been frequently used, the parallelism between them seems generally to have escaped notice. It is, however, of importance; for if w is a solution of (4) for a certain law of depth $h=\chi(x)$, then $w_1(\lambda)=-\partial w/\partial x$ is, when expressed in terms of λ, a solution of

$$\frac{\partial^2 w_1}{\partial t^2}=g\psi\frac{\partial^2 w_1}{\partial \lambda^2},$$

where $\psi(\lambda)=1/\chi(x)$; so that $w_1(x)$ is a solution when the law of depth is $h=\psi(x)$. A repetition of this process evidently leads to the original solution, so that the relation between w and w_1 is a reciprocal one. They will therefore be called *reciprocal* solutions.

2. *Wave Propagation in Two Dimensions.* Let the x- and z-axes be taken on the undisturbed surface, and the y-axis vertically upwards as before. If a, b and c are the initial coordinates of the point x, y, z, and ξ, η, ζ the corresponding displacements, the Lagrangian equation of continuity,

$$\partial(x, y, z)/\partial(a, b, c)=1,$$

becomes for long waves

$$\frac{\partial \eta}{\partial b}=-\left(\frac{\partial \xi}{\partial a}+\frac{\partial \zeta}{\partial c}\right).$$

Since ξ and ζ are supposed independent of the depth, we have, on integrating,

$$\eta=-(b+h)\left(\frac{\partial \xi}{\partial a}+\frac{\partial \zeta}{\partial c}\right)-h(a+\xi, c+\zeta)+h(a, c),$$

which, if $\alpha=h\xi$, $\gamma=h\zeta$, gives to a first approximation

$$\eta=-b\left(\frac{\partial \xi}{\partial a}+\frac{\partial \zeta}{\partial c}\right)-\frac{\partial \alpha}{\partial a}-\frac{\partial \gamma}{\partial c}. \tag{7}$$

To the same order of approximation the Lagrangian equations for a point on the surface are

$$\frac{\partial^2\xi}{\partial t^2}=-g\frac{\partial\eta}{\partial a},\qquad \frac{\partial^2\zeta}{\partial t^2}=-g\frac{\partial\eta}{\partial c}$$

or

$$\frac{\partial^2\alpha}{\partial t^2}=-gh\frac{\partial\eta}{\partial a},\qquad \frac{\partial^2\gamma}{\partial t^2}=-gh\frac{\partial\eta}{\partial c}. \tag{8}$$

Eliminating η by means of (7),

$$\frac{\partial^2\alpha}{\partial t^2}=gh\Big(\frac{\partial^2\alpha}{\partial a^2}+\frac{\partial^2\gamma}{\partial a\partial c}\Big),\qquad \frac{\partial^2\gamma}{\partial t^2}=gh\Big(\frac{\partial^2\alpha}{\partial a\partial c}+\frac{\partial^2\gamma}{\partial c^2}\Big), \tag{9}$$

or eliminating α and γ,

$$\frac{\partial^2\eta}{\partial t^2}=g\Big(\frac{\partial}{\partial a}h\frac{\partial\eta}{\partial a}+\frac{\partial}{\partial c}h\frac{\partial\eta}{\partial c}\Big). \tag{10}$$

If the motion is symmetrical about the y-axis, we may set

$$\alpha=R\frac{a}{r},\qquad \gamma=R\frac{c}{r},$$

where R is a function of $r=\sqrt{a^2+c^2}$ alone. On making this substitution we find

$$\left.\begin{aligned}\frac{\partial^2 R}{\partial t^2}&=gh\Big(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}-\frac{1}{r^2}R\Big),\\ \frac{\partial^2\eta}{\partial t^2}&=g\Big(\frac{\partial}{\partial r}h\frac{\partial\eta}{\partial r}+\frac{h}{r}\frac{\partial\eta}{\partial r}\Big),\end{aligned}\right\} \tag{11}$$

which may also be put in the form

$$\frac{\partial^2 rR}{\partial t^2}=4ghr^2\frac{\partial^2 rR}{\partial(r^2)^2},\qquad \frac{\partial^2\eta}{\partial t^2}=g\frac{\partial}{\partial r^2}\Big(4hr^2\frac{\partial\eta}{\partial r^2}\Big), \tag{12}$$

which are of the same analytical form as the equations for one-dimensional waves. This latter form of the equations is derived directly if we consider the basin replaced by an indefinite number of canals each lying between two vertical planes which pass through the central axis, the angle between the planes being infinitesimal.

Besides the reciprocal transformation suggested by (6), the equations (12) are susceptible of another transformation which leaves the form unchanged. Let $s=1/r$; then

$$\frac{\partial^2 R}{\partial t^2}=gh\Big(\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}-\frac{1}{r^2}R\Big)=ghs^4\Big(\frac{\partial^2 R}{\partial s^2}+\frac{1}{s}\frac{\partial R}{\partial s}-\frac{1}{s^2}R\Big).$$

This suggests the more general homographic transformation discussed in the next paragraph.

3. Generalizing the transformation indicated above by combining with it a change of scale and origin, let us set

$$x'=\frac{ax+b}{cx+d},\qquad w'=\frac{1}{cx+d}w;$$

then, if $\Delta = ad - bc$,

$$\frac{\partial}{\partial x} = \frac{\Delta}{(cx+d)^2}\frac{\partial}{\partial x'}, \qquad \frac{\partial^2}{\partial x^2} = \frac{\Delta^2}{(cx+d)^4}\frac{\partial^2}{\partial x'^2} - \frac{2c\Delta}{(cx+d)^3}\frac{\partial}{\partial x'},$$

and therefore

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{\partial^2}{\partial x^2}(cx+d)w' = 2c\frac{\partial w'}{\partial x} + (cx+d)\frac{\partial^2 w'}{\partial x^2} \\ &= \frac{2c\Delta}{(cx+d)^2}\frac{\partial w'}{\partial x'} - \frac{2c\Delta}{(cx+d)^2}\frac{\partial w'}{\partial x'} + \frac{\Delta^2}{(cx+d)^3}\frac{\partial^2 w'}{\partial x'^2} \\ &= \frac{1}{\Delta}(a - cx')^3 \frac{\partial^2 w'}{\partial x'^2}; \end{aligned}$$

so that the equation becomes

$$\frac{\partial^2 w'}{\partial t^2} = h'\frac{\partial^2 w'}{\partial x'^2}, \tag{13}$$

where

$$h' = \frac{(cx'-a)^4}{\Delta^2}h, \tag{14}$$

h being expressed in terms of x'. The transformed equation (13) being of the same form as the original equation, we can derive by this method the solution for a considerable range of forms of canal from one particular case.

4. *Certain Soluble Cases.* Generalizing the solution for the case of constant depth, we shall now inquire under what conditions (4) has a solution of the form

$$w = f(x)F[\theta(x) \pm \sigma t],$$

F being an arbitrary function, but f and θ definite functions of their arguments. Substituting this value of w in (4) we get

$$ghf''F + gh(2f'\theta' + f\theta'')F' + f(gh\theta'^2 - \sigma^2)F'' = 0, \tag{15}$$

whence, if F is arbitrary,

$$f'' = 0, \qquad 2f'\theta' + f\theta'' = 0, \qquad gh\theta'^2 - \sigma^2 = 0, \tag{15$'$}$$

which on being integrated give

$$f = ax + b, \qquad \theta = 1/a(ax+b), \qquad h = \sigma^2(ax+b)^4/g, \tag{16}$$

where a is supposed not zero and one unnecessary constant of integration is suppressed.

When $a=1$ and $b=0$, the corresponding values of ξ and η are

$$\left.\begin{aligned} \xi &= \frac{g}{\sigma^2}x^{-3}F\left(\frac{1}{x} \pm \sigma t\right), \\ \eta &= F\left(\frac{1}{x} \pm \sigma t\right) - \frac{1}{x}F'\left(\frac{1}{x} \pm \sigma t\right). \end{aligned}\right\} \tag{17}$$

The reciprocal solution is more interesting, as it gives a law of depth more nearly realized in practice. Here

$$
\begin{aligned}
\lambda &= \int dx/h = g\int dx/\sigma^2x^4 = -g/3\sigma^2x^3,\\
\psi(\lambda) &= g/\sigma^2x^4 = (3^4\sigma^2\lambda^4/g)^{\frac{1}{3}},\\
\theta(x) &= 1/x = -(3\sigma^2\lambda/g)^{\frac{1}{3}}.
\end{aligned}
$$

Hence, with a slight change of notation, we have the solution

$$
\left.
\begin{aligned}
\xi &= x^{-\frac{4}{3}}F(x^{\frac{1}{3}}\pm\sigma t) - x^{-1}F'(x^{\frac{1}{3}}\pm\sigma t),\\
\eta &= \tfrac{1}{3}x^{-\frac{1}{3}}F''(x^{\frac{1}{3}}\pm\sigma t),
\end{aligned}
\right\} \tag{18}
$$

when $h=9\sigma^2x^{\frac{4}{3}}/g$.

The form of (18) suggests assuming a solution of the form

$$w=\Sigma f_nF^{(n)}(\theta\pm\sigma t). \tag{19}$$

As before, this leads to the following equations:

$$
\left.
\begin{aligned}
f_0'' &= 0,\\
f_1''+2f_0'\theta'+f_0\theta'' &= 0,\\
\dots\dots\dots\dots &,\\
\dots\dots\dots\dots &,\\
f_n''+2f_{n-1}'\theta'+f_{n-1}\theta'' &= f_{n-2}\left(\frac{\sigma^2}{gh}-\theta'^2\right),\\
\dots\dots\dots\dots\dots\dots &,
\end{aligned}
\right\} \tag{20}
$$

Here θ may obviously be chosen arbitrarily; but, if the series (19) is to terminate, we must have $\theta'^2=\sigma^2/gh$.

We shall now investigate some interesting particular cases in which the series does terminate.

The first equation of (20) gives $f_0=ax+b$. There are two cases according as a is or is not zero; and it is interesting to find that these two cases are reciprocal.

Let us first suppose that $a\neq 0$. Then, by a simultaneous change of origin and scale, we may set $f_0=x$. If now we assume $\theta=x^n$, it is easily found that

$$f_r=k_rx^{rn+1},$$

k_r being calculated by means of the recursion formula

$$r(rn+1)k_r=-[(2r-1)n+1]k_{r-1}, \qquad k_0=1.$$

In order that f_{s+1} may be zero, we must have

$$0=2f_s'\theta'+f_s\theta''=k_s[2(sn+1)n+n(n-1)]x^{(s+1)n-1},$$

which is satisfied if $n=-1/(2s+1)$. This gives

$$h=\frac{\sigma^2}{g}(2s+1)^2x^m, \tag{21}$$

where $m=4(s+1)/(2s+1)$. The following table gives the values of n and m for $s=1, 2, \ldots, 5$.

s	m	$-n$
0	4	1
1	$\frac{8}{3}$	$\frac{1}{3}$
2	$\frac{12}{5}$	$\frac{1}{5}$
3	$\frac{16}{7}$	$\frac{1}{7}$
4	$\frac{20}{9}$	$\frac{1}{9}$
5	$\frac{24}{11}$	$\frac{1}{11}$

If $a=0$, we may assume $f_0=1$, when we readily find that $f_r=k_r x^{rn}$; and the series terminates after s terms if $n=1/(2s+1)$. This gives

$$h=\frac{\sigma^2}{g}(2s+1)^2 x^m, \qquad m=\frac{4s}{2s+1}. \tag{22}$$

s	m	n
0	0	1
1	$\frac{4}{3}$	$\frac{1}{3}$
2	$\frac{8}{5}$	$\frac{1}{5}$
3	$\frac{12}{7}$	$\frac{1}{7}$
4	$\frac{16}{9}$	$\frac{1}{9}$
5	$\frac{20}{11}$	$\frac{1}{11}$

In the first case the series of values of m approach the limit 2 from above, and in the second case they approach the same value from below. For $m=2$, it is easily seen that the two reciprocal solutions coincide.

By the transformations given in § 3, we can derive the solution from these results when

$$h=(ax+b)^4(cx+d)^m/(ex+f)^m, \tag{23}$$

the coefficients being arbitrary and m having the same values as above; and also, as a particular case, the solutions in the first case can be derived from those of the second case and *vice versa.*

5. *The Normal Form of the Differential Equation.* Though by no means necessary, it is convenient to bring equation (4) to its normal form.

Let $\theta=\int dx/\sqrt{gh}$, and put

$$r=\theta+t, \qquad s=\theta-t, \qquad k(r+s)=-\theta''/4\theta'^2;$$

then

$$\frac{\partial^2 w}{\partial r\partial s}=k\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right). \tag{24}$$

It is interesting to note that this equation is the same as that used by Riemann in the theory of waves in air of *finite* amplitude.

To obtain a solution of this equation we assume

$$w=\sum_{n=0}^{\infty} m_n[F^{(n)}(r)+G^{(n)}(s)], \tag{25}$$

where $F(r)$ and $G(s)$ are arbitrary functions of r and s respectively, and m is a function of $r+s$. Substituting this value of w in (24), we get

$$m_0''-2km_0'=0,$$
$$\cdots\cdots\cdots\cdots\cdots,$$
$$m_n''-2km_n'=-m_{n-1}'+km_{n-1},$$

whence, if we set $m=\Sigma m_n z^n$, m satisfies the differential equation

$$m''+(z-2k)m'-kzm=0, \tag{26}$$

where the independent variable is $r+s$ and z is an arbitrary parameter.

In discussing the initial conditions in § 9 below, we shall find that it is convenient to replace $r+s$ by a variable $q=(r+s)/2$. We therefore set* $m(r+s)=v(q)$, whereupon the differential equation becomes

$$\frac{d^2v}{dq^2}+2(z-\varkappa)\frac{dv}{dq}-2z\varkappa v=0, \tag{27}$$

where

$$\varkappa(q)=2k(r+s).$$

6. *A Study of v as a Function of z.* In discussing the solution of equation (27) we shall assume that q is a real variable lying in a finite range $q_0<q<q_0+Q$, and that $\varkappa$ is a real function of q alone which remains finite and in absolute value less than K in the range considered, while z is an arbitrary parameter which may be imaginary. Under these conditions it is always possible to expand v as a power-series in z, and we shall now show that for certain determinations of the initial conditions this series represents an integral function of z.

Let $v=\Sigma v_n z^n$; then

$$v_0''-2\varkappa v_0'=0, \qquad v_n''-2\varkappa v_n'=-2(v_{n-1}'-\varkappa v_{n-1}); \tag{28}$$

---

*When it is desired to emphasize the fact that v depends on z, it will be denoted by $v(q, z)$.

therefore

$$v_0' = e^{\int 2\kappa dq}, \qquad v_0 = \int e^{\int 2\kappa dq} dq,$$

$$v_n' = -2e^{\int 2\kappa dq} \int e^{-\int 2\kappa dq}(v_{n-1}' - \kappa v_{n-1})dq = -2e^{\int 2\kappa dq} \int e^{-\int \kappa dq} \frac{d}{dq}(e^{-\int \kappa dq} v_{n-1})dq.$$

By a particular choice of the constants of integration we have

$$v_0' = be^{\int_{q_0} 2\kappa dq} \equiv b\alpha, \qquad v_0 = \int_q v_0' dq + a,$$

where $\alpha = e^{\int_{q_0} 2\kappa dq}$, and

$$v_1' = -2\alpha \int_{q_0} \frac{1}{\sqrt{\alpha}} \frac{d}{dq}(v_0/\sqrt{\alpha})dq + c, \qquad v_1 = \int_{q_0} v_1' dq, \tag{29}$$

$$v_n' = -2\alpha \int_{q_0} \frac{1}{\sqrt{\alpha}} \frac{d}{dq}(v_{n-1}/\sqrt{\alpha})dq, \qquad v_n = \int_{q_0} v_n' dq,$$

where the constants are so chosen that $v_n = 0$, $(n \geq 1)$, when $q = q_0$.

Let now $\bar{\alpha}$ denote a mean among the values of α; then

$$v_1' = -2\frac{\sqrt{\alpha} v_0}{\sqrt{\bar{\alpha}} v_0'} + 2a\frac{\alpha}{\sqrt{\bar{\alpha}}} + c = 2\sqrt{\frac{\alpha}{\bar{\alpha}}}(-v_0 + a\sqrt{\alpha}) + c, \tag{30}$$

and similarly for $n > 1$,

$$v_n' = -2\sqrt{\frac{\alpha}{\bar{\alpha}}}\, v_{n-1}, \tag{30'}$$

where of course the value of $\bar{\alpha}$ depends on n.

Let $\bar{N}_0$ and N_0 be the smallest and largest values of $|v_0'|$, and set $L = 2\sqrt{N_0/\bar{N}_0}$; and let N_n and M_n denote the maximum values of $|v_n'|$ and $|v_n|$ respectively. It must be noted that L is independent of b, so that $2\sqrt{\alpha/\bar{\alpha}} \leq L$. Using this notation, we readily derive from (29) and (30) that

$$M_0 \leq N_0 |q - q_0| + |a|,$$

$$N_1 \leq L\left(M_0 + |a\alpha^{\frac{1}{2}}| + \frac{|c|}{L}\right), \qquad M_1 \leq L\left(M_0 + |a\alpha^{\frac{1}{2}}| + \frac{|c|}{L}\right)|q - q_0|,$$

$$N_n \leq LM_{n-1}, \qquad M_n \leq LM_{n-1}|q - q_0|,$$

whence

$$M_n \leq L^{n-1} M_1 |q - q_0|^{n-1} \leq L^n M |q - q_0|^n,$$

where

$$M = \left(M_0 + |a\alpha^{\frac{1}{2}}| + \frac{|c|}{L}\right).$$

Therefore

$$N_n \leq L^n M |q - q_0|^{n-1},$$

which gives

$$M_n \leq L^n M |q - q_0|^n / n,$$

which in turn leads to

$$N_n \leq L^n M\,|q-q_0|^{n-1}/(n-1),$$
$$M_n \leq L^n M\,|q-q_0|^{n}/n(n-1),$$

and finally

$$N_n \leq L^n M\,|q-q_0|^{n-1}/(n-1)\,!, \qquad M_n \leq L^n\,|q-q_0|^{n}/n\,!.$$

We therefore have

$$|v| \leq M\Sigma L^n\,|q-q_0|^n\,|z|^n/n\,! = Me^{L|q-q_0||z|}. \tag{31}$$

Hence $Me^{L|q-q_0|z}$ is a dominant function for v which therefore converges for all values of z, and consequently represents an integral function of that variable.

The choice of initial conditions made above gives $v=a$, $v'=b+cz$ when $q=q_0$. The reason for this choice appears when we consider the equation when $\varkappa$ is constant. The solution in this case is

$$v=e^{(\kappa-z)q}\left[Ae^{q\sqrt{z^2+\kappa^2}}+Be^{-q\sqrt{z^2+\kappa^2}}\right],$$

where for the sake of simplicity $q_0=0$. The simplest way in which this value of v can be made uniform is by putting $A=B$, which leads to the same kind of initial conditions as the above. There are, of course, other ways in which v may be made uniform.

For reasons which will appear later we make $c=-a$, so that, when $q=q_0$,

$$v_0'=b, \quad v_0=a, \quad v_1'=-a, \quad v_1=0, \quad v_n'=0=v_n, \qquad (n>1),$$
$$v'=b-az, \quad v=a. \tag{32}$$

7. *The Asymptotic Expression for v.* The asymptotic expression of the solution of a differental equation, regarded as a function of a parameter, has been the subject of several important memoirs;* but the form in which the results are obtained does not appear to enable one to identify the various possible asymptotic forms with solutions which are defined with reference to initial conditions of the type here employed. The method used below avoids this difficulty, but it is not applicable to the most general form of equation.

Let $p=qz$; then equation (27) becomes

$$\frac{d^2v}{dp^2}+2\left(1-\frac{\varkappa}{z}\right)\frac{dv}{dp}-\frac{2\varkappa v}{z}=0. \tag{33}$$

For sufficiently large values of z, v can be expanded in the form

$$v(p,z)=\sum_{n=0}^{\infty}\frac{v_n}{z^n}, \tag{34}$$

where

$$v_0''+2v_0'=0, \qquad v_n''+2v_n'=2\varkappa(v_{n-1}'+v_{n-1}), \tag{35}$$

---

* Cf., for instance, Birkhoff, *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 219, where full references are given.

the dashes now indicating differentiation with regard to p. Solving these equations, we get

$$v_0' = -2\alpha e^{-2p}, \qquad v_0 = \alpha e^{-2p} + \beta,$$

α and β being arbitrary constants; and also

$$v_n' = 2e^{-2p}\int e^{2p}\varkappa(v_{n-1}' + v_{n-1})\,dp,$$
$$v_n = 2\int e^{-2p}\int e^{2p}\varkappa(v_{n-1}' + v_{n-1})\,dp\,dp.$$

We shall first consider the equation which results from putting for $\varkappa$ the maximum value, K, of $|\varkappa|$. The solution of this equation is

$$V(p, z) = e^{\left(\frac{K}{z}-1\right)(p-p_0)}\left[Ae^{(p-p_0)\sqrt{1+K^2/z^2}} + Be^{-(p-p_0)\sqrt{1+K^2/z^2}}\right]. \tag{36}$$

If $A+B$ and $(A-B)/\sqrt{1+K^2/z^2}$ are uniform integral functions of $1/z$, then V is also uniform and integral, so that we may write $V = \Sigma V_n/z^n$, and we can determine the limits of integration so that V is derived from

$$\begin{aligned} &v_0' = -2ae^{-2(p-p_0)}, && v_0 = ae^{-2(p-p_0)} + b, \\ &v_1' = ce^{-2(p-p_0)} + 2e^{-2p}\int_{p_{11}} e^{2p}\varkappa(v_0' + v_0)\,dp, && v_1 = \int_{p_{12}} v_1'\,dp, \\ &v_n' = 2e^{-2p}\int_{p_{n1}} e^{2p}\varkappa(v_{n-1}' + v_{n-1})\,dp, && v_n = \int_{p_{n2}} v_n'\,dp \end{aligned} \tag{37}$$

by putting K for $\varkappa$.

If the constants of integration are real, then V is real when p and z are real; we shall assume that p_{n1} and p_{n2} do not increase without limit as n increases, so that we can always choose a real value of p so large that V_n and V_n' have the same sign as $V_{n-1}' + V_{n-1}$. For instance, we can put A and B equal to the same real constant a, which gives

$$V(p_0, z) = 2a, \qquad V'(p_0, z) = \left(\frac{K}{z} - 1\right)2a; \tag{38}$$

while, if $p_{n2} = p_0$ $(n \gtrless 1)$, $p_{n1} = p_0$ $(n \gtrless 1)$, we get from (37)

$$V(p_0, z) = a + b,$$

$$V'(p_0, z) = -2a + \frac{c}{z},$$

which is identical with (38) if $b = a$ and c is chosen equal to $2aK$. If we therefore give c this value, the function V, defined by (37) when $\varkappa = K$, is identical with the function given by (36). Now

$$V_0 = b + ae^{-2(p-p_0)}, \qquad V_0' + V_0 = b - ae^{-2(p-p_0)};$$

therefore, if a and b are positive and $a \leq b$, $p \gtrless p_0$ and $p > p_{nr}$, $(r = 1, 2)$, for every n, then cV_0 and $V_0' + V_0$ are positive, and hence V_n' and V_n are positive for every value of n, except $n = 0$, when V_0' is negative.

Returning to the general case, we shall suppose the limits of integration and the arbitrary constants identical with those just chosen for V. The range of values of p with which we are concerned is the range of qz, q being real and lying between fixed limits, while z is constant but so large as to make $|p|$ greater than p_{n1} and p_{n2} for all values of n. Let $p=qz=re^{i\theta}$; then

$$|v_n'| \leq 2Ke^{-2r\cos\theta}\int_{r_{n1}} e^{2r\cos\theta}|v_{n-1}'+v_{n-1}|\,dr.$$

Assume now that

$$|v_{n-1}'+v_{n-1}| \leq l_{n-1}[V_{n-1}'(r\cos\theta)+V_{n-1}(r\cos\theta)], \qquad (n \geqq 1),$$

l_{n-1} being a positive constant, $(l_0 \geqq 1)$, and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. It follows that

$$|v_n'| \leq \frac{l_{n-1}}{\cos\theta}V_n'(r\cos\theta), \qquad |v_n| \leq \frac{l_{n-1}}{\cos^2\theta}V_n(r\cos\theta).$$

Therefore

$$|v_n'+v_n| \leq \frac{l_{n-1}}{\cos^2\theta}[V_n'(r\cos\theta)+V_n(r\cos\theta)]$$

if $r\cos\theta$ is so large as to make V_n' and V_n positive when $n>0$; we therefore set $l_n=l_{n-1}/\cos^2\theta$, which gives

$$l_n=\frac{l_0}{\cos^{2n}\theta}.$$

To determine l_0 we have

$$|v_0+v_0'|=|b-ae^{-2(p-p_0)}| \leq l_0(b-ae^{-2(r-r_0)\cos\theta});$$

if $b>a$ we may therefore put $l_0=(a+b)/(a-b)$, and if $a=b$ a short calculation shows that it is sufficient to put $l_0=1+2|\tan\theta|$. This leads to

$$|v| \leq \Sigma|v_n|/|z|^n \leq l_0\Sigma V_n(r\cos\theta)/|z\cos^2\theta|^n \leq l_0 V(r\cos\theta,\ |z\cos^2\theta|), \qquad (40)$$

since the series for V converges for all values of $1/z$ different from zero. Similarly, for v' we have

$$|v'| \leq l_0\cos\theta\,\Sigma V_n'(r\cos\theta)/|z\cos^2\theta|^n \leq l_0\cos\theta\,V'(r\cos\theta,\ |z\cos^2\theta|).$$

Now, if $z=\rho e^{i\theta}$,

$$V(r\cos\theta,\ |z\cos^2\theta|)=e^{(K/\rho\cos^2\theta-1)\rho(q-q_0)\cos\theta}\left[Ae^{\rho(q-q_0)\cos\theta\sqrt{1+K^2/\rho^2\cos^4\theta}} + Be^{-\rho(q-q_0)\cos\theta\sqrt{1+K^2/\rho^2\cos^4\theta}}\right],$$

$$\rho\cos^2\theta V'(r\cos\theta,\ |z\cos^2\theta|)=$$
$$e^{(K/\rho\cos^2\theta-1)\rho(q-q_0)\cos\theta}\left[(K-\rho\cos^2\theta+\sqrt{\rho^2\cos^4\theta+K^2})Ae^{\rho(q-q_0)\cos\theta\sqrt{1+K^2/\rho^2\cos^4\theta}} + (K-\rho\cos^2\theta-\sqrt{\rho^2\cos^4\theta+K^2})Be^{-\rho(q-q_0)\cos\theta\sqrt{1+K^2/\rho^2\cos^4\theta}}\right].$$

These are respectively asymptotic to

$$Ae^{K(q-q_0)/\cos\theta} \quad \text{and} \quad AKe^{K(q-q_0)/\cos\theta},$$

and therefore $|v|$ and $|dv/dq| \equiv |zdv/dp|$ remain finite as z approaches infinity in the region S defined by $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

When $q=q_0$, $V(r\cos\theta,\ |z\cos^2\theta|)$ is asymptotic to $A+B$; and if, in addition, $A=B$, then $V'(r\cos\theta,\ |z\cos^2\theta|)$ is asymptotic to $2AK/\rho\cos^2\theta-2A$.

From (40), $l_0V(r\cos\theta,\ z\cos^2\theta)$ is a dominant function for $v(p,\ z)$ as regards z, and $l_0\cos\theta V'(r\cos\theta,\ z\cos^2\theta)$ is a dominant function for $v'(p,\ z)$, so that

$$\left|\frac{\partial v}{\partial z}\right| \leq l_0|\partial V(r\cos\theta,\rho\cos^2\theta)/\partial\rho|,$$

$$\left|\frac{\partial^2 v}{\partial p\partial z}\right| \leq l_0\cos\theta|\partial^2 V(r\cos\theta,\ \rho\cos^2\theta)/\partial p\partial\rho|,$$

$$\left|\frac{\partial^2 v}{\partial z^2}\right| \leq l_0|\partial^2 V(r\cos\theta,\ \rho\cos^2\theta)/\partial\rho^2|;$$

and from these three relations we find, after a short calculation, that $z\partial v/\partial p$, $z^2\partial^2 v/\partial p\partial z$ and $z^2\partial^2 v/\partial z^2$ remain finite as ρ approaches infinity.

If we now assume that $A=B=a=b$, we have, when $p=p_0$ or $q=q_0$,

$$v(p_0,\ z)=2a, \qquad \frac{dv(p_0,\ z)}{dp}=-2a+\frac{c}{z}.$$

But $dv/dp=dv/zdq$, and therefore

$$\frac{dv}{dq}=c-2az, \qquad (q=q_0).$$

These initial conditions are of the same form as in (32), b being replaced by c and a by $2a$. It follows that v is an integral function of z when p is replaced by qz.

Now v, regarded as a function of z, remains finite in the region S defined by $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, and approaches a definite limit as z approaches infinity; hence in the same region zdv/dz and z^2d^2v/dz^2 remain less than some finite quantity. But

$$\frac{d^2v}{dz^2}=q^2\frac{\partial^2 v}{\partial p^2}+2q\frac{\partial^2 v}{\partial p\partial z}+\frac{\partial^2 v}{\partial z^2},$$

and in this equation z^2d^2v/dz^2, $qz^2\partial^2v/\partial p\partial z$ and $z^2\partial^2 v/\partial z^2$ have been shown to be less than some finite quantity in the region S; hence the same is also true of $z^2\partial^2 v/\partial p^2$, and therefore of $\partial^2 v/\partial q^2$. Now

$$\frac{\partial^2 v}{\partial q^2}-2\kappa\frac{\partial v}{\partial q}+2z\left(\frac{\partial v}{\partial q}-\kappa v\right)=0,$$

and we may therefore set

$$\frac{\partial v}{\partial q}-\kappa v=\left(\frac{\partial^2 v}{\partial q^2}-2\kappa\frac{\partial v}{\partial q}\right)/z=\frac{U}{z},$$

where U remains finite as z approaches infinity in S. Hence

$$v=ae^{\int_{q_0}\kappa dq}+\frac{e^{\int_{q_0}\kappa dq}}{z}\int_{q_0}e^{-\int_{q_0}\kappa dq}Udq.$$

In this expression we know that U is finite in S even when $q=q_0$; hence v is asymptotic to

$$ae^{\int_{q_0}\kappa dq}$$

in* S.

8. *On an Expansion of an Arbitrary Function in Terms of v.* We shall now regard $v(q, z)$ as a function of q and z alone, choosing the arbitrary constant so that

$$v=1, \qquad \frac{dv}{dq}=b-z$$

when $q=q_0$, so that in S

$$v(q, z)\sim e^{\int_{q_0}\kappa dq}.$$

Now it is readily shown that

$$u\equiv e^{-2z(q-q_0)}v(q, -z)$$

satisfies the same differential equation as v. Differentiating, we have

$$u'=e^{-2z(q-q_0)}[v'(q, -z)-2zv(q, -z)],$$

so that when $q=q_0$

$$u=1, \qquad u'=b+z-2z=b-z.$$

Hence $u\equiv v$, or

$$v(q, -z)=e^{2z(q-q_0)}v(q, z). \tag{41}$$

The rest of this paragraph follows closely the treatment of one of Cauchy's proofs of Fourier's development, which is given in Picard's *Traité d'Analyse*, t., II, p. 180, to which we refer the reader for some of the details of the argument. The method depends on the integral

$$\frac{r}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{1}{2}[\mathfrak{F}(re^{\phi i})-\mathfrak{F}(-re^{\phi i})]e^{\phi i}d\phi, \tag{42}$$

which is equal to the sum of the residues of the analytic function $\mathfrak{F}(z)$ inside the circle with center at $z=0$ and radius r. A special form is chosen for $\mathfrak{F}(z)$, and then the limit of (42) is investigated as r approaches infinity over a sequence of values so chosen that no circle passes through a singularity of the integrand. When this limit exists, it is equal to the sum of the residues of $\mathfrak{F}(z)$.

Cauchy sets

$$\mathfrak{F}(z)=\frac{\psi(z)f(z, q)}{\pi(z)},$$

where ψ, π and f are integral functions of z, f alone depending on the parameter q. The main difference in the investigation given below is that ψ and π are also taken to depend on q; otherwise the conditions are very much the same as in Cauchy's development, being varied only by the special form assumed for ψ.

* That this is also true on the boundaries of S is shown by Birkhoff, *Transactions of the American Mathematical Society*, Vol. IX, p. 219.

We shall first of all put

$$f(z, q) = \int_{q_1}^{q} e^{z(\mu-q)} f(\mu)\, d\mu,$$

f being a real integrable function which satisfies Dirichlet's conditions.

A sufficient condition for the convergence of the integral (42) is that

$$\tfrac{1}{2} z \frac{\psi(z)}{\pi(z)} \int_{q_1}^{q} e^{z(\mu-q)} f(\mu)\, d\mu - \tfrac{1}{2} z \frac{\psi(-z)}{\pi(-z)} \int_{q_1}^{q} e^{-z(\mu-q)} f(\mu)\, d\mu \tag{43}$$

shall approach a definite limit as z increases, so that its real part remains positive and its modulus takes on the increasing sequence of values $r_1, r_2, \ldots\ldots$

Departing slightly from Cauchy's conditions, we suppose that $\psi(z)/\pi(z)$ approaches a definite limit c, while $e^{bz}\psi(-z)/\pi(-z)$, $(b=q-q_1)$, tends in general to zero. The second part of (43) can then be written

$$\tfrac{1}{2} \frac{\psi(-z)}{\pi(-z)} e^{z(q-q_1)} z \int_{q_1}^{q} e^{-z(\mu-q_1)} f(\mu)\, d\mu,$$

and therefore approaches zero under the given conditions.* The first part approaches†

$$\tfrac{1}{2} c \lim_{z=\infty} z \int_{q_1}^{q} e^{-z(q-\mu)} f(\mu)\, d\mu = \tfrac{1}{2} c f(q).$$

Hence

$$\tfrac{1}{2} c f(q) = \Sigma \frac{\psi(\lambda_n)}{\pi'(\lambda_n)} \int_{q_1}^{q} e^{z(\mu-q)} f(\mu)\, d\mu, \tag{44}$$

where λ_n is a root of $\pi(z)$, and the summation extends over all the roots.

Similarly, if $b=q_2-q$ and

$$f(z, q) = \int_{q}^{q_2} e^{z(q-\mu)} f(\mu)\, d\mu,$$

and C is put for c, the same conditions lead to

$$\tfrac{1}{2} C f(q) = \Sigma \frac{\psi(\lambda_n)}{\pi'(\lambda_n)} \int_{q}^{q_2} e^{z(q-\mu)} f(\mu)\, d\mu, \tag{45}$$

where of course ψ and π are not necessarily the same as before.

Let us first set

$$\psi(z) = e^{z(a+q-q_0)} v(q, z) = e^{z(a-q+q_0)} v(q, -z),$$
$$\pi(z) = e^{z(q-q_0)} (e^{az}-1),$$

Then, in S,

$$c = \lim_{n=\infty} \frac{\psi(z)}{\pi(z)} = \lim_{n=\infty} \frac{e^{z(a+q-q_0)} v(q, z)}{e^{z(q-q_0)} (e^{az}-1)} = e^{\int_{q_0} \kappa dq},$$

$$\lim_{n=\infty} \frac{\psi(-z)}{\pi(-z)} e^{bz} = \lim_{n=\infty} \frac{e^{z(q-q_0-a+b)} v(q, z)}{e^{-z(q-q_0)} (e^{-az}-1)} = 0,$$

provided

$$2(q-q_0)+b < a.$$

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*Cf. Picard, *loc. cit.*, p. 182–183. †Cf. Picard, *loc. cit.*, p. 183–184.

Hence, if $\lambda_n = 2\pi i n/a$, $(n=0, \pm 1, \pm 2, \ldots\ldots)$, denote the roots $\pi(z)$, we have

$$\tfrac{1}{2} e^{\int_{q_0} \kappa dq} f(q) = \frac{1}{a} \sum_{-\infty}^{\infty} v(q, \lambda_n) \int_{q_1}^{q} e^{\lambda_n (q-\mu)} f(\mu)\, d\mu. \tag{46}$$

Using now the second form of $f(z, q)$, we shall set

$$\psi(z) = e^{z(q-q_0)} v(q, z) = e^{-z(q-q_0)} v(q, -z),$$

$$\pi(z) = e^{z(q-q_0)} (1 - e^{-az}).$$

Then

$$C = \lim_{n=\infty} \frac{\psi(z)}{\pi(z)} = e^{\int_{q_0} \kappa dq} = c$$

as before, and

$$\lim_{n=\infty} \frac{\psi(-z)}{\pi(-z)} e^{bz} = \lim_{n=\infty} \frac{e^{z(b+q-q_0)} v(q, z)}{e^{-z(q-q_0)} (1-e^{az})} = 0,$$

provided

$$2(q-q_0) + b < a.$$

Hence

$$\tfrac{1}{2} e^{\int_{q_0} \kappa dq} f(q) = \frac{1}{a} \sum_{-\infty}^{\infty} v(q, \lambda_n) \int_{q}^{q_2} e^{\lambda_n (q-\mu)} f(\mu). \tag{47}$$

Adding (46) and (47), we arrive at the final form

$$e^{\int_{q_0} \kappa dq} f(q) = \frac{1}{a} \sum_{-\infty}^{\infty} v(q, \lambda_n) \int_{q_1}^{q_2} e^{\lambda_n (q-\mu)} f(\mu)\, d\mu, \tag{48}$$

provided

$$2(q-q_0) + q - q_1 < a,$$

$$2(q-q_0) + q_2 - q < a.$$

The form of these inequalities shows that we may put $q_0 = 0$ without loss of generality; and having done so, we easily find that, if q is any number in the interval $q_1 < q < q_2$, then the maximum value of q_2 is $(2a - q_1)/3$, while $q_1 < a/2$. We may, for instance, put $q_1 = 0$, which gives $q_2 = 2a/3$.

It should be noticed that if q_1 and q_2 lie between 0 and a, then the series given in (48) still converges if the factor $v(q, \lambda_n)$ is omitted from each term, as is easily seen by comparing (48) with Fourier's expansion.

9. *Initial Conditions.* The only type of initial conditions which we shall consider is the case where the profile of the wave and the horizontal velocity of particles on the surface are given as functions of x when $t=0$. Expressed in terms of w, this means that $\partial w/\partial x$ and $\partial w/\partial t$ are given as functions of x when $t=0$.

Now

$$\frac{\partial w}{\partial x} = \theta' \left(\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \right), \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial s};$$

we may therefore set

$$\left.\begin{aligned}\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}&=\frac{d\phi(q)}{dq},\\ \frac{\partial w}{\partial r}-\frac{\partial w}{\partial s}&=\psi(q),\end{aligned}\right\}\quad (r=s=q), \tag{49}$$

where ψ is a given function of q and ϕ is determined to an additive constant.

When $r=q=s$, the form assumed for w in (25) becomes

$$\begin{aligned} w&=\sum_{n=0}^{\infty} v_n(q)\,[F^{(n)}(q)+G^{(n)}(q)]\\ &=\sum_{n=0}^{\infty} v_n(q)H^{(n)}(q), \end{aligned} \tag{50}$$

where $H(q)=F(q)+G(q)$. Inserting this value of w in the first equation of (49), we get

$$\phi'=\sum_{n=0}^{\infty}(v_n'+v_{n-1})H^{(n)}(q)$$

or

$$\phi=\sum_{n=0}^{\infty} v_n(q)H^{(n)}(q); \tag{51}$$

and similarly, if $K(q)=F'(q)-G'(q)$,

$$\psi=\sum_{n=0}^{\infty} v_{n-1}[F^{(n)}(q)-G^{(n)}(q)]=\sum_{n=0}^{\infty} v_n(q)K^{(n)}(q). \tag{52}$$

We shall now proceed to make use of the developments of the preceding paragraph in showing that we can in general determine H and K to satisfy the initial conditions.

Assuming that H and K are expansible in a Fourier series

$$H(q)=\sum_{-\infty}^{\infty}\alpha_n e^{\lambda_n q},\qquad K(q)=\sum_{-\infty}^{\infty}\beta_n e^{\lambda_n q}, \tag{53}$$

and that these series can be differentiated term by term, we have, from (51),

$$\begin{aligned}\phi&=\sum_{r=0}^{\infty} v_r(q)\sum_{n=-\infty}^{\infty}\alpha_n\lambda_n^r e^{\lambda_n q}=\sum_{n=-\infty}^{\infty}\alpha_n\sum_{r-0}^{\infty} v_r(q)\lambda_n^r e^{\lambda_n q}\\ &=\sum_{-\infty}^{\infty}\alpha_n v(q,\lambda_n)e^{\lambda_n q}.\end{aligned} \tag{54}$$

This expansion is of the same form as (48), where, if

$$\phi_1=e^{-\int_{q_0}\kappa dq}\phi, \tag{55}$$

we have

$$\left.\begin{aligned}\phi&=\sum_{-\infty}^{\infty}\alpha_n v_n(q,\lambda_n)e^{\lambda_n q},\\ \alpha_n&=\frac{1}{a}\int_{q_1}^{q_2}e^{-\lambda_n\mu}\phi_1(\mu)\,d\mu.\end{aligned}\right\} \tag{56}$$

Giving α_n this value in (53), the series derived for H satisfies the required conditions if these same conditions are satisfied by the Fourier series of the given function ϕ.

Similarly, for K we obtain

$$\beta_n = \frac{1}{a}\int_{q_1}^{q_2} e^{-\lambda_n\mu}\psi_1(\mu)\,d\mu, \tag{57}$$

where

$$\psi_1 = e^{-\int_{q_0}\kappa dq}\psi. \tag{58}$$

From this we easily deduce

$$\begin{aligned} F(q) &= \tfrac{1}{2}[H(q)+\int^q K(\mu)\,d\mu], \\ G(q) &= \tfrac{1}{2}[H(q)-\int^q K(\mu)\,d\mu]. \end{aligned} \tag{59}$$

The form of (56) suggests another form for w which makes it unnecessary to introduce functions having derivatives of every order.

Assume

$$w = \Sigma v_n[\tfrac{1}{2}(r+s),\,\lambda_n]\,[\alpha_n e^{\lambda_n r} + \beta_n e^{\lambda_n s}], \tag{60}$$

where the summation extends over any sequence of values λ_n for which the series converges together with its first and second derivatives. It is easily verified that this series satisfies the differential equation for w, and the discussion of the initial conditions then proceeds as before.

10. *Some Special Methods.* We have seen in the preceding sections how to find a solution for any case in the form of an infinite series, but in some particular cases there are more convenient methods.

Let $h=kx$; the equation for η then is

$$\frac{\partial^2\eta}{\partial t^2} = kg\frac{\partial}{\partial x}\left(x\frac{\partial\eta}{\partial x}\right),$$

a particular integral of which is*

$$\eta = AJ_0(nx^{\frac{1}{2}})\cos n\sigma t,$$

where n is arbitrary and $\sigma = \frac{1}{2}\sqrt{gk}$. This solution may be written

$$\begin{aligned} \eta &= B\int_0^{\frac{\pi}{2}} \cos\,(nx^{\frac{1}{2}}\sin\theta)\cos n\sigma t\,d\theta \\ &= \frac{B}{2}\int_0^{\frac{\pi}{2}}[\cos n(x^{\frac{1}{2}}\sin\theta+\sigma t)+\cos n(x^{\frac{1}{2}}\sin\theta-\sigma t)]\,d\theta. \end{aligned}$$

Hence we have as a solution

$$\eta = \int_0^{\frac{\pi}{2}}[F(x^{\frac{1}{2}}\sin\theta+\sigma t)+F(x^{\frac{1}{2}}\sin\theta-\sigma t)]\,d\theta, \tag{61}$$

F being any even function which can be expanded as series of cosines.

This gives two wave systems travelling in opposite directions. Each part is not, however, by itself a solution. For if we put

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* Cf. Lamb, *loc. cit.*, p. 259.

$$\eta=\int_0^{\frac{\pi}{2}}F(x^{\frac{1}{2}}\sin\theta+\sigma t)\,d\theta,$$

we get

$$\frac{\partial^2\eta}{\partial t^2}-gk\frac{\partial}{\partial x}\left(x\frac{\partial\eta}{\partial x}\right)=-\sigma^2x^{-\frac{1}{2}}F'(\sigma t).$$

This shows, however, that if G is any odd function,

$$\eta=\int_0^{\frac{\pi}{2}}[G(x^{\frac{1}{2}}\sin\theta+\sigma t)-G(x^{\frac{1}{2}}\sin\theta+\sigma t)]\,d\theta$$

is a solution, and since any function can be expressed as the sum of an even and an odd function, we readily derive a solution in the form

$$\eta=\int_0^{\frac{\pi}{2}}[F(x^{\frac{1}{2}}\sin\theta+\sigma t)+F(-x^{\frac{1}{2}}\sin\theta+\sigma t)]\,d\theta.$$

In certain cases it is possible to determine F in (61) to satisfy given initial conditions. To show this let $\psi=x\sin^2\theta$; then, if η_0 is the value of η for $t=0$,

$$\eta_0=2\int_0^{\frac{\pi}{2}}F(x^{\frac{1}{2}}\sin\theta)\,d\theta=\int_0^x\frac{F(\psi^{\frac{1}{2}})}{\psi^{\frac{1}{2}}\sqrt{x-\psi}}\,d\psi.$$

A solution of this integral equation was obtained by Abel* in the form

$$F(\psi^{\frac{1}{2}})=\frac{\psi^{\frac{1}{2}}}{\pi}\int_0^{\psi}\frac{\frac{\partial\eta_0}{\partial x}}{\sqrt{\psi-x}}\,dx,$$

when η satisfies the following very general conditions: namely, η_0 (i) is continuous, (ii) has a finite derivative in the region under consideration, and (iii) vanishes for $x=0$.

This is a solution for which $\partial\eta/\partial t$ (and therefore also $\partial\xi/\partial t$) is zero when $t=0$.

Instead of forming the reciprocal solution, we shall investigate independently under what conditions there is a solution of the form

$$w=f\int_0^{\frac{\pi}{2}}[F(\phi\sin\theta+\sigma t)+F(\phi\sin\theta-\sigma t)]\,d\theta,$$

where f and ϕ are functions of x. Let

$$w_1=f\int_0^{\frac{\pi}{2}}F(\phi\sin\theta+\sigma t)\,d\theta;$$

substituting in (4), we get

$$\sigma^2f\int_0^{\frac{\pi}{2}}F''\,d\theta=gh\left[f''\int_0^{\frac{\pi}{2}}F\,d\theta+(2f'\phi'+f\phi''-f\phi'^2/\phi)\int_0^{\frac{\pi}{2}}F'\sin\theta\,d\theta+f\phi'^2\int_0^{\frac{\pi}{2}}F''\,d\theta\right]$$
$$+\frac{ghf\phi'^2}{\phi}F'(\sigma t),$$

* See, for example, Bôcher, "Introduction of the Study of Integral Equations," 1909, p. 6.

whence

$$f''=0,$$

$$2f'\phi'+f\phi''-\frac{f\phi'^2}{\phi}=0, \tag{62}$$

$$gh\phi'^2-\sigma^2=0,$$

and F' is an odd function.

Integrating the first two equations, we have

$$f=ax+b, \qquad \frac{\phi'}{\phi}=kf^{-2}, \qquad (a\neq 0),$$
$$=k, \qquad (a=0).$$

Taking first the case $a=0$, let $f=1$; then $\phi=e^{kx}$, the constant of integration being absorbed in σ, and

$$h=\frac{\sigma^2}{gk^2}e^{-2kx},$$

$$\xi=\frac{gk^2}{\sigma^2}e^{2kx}\int_0^{\frac{\pi}{2}}[F(e^{kx}\sin\theta+\sigma t)+F(e^{kx}\sin\theta-\sigma t)]d\theta, \tag{63}$$

$$\eta=-ke^{kx}\int_0^{\frac{\pi}{2}}[F'(e^{kx}\sin\theta+\sigma t)+F'(e^{kx}\sin\theta-\sigma t)]\sin\theta d\theta.$$

This solution is the reciprocal of (61) except that k must be replaced by $\frac{1}{2}k$. If $a\neq 0$, we may by a change of scale and origin set $f=x$, which gives

$$\phi=e^{-k/x}, \qquad h=\sigma^2x^4e^{2k/x}/gk^2.$$

We have already found that there is a solution $w=f(x)F(\theta\pm\sigma t)$, F being arbitrary, only when $h=(ax+b)^4$. If, however, F is not arbitrary, a solution of this type may be found for any form of canal.

Since the coefficients in (15) are functions of x alone, while F is a function of $\theta+\sigma t$, this equation leads to

$$\frac{ghf\theta'^2-\sigma^2f}{ghf''}=\frac{1}{\alpha} \tag{64}$$

and

$$\frac{gh(2f'\theta'+f\theta'')}{ghf^2\theta'^2-\sigma^2f}=\beta, \tag{65}$$

α and β being constants. F must therefore satisfy the equation

$$F''+\beta F'+\alpha F=0.$$

We shall first consider the case $\beta=0$. In this case we have

$$F=C\cos\mu(\theta\pm\sigma t), \qquad \mu^2=\alpha;$$

also

$$2f'\theta'+f\theta''=0,$$

whence

$$\theta' f^2 = k,$$

and, from (64),

$$h = \frac{\mu^2\sigma^2}{g}\,\frac{f^4}{\mu^2k^2 - f''f^3}, \tag{66}$$

so that when h is given f is determined. In practice this equation for f is intractable, but interesting cases arise on giving f particular forms. For instance, if $f = x^n$, we get

$$\left.\begin{aligned} h &= \frac{\mu^2\sigma^2x^{4n}}{g[\mu^2 - n(n-1)x^{4n-2}]}, \\ \theta &= -\frac{x^{1-2n}}{2n-1}, \qquad (n \neq \tfrac{1}{2}), \end{aligned}\right\} \tag{67}$$

and for $n = \frac{1}{2}$

$$\theta = \log x.$$

The latter case is interesting as giving a particular solution of the self-reciprocal case previously referred to.

Another soluble case is obtained by setting $f = (x^2 + bx + c)^n$.

If instead of putting $\beta = 0$ we assume $gh\theta'^2 - \sigma^2 = 0$, or $1/\alpha = 0$, the equation to be satisfied by F becomes

$$F' + \frac{f''}{2f'\theta' + f\theta''}F = 0,$$

and

$$\frac{f''}{2f'\theta' + f\theta''} = \text{constant} = \gamma$$

or

$$f'' - 2\gamma f'\theta' - \gamma f\theta'' = 0,$$

the normal form of which is

$$v'' + \frac{\gamma^2\sigma^2}{gh}v = 0, \qquad f = e^{\gamma\theta}v.$$

Hence

$$w = ve^{\pm\gamma\sigma t},$$

which is equivalent to the ordinary harmonic solution of (4). The equation can also be normalized by changing the independent variable x to $x_1 = \int e^{2\gamma\theta}dx$, which gives the solution

$$w = v\left(\int e^{2\gamma\theta}dx\right)e^{-\gamma(\theta\pm\sigma t)}.$$

PRINCETON UNIVERSITY, *June*, 1912.

On a Certain Completely Integrable System of Linear Partial Differential Equations.

By E. J. Wilczynski.

Introduction.

From the point of view of projective differential geometry, developables and ruled surfaces require a special discussion, on account of their many exceptional properties. These exceptional properties manifest themselves analytically by the fact that, for these surfaces, the differential invariants of lowest order are equal to zero. If a surface is neither a developable nor a ruled surface, its absolute invariants of lowest order are those which have been denoted by the author by I and J. In a certain sense, then, those surfaces for which I and J are identically equal to zero, are to be regarded as constituting an especially simple and fundamental class.

I have shown recently,* that these surfaces are the integral surfaces of the completely integrable system of partial differential equations

$$(S) \quad \begin{cases} \dfrac{\partial^2 y}{\partial u^2} + 2\dfrac{\partial y}{\partial v} + (c_0 u + c_1)\, y = 0, \\[2ex] \dfrac{\partial^2 y}{\partial v^2} + 2\dfrac{\partial y}{\partial u} + (c_0 v + c_2)\, y = 0, \end{cases}$$

where c_0, c_1 and c_2 are constants, and I have studied in detail the case $c_0 = 0$. The system (S) is then capable of integration by means of elementary functions, and the corresponding surfaces, for which, of course, $I = J = 0$, have the further property of being invariant under a continuous group of projective transformations. If $c_0 \neq 0$, no such group exists, which leaves the surface invariant, and the problem of integrating (S) can not be solved in such a simple manner.

The directrix curves† of any integral surface of (S) form a conjugate net, whether c_0 vanishes or not. Upon this remark may be based a transformation theory of system (S) which we hope to discuss on some future occasion. The

* "On a Certain Class of Self-projective Surfaces," *Transactions of the American Mathematical Society*, October, 1913.

† For the notion of directrix curves, cf. a paper of mine in the *Transactions of the American Mathematical Society*, Vol. IX (1908), pp. 114–120.

present paper, however, is merely devoted to the integration of system (S) in the case $c_0 \neq 0$.

In order to accomplish this integration, we find it necessary to solve the following problem: To integrate a certain completely integrable system of non-homogeneous linear partial differential equations, when the solutions of the corresponding homogeneous system are known. Since I have been unable to find any discussion of this question in the literature, the first two sections of this paper are devoted to a solution of this problem in so far as it is needed for what follows. It then becomes possible to prove the existence of various kinds of solutions of system (S). The most interesting of these can be written as an exponential function, multiplied by a series of positive integral powers of c_0, the coefficients of this power-series being *polynomials* in u and v. A number of methods will be given for calculating these polynomials and for studying some of their properties, but I have been able to find a perfectly explicit formula for only some of them.

The solutions of (S) are found to satisfy integral equations of a peculiar kind, involving integrals of exact differentials. A more detailed discussion of such integral equations will be given in a separate paper.

§ 1. *Integrability Conditions for a Non-Homogeneous System of Linear Partial Differential Equations of the Second Order.*

Let us consider the non-homogeneous system of partial differential equations

$$(L) \quad \begin{cases} y_{uu} + 2\,a\,y_u + 2\,b\,y_v + c\,y + d = 0, \\ y_{vv} + 2\,a'y_u + 2\,b'y_v + c'y + d' = 0, \end{cases}$$

where

$$y_u = \frac{\partial y}{\partial u}, \quad y_v = \frac{\partial y}{\partial v}, \quad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad y_{vv} = \frac{\partial^2 y}{\partial v^2},$$

and where $a, b, \ldots, d'$ are analytic functions of u and v. By differentiating these equations, we obtain, for the third-order derivatives of y, the following unique expressions:

$$\begin{aligned} y_{uuu} &= p_1 y_{uv} + p_2 y_u + p_3 y_v + p_4 y + p_5, \\ y_{uuv} &= q_1 y_{uv} + q_2 y_u + q_3 y_v + q_4 y + q_5, \\ y_{uvv} &= r_1 y_{uv} + r_2 y_u + r_3 y_v + r_4 y + r_5, \\ y_{vvv} &= s_1 y_{uv} + s_2 y_u + s_3 y_v + s_4 y + s_5, \end{aligned}$$

where

$$q_1 = -2a, \quad q_2 = 4a'b - 2a_v, \quad q_3 = 4bb' - 2b_v - c, \quad q_4 = 2bc' - c_v, \quad q_5 = 2bd' - d_v,$$
$$r_1 = -2b', \quad r_2 = 4aa' - 2a'_u - c', \quad r_3 = 4a'b - 2b'_u, \quad r_4 = 2a'c - c'_u, \quad r_5 = 2a'd - d'_u,$$

and where the values of p_i and s_i are immaterial for our present purpose.

If we form the fourth-order derivatives of y, the expressions for y_{uuuu} and y_{vvvv} are unique, and the two expressions found for y_{uuuv}, viz., $\dfrac{\partial y_{uuu}}{\partial v}$ and $\dfrac{\partial y_{uuv}}{\partial u}$, will be identical without imposing any further condition, since each of them is the result of differentiating the first equation of (L) once with respect to u and once with respect to v. Similarly, the equation

$$\frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}$$

will be satisfied without imposing any conditions on the coefficients of (L). The equality of the two expressions $\dfrac{\partial y_{uuv}}{\partial v}$ and $\dfrac{\partial y_{uvv}}{\partial u}$, however, gives rise to such conditions. In fact we find

$$\begin{aligned}
\frac{\partial y_{uuv}}{\partial v} &= \left(\frac{\partial q_1}{\partial v} + q_1 r_1 + q_2\right) y_{uv} + \left(\frac{\partial q_2}{\partial v} + q_1 r_2 - 2 a' q_3\right) y_u \\
&\quad + \left(\frac{\partial q_3}{\partial v} + q_1 r_3 - 2b' q_3 + q_4\right) y_v + \left(\frac{\partial q_4}{\partial v} + q_1 r_4 - c' q_3\right) y + \frac{\partial q_5}{\partial v} + q_1 r_5 - d' q_3, \\
\frac{\partial y_{uvv}}{\partial u} &= \left(\frac{\partial r_1}{\partial u} + r_1 q_1 + r_3\right) y_{uv} + \left(\frac{\partial r_2}{\partial u} + r_1 q_2 - 2 a r_2 + r_4\right) y_u \\
&\quad + \left(\frac{\partial r_3}{\partial u} + r_1 q_3 - 2 b r_2\right) y_v + \left(\frac{\partial r_4}{\partial u} + r_1 q_4 - c r_2\right) y + \frac{\partial r_5}{\partial u} + r_1 q_5 - d r_2.
\end{aligned}$$

In order that system (L) may have four linearly independent solutions, the corresponding coefficients of y_{uv}, y_u, y_v, y and y^0 in these two expressions must be equal.* *We obtain in this way the following five integrability conditions:*

$$(I) \quad \begin{cases}
a_v - b'_u = 0, \\
a'_{uu} + c'_u - 2 a' a_u - 2 a a'_u - (a_{vv} + 2 b' a_v - 2 b a'_v - 4 a' b_v) = 0, \\
b'_{uu} + 2 a b'_u - 2 a' b_u - 4 b a'_u - (b_{vv} + c_v - 2 b b'_v - 2 b' b_v) = 0, \\
c'_{uu} - 4 c a'_u - 2 a' c_u + 2 a c'_u - (c_{vv} - 4 c' b_v - 2 b c'_v + 2 b' c_v) = 0, \\
d'_{uu} + 2 a d'_u - 2 a' d_u - 4 a'_u d - c' d - (d_{vv} + 2 b' d_v - 2 b d'_v - 4 b_v d' - c d') = 0.
\end{cases}$$

Thus, the integrability conditions for the non-homogeneous system (L) differ from those for the corresponding homogeneous system only by the presence of the last of these five conditions. †

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* Otherwise y would also satisfy an equation of the form

$$y_{uv} + 2 a'' y_u + 2 b'' y_v + c'' y + d'' = 0.$$

The system obtained by adding this equation to (L) has at most three linearly independent solutions.

† E. J. Wilczynski, "Projective Differential Geometry of Curved Surfaces" (First Memoir), *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 245.

§ 2. *Integration by Quadratures of the Non-Homogeneous System in Terms of the Solutions of the Corresponding Homogeneous System.*

If the conditions (I) are satisfied, (L) is a completely integrable system, and the corresponding homogeneous system

$$(H)\quad \begin{cases} y_{uu} + 2\,a\,y_u + 2\,b\,y_v + c\,y = 0, \\ y_{vv} + 2\,a'y_u + 2\,b'y_v + c'y = 0 \end{cases}$$

will also be completely integrable. Therefore, (H) will possess precisely four linearly independent solutions. Let $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ be four such linearly independent solutions of (H), so that

$$c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)} + c_4 y^{(4)}$$

will be its general solution if $c_1, \ldots, c_4$ are arbitrary constants. Let y be the general, and Y any particular solution of (L). Then $y - Y$ will be a solution of (H), so that

$$y = Y + c_1 y^{(1)} + c_2 y^{(2)} + c_3 y^{(3)} + c_4 y^{(4)}.$$

Consequently, if we assume the solutions of (H) as known, it suffices to find any particular solution of (L) in order to integrate (L) completely.

We use the method of variation of constants to find such a particular solution of (L). Let us write Y in the form

$$Y = \sum_{k=1}^{4} l^{(k)}\, y^{(k)}, \tag{1}$$

where $l^{(1)}, \ldots, l^{(4)}$ are functions, yet to be determined, of u and v. We shall have

$$Y_u = \Sigma\, l^{(k)}\, y_u^{(k)}, \qquad Y_v = \Sigma\, l^{(k)}\, y_v^{(k)},$$

if we impose upon $l^{(1)}, \ldots, l^{(4)}$ the conditions

$$\Sigma\, l_u^{(k)}\, y^{(k)} = \Sigma\, l_v^{(k)}\, y^{(k)} = 0,$$

or

$$\Sigma\, y^{(k)}\, d\,l^{(k)} = 0. \tag{2}$$

We then find

$$Y_{uu} = \Sigma\,(l^{(k)}\, y_{uu}^{(k)} + l_u^{(k)}\, y_u^{(k)}), \qquad Y_{vv} = \Sigma\,(l^{(k)}\, y_{vv}^{(k)} + l_v^{(k)}\, y_v^{(k)}),$$

so that Y will be a solution of (L), if, and only if,

$$\Sigma\, l_u^{(k)}\, y_u^{(k)} = -\,d, \qquad \Sigma\, l_v^{(k)}\, y_v^{(k)} = -\,d'. \tag{3}$$

We may write

$$Y_{uv} = \Sigma\, l^{(k)}\, y_{uv}^{(k)},$$

if we impose the further conditions

$$\Sigma\, l_v^{(k)}\, y_u^{(k)} = 0, \qquad \Sigma\, l_u^{(k)}\, y_v^{(k)} = 0. \tag{4}$$

But equations (3) and (4) are equivalent to the following:

$$\Sigma y_u^{(k)} d l^{(k)} = - d \cdot d u, \quad \Sigma y_v^{(k)} d l^{(k)} = - d' \cdot d v, \tag{5}$$

so that we have in (2) and (5) a system of three equations for the four differentials $d l^{(k)}$. We add to these equations a fourth, viz.,

$$\Sigma y_{uv}^{(k)} d l^{(k)} = -(A\, d u + B\, d v),$$

where we shall determine the (as yet unknown) functions A and B of u and v in such a way that the values of $d l^{(k)}$ obtained from these four equations shall be exact differentials.

Let us write these equations as follows:

$$\left.\begin{aligned} &\Sigma y_{uv}^{(k)} d l^{(k)} = -(A\, d u + B\, d v), \\ &\Sigma y_u^{(k)} d l^{(k)} = - d \cdot d u, \\ &\Sigma y_v^{(k)} d l^{(k)} = - d' \cdot d v, \\ &\Sigma y^{(k)} d l^{(k)} = 0, \end{aligned}\right\} \tag{6}$$

and denote the determinant of the left members by Δ, so that

$$\Delta = | y_{uv}, y_u, y_v, y |.$$

Then Δ is not identically equal to zero. For, if it were, (H) would have at most three linearly independent solutions, and $y^{(1)}, \ldots, y^{(4)}$ could not be linearly independent.

We may write

$$\Delta = \Sigma y_{uv}^{(k)} \lambda^{(k)} = \Sigma y_u^{(k)} \mu^{(k)} = \Sigma y_v^{(k)} \nu^{(k)} = \Sigma y^{(k)} \rho^{(k)},$$

where $\lambda^{(k)}$, $\mu^{(k)}$, etc., are the third-order minors of Δ, and may be written as follows:

$$\lambda^{(1)} = + | y_u, y_v, y |, \quad \mu^{(1)} = - | y_{uv}, y_v, y |,$$
$$\nu^{(1)} = + | y_{uv}, y_u, y |, \quad \rho^{(1)} = - | y_{uv}, y_u, y_v |,$$

etc., the notation adopted being self-explanatory.

The solution of (6) for $dl^{(k)}$ gives

$$\Delta\, d l^{(k)} = - \lambda^{(k)} (A\, d u + B\, d v) - \mu^{(k)} d \cdot d u - \nu^{(k)} d' \cdot d v.$$

But we find

$$\begin{aligned} \lambda_u^{(1)} &= | y_{uu}, y_v, y | + | y_u, y_{uv}, y | + | y_u, y_v, y_u | \\ &= | -2 a y_u - 2 b y_v - c y, y_v, y | + | y_u, y_{uv}, y | = -2 a \lambda^{(1)} - \nu^{(1)}, \end{aligned}$$

and similarly,

$$\lambda_u^{(k)} = -2 a \lambda^{(k)} - \nu^{(k)}, \quad \lambda_v^{(k)} = -2 b' \lambda^{(k)} - \mu^{(k)}, \tag{7}$$

whence

$$\mu^{(k)} = - \lambda_v^{(k)} - 2 b' \lambda^{(k)}, \quad \nu^{(k)} = - \lambda_u^{(k)} - 2 a \lambda^{(k)}.$$

Consequently we may write

$$\Delta\, d l^{(k)} = L^{(k)} d u + M^{(k)} d v, \qquad (k = 1, 2, 3, 4), \tag{8}$$

if we put

$$\left.\begin{aligned} L^{(k)} &= d\cdot\lambda_v^{(k)} + (2\,b'\,d - A)\,\lambda^{(k)}, \\ M^{(k)} &= d'\cdot\lambda_u^{(k)} + (2\,a\,d' - B)\,\lambda^{(k)}. \end{aligned}\right\} \tag{9}$$

In order that the expressions for $d\,l^{(k)}$ may be exact differentials, we must have

$$\frac{\partial}{\partial v}\left(\frac{L^{(k)}}{\Delta}\right) = \frac{\partial}{\partial u}\left(\frac{M^{(k)}}{\Delta}\right).$$

If we make use of the equations (7) and the equations

$$\frac{\partial\Delta}{\partial u} = -4\,a\,\Delta, \quad \frac{\partial\Delta}{\partial v} = -4\,b'\,\Delta,\text{*}$$

we find that these integrability conditions give rise to a system of four equations of the form

$$\begin{aligned} &(B - d'_u + 2\,a'\,d)\,\lambda_u^{(k)} - (A - d_v + 2\,b\,d')\,\lambda_v^{(k)} \\ &\quad + [-A_v + B_u + 2\,b'\,d_v - 2\,a\,d'_u - d\,(2\,a'_u + c' - 8\,a\,a') \\ &\qquad + d'\,(2\,b_v + c - 8\,b\,b') - 4\,b'\,A + 4\,a\,B]\,\lambda^{(k)} = 0, \qquad (10) \\ &\qquad\qquad (k = 1, 2, 3, 4). \end{aligned}$$

If we interpret $y^{(1)}, \ldots, y^{(4)}$ as the homogeneous coordinates of a point P_y in space, the point P_y describes a surface as u and v pass through the values of their respective ranges. Moreover, this surface can not be a developable, for a system of equations of the form (H).† But the quantities $\lambda^{(k)}$ are the homogeneous coordinates of the plane, tangent to this surface at P_y, and therefore can not satisfy a system of equations of the form (10) with non-vanishing coefficients. For, if they did, their ratios would be functions of a single variable; *i. e.*, the surface would be a developable. Consequently, the coefficients of (10) must be equal to zero. If we equate to zero the first two, we find

$$A = d_v - 2\,b\,d', \quad B = d'_u - 2\,a'\,d,$$

and these values of A and B also cause the remaining coefficient to vanish, as a consequence of the last of the integrability conditions (I).

We have obtained the following theorem:

Let $y^{(1)}, \ldots, y^{(4)}$ be four linearly independent solutions of the homogeneous system (H), and let

$$\Delta = \begin{vmatrix} y_{uv}^{(1)} & y_{uv}^{(2)} & y_{uv}^{(3)} & y_{uv}^{(4)} \\ y_u^{(1)} & y_u^{(2)} & y_u^{(3)} & y_u^{(4)} \\ y_v^{(1)} & y_v^{(2)} & y_v^{(3)} & y_v^{(4)} \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \end{vmatrix} = \sum_{k=1}^{4} y_{uv}^{(k)}\,\lambda^{(k)}, \tag{11}$$

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* *Loc. cit.*, p. 258. † *Ibid.*, § 2.

so that $\lambda^{(k)}$ is the co-factor of $y^{(k)}_{uv}$ in this determinant, which, moreover, is not identically equal to zero. If we put

$$\left.\begin{aligned} L^{(k)} &= d\,\lambda^{(k)}_v + (2\,b\,d' + 2\,b'\,d - d_v)\,\lambda^{(k)}, \\ M^{(k)} &= d'\lambda^{(k)}_u + (2\,a\,d' + 2\,a'\,d - d'_u)\,\lambda^{(k)}, \end{aligned}\right\}\ (k = 1, 2, 3, 4), \tag{12}$$

the quantities

$$d\,l^{(k)} = \frac{1}{\Delta}\,(L^{(k)}\,d\,u + M^{(k)}\,d\,v) \tag{13}$$

will be exact differentials, and the function

$$Y = \sum_{k=1}^{4} l^{(k)}\,y^{(k)} = \sum_{k=1}^{4} y^{(k)} \int_{(a_k,\,b_k)}^{(u,\,v)} \frac{1}{\Delta}\,(L^{(k)}\,d\,u + M^{(k)}\,d\,v), \tag{14}$$

where the lower limits of the integrals are arbitrary constants, will be a solution of the non-homogeneous system (L).

If we choose the four lower limits equal, so that

$$a_k = a, \quad b_k = b, \qquad (k = 1, 2, 3, 4),$$

we obtain that particular solution of (L) which corresponds to the initial conditions

$$Y = Y_u = Y_v = Y_{uv} = 0 \ \text{ for } \ u = a,\ v = b.$$

The quantities $\lambda^{(k)}$ satisfy a system of partial differential equations of the same type as (H). If we put

$$Y^{(k)} = \frac{\lambda^{(k)}}{\sqrt{\Delta}}, \quad (k = 1, 2, 3, 4),$$

$Y^{(1)}, \ldots, Y^{(4)}$ are solutions of the adjoined system of (H), viz.:

$$(H') \quad \begin{cases} Y_{uu} + 2\,a\,Y_u - 2\,b\,Y_v + (c + 2\,b_v - 4\,b\,b')\,Y = 0, \\ Y_{vv} - 2\,a'Y_u + 2\,b'Y_v + (c' + 2\,a'_u - 4\,a\,a')\,Y = 0.^{*} \end{cases}$$

In terms of these quantities we find

$$\left.\begin{aligned} L^{(k)} &= \sqrt{\Delta}\,[d\,Y^{(k)}_v + (2\,bd' - d_v)\,Y^{(k)}], \\ M^{(k)} &= \sqrt{\Delta}\,[d'Y^{(k)}_u + (2\,a'd - d'_u)\,Y^{(k)}], \end{aligned}\right\} \tag{12'}$$

and these formulæ are sometimes more convenient than (12). In all applications of these formulæ it is good to remember that, since

$$\frac{\partial\Delta}{\partial u} = -4\,a\,\Delta, \quad \frac{\partial\Delta}{\partial v} = -4\,b'\,\Delta,$$

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* *Loc. cit.*, p. 259. It should be remarked that system (H') is not in general equivalent to the system obtained from (H) by taking the Riemannian adjoint of each of its equations. There are two distinct methods of generalizing the Lagrange adjoint of an ordinary linear differential equation. The above method, based on the principle of duality, may be called the geometric method, as distinguished from the analytic method of Riemann.

we shall have

$$\Delta = C e^{-4p}, \quad C = \text{const.}, \tag{15}$$

where p is determined by the conditions

$$\frac{\partial p}{\partial u} = a, \quad \frac{\partial p}{\partial v} = b',$$

which are consistent, according to the first of the integrability conditions (I). The constant C may, of course, be reduced to unity, by a proper choice of the fundamental system $y^{(1)}, \ldots, y^{(4)}$.

§ 3. *Integration of System* (S) *by Series Proceeding according to Positive Integral Powers of* c_0.

If the coefficients of an ordinary linear homogeneous differential equation

$$\frac{d^n y}{d x^n} + P_1 \frac{d^{n-1} y}{d x^{n-1}} + \ldots + P_n y = 0$$

are integral rational functions of a parameter μ, and if, for an initial value of the independent variable $x = x_0$ independent of μ, initial values be prescribed for $y, y', \ldots, y^{(n-1)}$ which are either independent of μ or integral (rational or transcendental) functions of μ, the corresponding solution of the differential equation will itself be an integral (rational or transcendental) function of μ. This theorem, due to Poincaré, Günther and Horn,* may be applied to our system, in so far as we may by simple processes find from (S) an ordinary differential equation of the fourth order for y as a function of u, the coefficients of this equation being rational integral functions of c_0 and v, as well as of u. A similar equation of course exists for y as function of v. The actual determination of the solution might be accomplished on this basis, but manifestly the lack of symmetry in the treatment accorded to the two independent variables would be a disadvantage. We therefore prefer to start anew, without making any use of the above theorem except in so far as it assures us *à priori* of the existence of solutions of the form

$$y_r = y_r^{(0)} + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \ldots, \tag{16}$$

and even this is unnecessary, as we shall establish the existence of such solutions directly *à posteriori*.

Let us then assume for y in system (S) a development of form (16). We find at once the following conditions:

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* Poincaré, *Acta Mathematica*, Vol. IV (1884), p. 212. Günther, *Crelle's Journal*, Vol. CVII (1889) p. 312. Horn, *Mathematische Annalen*, Vol. LII (1898), p. 343.

$$\left.\begin{aligned}\frac{\partial^2 y_r^{(0)}}{\partial u^2} + 2\frac{\partial y_r^{(0)}}{\partial v} + c_1 y_r^{(0)} = 0,\\ \frac{\partial^2 y_r^{(0)}}{\partial v^2} + 2\frac{\partial y_r^{(0)}}{\partial u} + c_2 y_r^{(0)} = 0,\end{aligned}\right\} \tag{17a}$$

and

$$\left.\begin{aligned}\frac{\partial^2 y_r^{(k)}}{\partial u^2} + 2\frac{\partial y_r^{(k)}}{\partial v} + c_1 y_r^{(k)} + u\, y_r^{(k-1)} = 0,\\ \frac{\partial^2 y_r^{(k)}}{\partial v^2} + 2\frac{\partial y_r^{(k)}}{\partial u} + c_2 y_r^{(k)} + v\, y_r^{(k-1)} = 0,\end{aligned}\right\} (k = 1, 2, 3, \ldots.), \tag{17b}$$

which, together with the proper convergence conditions, are necessary and sufficient for the existence of a solution of (S) of the form (16).

According to (17a) we may take for $y_r^{(0)}$ the exponential $e^{\alpha_r u + \beta_r v}$, provided the constants α_r and β_r satisfy the equations

$$\left.\begin{aligned}\alpha^2 + 2\beta + c_1 = 0,\\ \beta^2 + 2\alpha + c_2 = 0.\end{aligned}\right\} \tag{18}$$

In general, there are four linearly independent functions of this kind,

$$y_i^{(0)} = e^{\alpha_i u + \beta_i v}, \quad (i = 1, 2, 3, 4), \tag{19}$$

and we shall, for the present, confine our discussion to this case. As we shall see at the end of this paragraph, this limitation does not essentially affect the generality of our argument and, moreover, may be easily removed afterward.*

The solution $y_r^{(0)} = e^{\alpha_r u + \beta_r v}$ of (17a) is characterized by the initial conditions

$$y_r^{(0)} = e^{\alpha_r a + \beta_r b}, \quad \frac{\partial y_r^{(0)}}{\partial u} = \alpha_r e^{\alpha_r a + \beta_r v}, \quad \frac{\partial y_r^{(0)}}{\partial v} = \beta_r e^{\alpha_r a + \beta_r b}, \quad \frac{\partial^2 y_r^{(0)}}{\partial u \partial v} = \alpha_r \beta_r e^{\alpha_r a + \beta_r b}$$

for $u = a$, $v = b$. We shall determine that solution of (S) which corresponds to these same initial conditions. If this solution is expressible by a series of form (16), we must satisfy the initial conditions

$$y_r^{(k)} = \frac{\partial y_r^{(k)}}{\partial u} = \frac{\partial y_r^{(k)}}{d v} = \frac{\partial^2 y_r^{(k)}}{\partial u \partial v} = 0, \qquad k \geqq 1, \tag{20}$$

for $u = a$, $v = b$, while

$$y_r^{(0)} = e^{\alpha_r u + \beta_r v}.$$

We proceed to calculate $y_r^{(k)}$ by applying the method of § 2 to (17b). In the notation of § 2 we have

$$\begin{aligned} a &= 0, & b &= 1, & c &= c_1, & d &= u\, y_r^{(k-1)},\\ a' &= 1, & b' &= 0, & c' &= c_2, & d' &= v\, y_r^{(k-1)}.\end{aligned}$$

Of the five integrability conditions (I), the first four are obviously satisfied. The left member of the last reduces to

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* A complete discussion of the various cases is to be found in my paper "On a Certain Class of Self-Projective Surfaces," to which I have already referred.

$$v\left(\frac{\partial^2 y_r^{(k-1)}}{\partial u^2}+2\frac{\partial y_r^{(k-1)}}{\partial v}+c_1 y_r^{(k-1)}\right)-u\left(\frac{\partial^2 y_r^{(k-1)}}{\partial v^2}+2\frac{\partial y_r^{(k-1)}}{d u}+c_2 y_r^{(k-1)}\right),$$

and this is equal to zero on account of (17b), if $k \geqq 2$, and on account of (17a), if $k=1$.

The homogeneous system, which corresponds to (17b), is (17a), and its solutions are determined by (18) and (19). Therefore, $\alpha_1, \ldots, \alpha_4$ are the four roots of the equation

$$\alpha^4+2c_1\alpha^2+8\alpha+c_1^2+4c_2=0, \tag{21}$$

so that

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=\beta_1+\beta_2+\beta_3+\beta_4=0.$$

Consequently we find

$$y_1^{(0)}\,y_2^{(0)}\,y_3^{(0)}\,y_4^{(0)}=1$$

and

$$\Delta=\begin{vmatrix}\alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_3 & \alpha_4\beta_4\\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\\ 1 & 1 & 1 & 1\end{vmatrix}=\frac{1}{4}\begin{vmatrix}\alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3\\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2\\ 1 & 1 & 1 & 1\end{vmatrix}, \tag{22}$$

or

$$4\Delta=-(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4).$$

We then find, for the co-factor of $\dfrac{\partial^2 y_i^{(0)}}{\partial u\,\partial v}$ in Δ, the value

$$\lambda^{(i)}=\frac{k_i}{2y_i^{(0)}}, \qquad (i=1,2,3,4),$$

where

$$\left.\begin{aligned}k_1&=(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4), & k_3&=(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_1-\alpha_2),\\ k_2&=-(\alpha_3-\alpha_4)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1), & k_4&=-(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3).\end{aligned}\right\} \tag{23}$$

If we substitute these values in (12), we find

$$L_{r,i}^{(k)}=\frac{k_i}{2}e^{-\alpha_i u-\beta_i v}\left[(-\beta_i u+2v)\,y_r^{(k-1)}-u\frac{\partial y_r^{(k-1)}}{\partial v}\right],$$

$$M_{r,i}^{(k)}=\frac{k_i}{2}e^{-\alpha_i u-\beta_i v}\left[(2u-\alpha_i v)\,y_r^{(k-1)}-v\frac{\partial y_r^{(k-1)}}{\partial u}\right],$$

where we have introduced a slight change of notation as compared with (12), the upper index k of (12) having been replaced by the lower index i, while the significance of the two new indices r and k, which appear in the last two equations, is obvious.

From the above values of Δ and k_1 we obtain the value of their quotient,

$$\frac{4\Delta}{k_1}=-(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4).$$

But, on account of (21), we have the relations

$$\Sigma \alpha_i = 0, \quad \Sigma \alpha_i \alpha_j = 2 c_1, \quad \Sigma \alpha_i \alpha_j \alpha_k = -8, \quad \alpha_1 \alpha_2 \alpha_3 \alpha_4 = c_1^2 + 4 c_2,$$

so that

$$\frac{4 \Delta}{k_1} = -4 (\alpha_1^3 + c_1 \alpha_1 + 2) = 8 (\alpha_1 \beta_1 - 1),$$

proving the first of the four equations

$$\frac{\Delta}{k_i} = 2 (\alpha_i \beta_i - 1), \qquad (i = 1, 2, 3, 4). \tag{24}$$

Clearly, $\alpha_i \beta_i - 1$ will be different from zero, since α_i is a simple root of (21).

From (14) we find, finally,

$$y_r^{(k)} = \sum_{i=1}^{4} e^{\alpha_i u + \beta_i v} \int_{a,b}^{u,v} \frac{e^{-\alpha_i u - \beta_i v}}{4 (\alpha_i \beta_i - 1)} \left[\left\{ (-\beta_i u + 2 v)\, y_r^{(k-1)} - u \frac{\partial y_r^{(k-1)}}{\partial v} \right\} d u \right.$$
$$\left. + \left\{ (2 u - \alpha_i v)\, y_r^{(k-1)} - v \frac{\partial y_r^{(k-1)}}{\partial u} \right\} d v \right]. \tag{25}$$

The lower limits of the four integrals which appear in this formula have been chosen equal to (a, b) so as to satisfy the conditions (20). That the expression (25) for $y_r^{(k)}$ actually satisfies these conditions may be seen as follows. We may write (25) in the form

$$y_r^{(k)} = \sum_{i=1}^{4} l_{r,i}^{(k)} y_i^{(0)}, \tag{25a}$$

where the quantities $l_{r,i}^{(k)}$ have been determined, in accordance with the method of § 2, in such a way that

$$\frac{\partial y_r^{(k)}}{\partial u} = \sum_{i=1}^{4} l_{r,i}^{(k)} \frac{\partial y_i^{(0)}}{\partial u}, \quad \frac{\partial y_r^{(k)}}{\partial v} = \sum_{i=1}^{4} l_{r,i}^{(k)} \frac{\partial y_i^{(0)}}{\partial v}, \quad \frac{\partial^2 y_r^{(k)}}{\partial u \partial v} = \sum_{i=1}^{4} l_{r,i}^{(k)} \frac{\partial^2 y_i^{(0)}}{\partial u \partial v}.$$

Consequently, all of these expressions will actually reduce to zero for $u = a$, $v = b$.

In order to establish the convergence of the expansion (16) for y_r, it will be convenient to write

$$\left. \begin{aligned} K_i (u, v;\, p, q) &= \frac{e^{\alpha_i (u-p) + \beta_i (v-q)}}{8 (\alpha_i \beta_i - 1)}, \\ S_i &= \int_{a,b}^{p,q} d (-\beta_i p^2 + 4 p q - \alpha_i q^2). \end{aligned} \right\} \quad (i = 1, 2, 3, 4), \tag{26}$$

Then we shall have

$$\left.\begin{aligned}
y_r^{(k)} &= \int_{a,b}^{u,v} \sum_{i=1}^{4} K_i \left\{ y_r^{(k-1)} d S_i - \frac{\partial y_r^{(k-1)}}{\partial q} d(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p} d(q^2) \right\}, \\
\frac{\partial y_r^{(k)}}{\partial u} &= \int_{a,b}^{u,v} \sum_{i=1}^{4} \alpha_i K_i \left\{ y_r^{(k-1)} d S_i - \frac{\partial y_r^{(k-1)}}{\partial q} d(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p} d(q^2) \right\}, \\
\frac{\partial y_r^{(k)}}{\partial v} &= \int_{a,b}^{u,v} \sum_{i=1}^{4} \beta_i K_i \left\{ y_r^{(k-1)} d S_i - \frac{\partial y_r^{(k-1)}}{\partial q} d(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p} d(q^2) \right\}, \\
\frac{\partial^2 y_r^{(k)}}{\partial u \partial v} &= \int_{a,b}^{u,v} \sum_{i=1}^{4} \alpha_i \beta_i K_i \left\{ y_r^{(k-1)} d S_i - \frac{\partial y_r^{(k-1)}}{d q} d(p^2) - \frac{\partial y_r^{(k-1)}}{\partial p} d(q^2) \right\}.
\end{aligned}\right\} \quad (27)$$

A simple induction proof, involving the use of equation (25), shows that, for every value of k, $y_r^{(k)}$ and all of its partial derivatives are continuous (and even analytic) functions of u and v, for all finite values of these variables. The path of integration from (a, b) to (u, v) may be chosen arbitrarily in the finite portion of the plane, since all of the differentials involved in (25) are exact. We may, moreover, without any essential restriction of generality, assume $a = b = 0$.

Let N be a positive number not less than the largest value assumed by $|e^{\alpha_r p + \beta_r q}|$ on the path of integration. We shall then have

$$|y_r^0| \leqq N, \quad \left|\frac{\partial y_r^0}{\partial p}\right| \leqq |\alpha_r|\, N, \quad \left|\frac{\partial y_r^{(0)}}{\partial q}\right| \leqq |\beta_r|\, N.$$

For any assigned finite values of u and v, there will exist a positive constant M_i, such that, for all values of p and q along the path of integration,

$$|K_i(u, v; p, q)| \leqq M_i.$$

Let

$$M = \sum_{i=1}^{4} M_i,$$

and denote by α and β two positive numbers, such that

$$|\alpha_i| \leqq \alpha, \quad |\beta_i| \leqq \beta, \qquad (i = 1, 2, 3, 4),$$

and consequently,

$$\left|\sum_{i=1}^{4} K_i(u, v; p, q)\right| \leqq M, \quad \left|\sum_{i=1}^{4} \alpha_i K_i\right| \leqq \alpha M, \quad \left|\sum_{i=1}^{4} \beta_i K_i\right| \leqq \beta M.$$

Introduce three positive functions by putting

$$P = \int_{0,0}^{p,q} |d(p^2)|, \quad Q = \int_{0,0}^{p,q} |d(q^2)|, \quad S = \int_{0,0}^{p,q} d(\beta |p|^2 + 4|p|\,|q| + \alpha |q|^2),$$

and we shall have

$$|dS_i| \leqq dS, \qquad (i = 1, 2, 3, 4).$$

If now we put $k = 1$ in (27), we find

$$|y_r^{(1)}| \leqq \int_{0,0}^{u,v} \sum_{i=1}^{4} M_i \{N\,dS + \beta N\,dP + \alpha N\,dQ\} \leqq N M T,$$

where

$$T = S + \beta P + \alpha Q,$$

and similarly,

$$\left|\frac{\partial y_r^{(1)}}{\partial u}\right| \leqq \alpha N M T, \qquad \left|\frac{\partial y_r^{(1)}}{\partial v}\right| \leqq \beta N M T, \qquad \left|\frac{\partial^2 y_r^{(1)}}{\partial u\,\partial v}\right| \leqq \alpha\beta N M T.$$

If we make use of these inequalities after putting $k = 2$ in (27), we find

$$|y_r^{(2)}| \leqq N\frac{M^2 T^2}{2}, \text{ etc.},$$

and, by induction, in general,

$$|y_r^{(k)}| \leqq N\frac{M^k T^k}{k!}, \qquad \left|\frac{\partial y_r^{(k)}}{\partial u}\right| \leqq \alpha N\frac{M^k T^k}{k!}, \qquad \left|\frac{\partial y_r^{(k)}}{\partial v}\right| \leqq \beta N\frac{M^k T^k}{k!}, \text{ etc.}$$

These inequalities assure us of the absolute and uniform convergence of our series for y_r, and of its term-by-term differentiability with respect to u and v taken either separately or together. It is easy to show, further, that $\frac{\partial^2 y_r}{\partial u^2}$ and $\frac{\partial^2 y_r}{\partial v^2}$ may also be obtained by repeated term-by-term differentiations from y_r as absolutely and uniformly convergent series. Consequently, we have obtained the following theorem:

The series

$$y_r = \sum_{k=0}^{\infty} y_r^{(k)} c_0^k,$$

for which $y_r^{(0)} = e^{\alpha_r u + \beta_r v}$, *and whose general term is determined by the recursion formula* (25), *is absolutely and uniformly convergent for all finite values of* u, v *and* c_0, *and represents that solution of the completely integrable system* (S), *whose initial values for* $u = a$, $v = b$ *coincide with those of the exponential function* $e^{\alpha_r u + \beta_r v}$.

We have computed the solutions y_r under the assumption that the four roots $\alpha_1, \ldots, \alpha_4$ of (21) are distinct. But this assumption, as we have already stated, does not really restrict the generality of our method. For, by means of a linear transformation of the independent variables

$$u = \bar{u} + h, \quad v = \bar{v} + k, \tag{28a}$$

system (S) may be transformed into a new system of the same kind, whose constants have the values

$$\bar{c}_0 = c_0, \quad \bar{c}_1 = c_1 + h c_0, \quad \bar{c}_2 = c_2 + k c_0. \tag{28b}$$

If $c_0 \neq 0$, we may therefore transform (S) into another system of the same form whose constants $\bar{c}_1$ and $\bar{c}_2$ have any desired values, for instance $\bar{c}_1 = \bar{c}_2 = 0$, in which case the four roots of (21) will actually be distinct. On the other hand, it would not be difficult to develop formulæ to take the place of (25) in the case of two or more equal roots.*

The linear transformation (28a) enables us to materially simplify the appearance of the differential equations (S), but we have made no use of this transformation in developing our theory up to this point, for two reasons. In the first place, there would have resulted no essential simplification in any of our formulæ sufficient to make up for the corresponding loss of symmetry. But, in the second place, the general formulæ have a real significance which might otherwise have escaped our notice, and which gives us a real insight into the nature of the integral surface.

In fact, the linear transformation (28a) may be interpreted as a change of position on the integral surface, instead of as a change of the parameters of reference. Now, if c_0 is small, our formulæ show that the surface whose parametric equations are

$$y_r = e^{\alpha_r u + \beta_r v}, \qquad (r = 1, 2, 3, 4), \tag{29}$$

may be regarded as projectively equivalent to a first approximation of our integral surface in the neighborhood of the point for which $u = a$, $v = b$. If $\bar{\alpha}_r$ and $\bar{\beta}_r$ are the values of α_r and β_r which correspond to the constants $\bar{c}_1$ and $\bar{c}_2$, the surface

$$y_r = e^{\bar{\alpha}_r \bar{u} + \bar{\beta}_r \bar{v}}, \qquad (r = 1, 2, 3, 4),$$

will, in the same way, be projectively equivalent to a first approximation of the integral surface in the vicinity of the point $u = a + h$, $v = b + k$. In other words, for small values of c_0, an integral surface Σ of (S) has associated with each of its points P, a self-projective surface of the form (29), whose exponents α_r, β_r change with P, and which may be regarded as a first approximation of Σ in the vicinity of P.

One further remark is of importance for our later developments. From the nature of our convergence proof by dominant functions, it is clear that the series

$$y_r^{(0)} + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \dots .$$

will still be a uniformly and absolutely convergent series for all finite values of u, v and c_0, if the recursion formula (25) be altered by writing in place of

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* Cf. foot-note on page 239.

(a, b), the common lower limit of all of the integrals, a different lower limit $(a_i^{(k)}, b_i^{(k)})$ for each of the integrals, provided that all of these limits are finite. Moreover, the function of u and v thus defined will still be a solution of (S), corresponding, however, to altered initial conditions.

§ 4. *Properties of the Coefficients of the Series Obtained in the Preceding Section.*

Let us make the generalization referred to at the end of the preceding section, and on that assumption calculate $y_r^{(1)}$. If we make use of the abbreviation involved in writing (25) in the form (25a), we find

$$l_{r,r}^{(1)} = \frac{1}{4(\alpha_r \beta_r - 1)} \Big[-\beta_r u^2 + 2uv - \alpha_r v^2 \Big]_{(a_r^{(1)}, b_r^{(1)})}^{(u,v)},$$

$$l_{r,i}^{(1)} = \left[\frac{e^{(\alpha_r - \alpha_i)u + (\beta_r - \beta_i)v}}{2(\alpha_i \beta_i - 1)(\alpha_r - \alpha_i)(\beta_r - \beta_i)} \{ (\alpha_r - \alpha_i)u + (\beta_r - \beta_i)v - 1 \} \right]_{(a_i^{(1)}, b_i^{(1)})}^{(u,v)}, \quad i \neq r.$$

Then

$$\begin{aligned} y_r^{(1)} = {} & e^{\alpha_r u + \beta_r v} \left[\frac{-\beta_r u^2 + 2uv - \alpha_r v^2}{4(\alpha_r \beta_r - 1)} + \sum_{i=1}^{4}{}' \frac{(\alpha_r - \alpha_i)u + (\beta_r - \beta_i)v - 1}{2(\alpha_i \beta_i - 1)(\alpha_r - \alpha_i)(\beta_r - \beta_i)} \right] \\ & - \sum_{i=1}^{4}{}' e^{\alpha_i u + \beta_i v} \frac{e^{(\alpha_r - \alpha_i) a_i^{(1)} (\beta_r - \beta_i) b_i^{(1)}}}{2(\alpha_i \beta_i - 1)(\alpha_r - \alpha_i)(\beta_r - \beta_i)} \{ (\alpha_r - \alpha_i) a_i^{(1)} + (\beta_r - \beta_i) b_i^{(1)} - 1 \} \\ & - e^{\alpha_r u + \beta_r v} \frac{-\beta_r a_r^{(1)2} + 2 a_r^{(1)} b_r^{(1)} - \alpha_r b_r^{(1)2}}{4(\alpha_r \beta_r - 1)}, \end{aligned} \tag{30}$$

where the stroke on the summation symbol indicates that the value $i = r$ is to be omitted. It will obviously be possible to find finite numbers $a_i^{(1)}$, $b_i^{(1)}$ and $a_r^{(1)}$, $b_r^{(1)}$ so as to cause the terms $e^{\alpha_i u + \beta_i v}$ $(i \neq r)$ to disappear, and to reduce to zero the constant term of the expression which is multiplied by $e^{\alpha_r u + \beta_r v}$. If this be done, we have

$$\begin{aligned} y_r^{(1)} & = e^{\alpha_r u + \beta_r v} \left[\frac{-\beta_r u^2 + 2uv - \alpha_r v^2}{4(\alpha_r \beta_r - 1)} + \sum_{i=1}^{4}{}' \frac{(\alpha_r - \alpha_i)u + (\beta_r - \beta_i)v}{2(\alpha_i \beta_i - 1)(\alpha_r - \alpha_i)(\beta_r - \beta_i)} \right] \\ & = e^{\alpha_r u + \beta_r v} P_r^{(1)}(u, v), \end{aligned} \tag{30a}$$

so that $y_r^{(1)}$ reduces to an exponential factor multiplied by a polynomial $P_r^{(1)}(u, v)$ of the second degree in u and v, whose constant term is equal to zero.

We may generalize this result. For every value of k there exists a solution $y_r^{(k)}$ of (17b), expressible in the form

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} \sum_{j=1}^{2k} A_{r,j}^{(k)} = e^{\alpha_r u + \beta_r v} P_r^{(k)}(u, v),$$

where $A_{r,j}^{(k)}$ is a homogeneous polynomial of degree j in u and v. In other words, $y_r^{(k)}$ is equal to $e^{\alpha_r u + \beta_r v}$ multiplied by a polynomial of degree $2k$ in u and v, without a constant term.

The theorem is true for $k=1$. We proceed to prove the general theorem by induction. Let us suppose, therefore, that

$$y_r^{(k-1)} = e^{\alpha_r u+\beta_r v} \sum_{j=1}^{2k-2} A_{r,j}^{(k-1)},$$

and consequently

$$\frac{\partial y_r^{(k-1)}}{\partial u} = e^{\alpha_r u+\beta_r v} \sum_{j=1}^{2k-2} \left(\frac{\partial A_{r,j}^{(k-1)}}{\partial u} + \alpha_r A_{r,j}^{(k-1)} \right),$$

$$\frac{\partial y_r^{(k-1)}}{\partial v} = e^{\alpha_r u+\beta_r v} \sum_{j=1}^{2k-2} \left(\frac{\partial A_{r,j}^{(k-1)}}{\partial v} + \beta_r A_{r,j}^{(k-1)} \right).$$

We shall find

$$l_{r,i}^{(k)} = \int_{a_i^{(k)},\, b_i^{(k)}}^{(u,v)} e^{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v} [\{ P_{r,i,1}^{(k-1)} + P_{r,i,2}^{(k-1)} + \dots + P_{r,i,2k-1}^{(k-1)} \} du + \{ Q_{r,i,1}^{(k-1)} + \dots + Q_{r,i,2k-1}^{(k-1)} \} dv], \quad (31)$$

where

$$\left.\begin{aligned} P_{r,i,j}^{(k-1)} &= \frac{1}{4(\alpha_i\beta_i-1)} \left[\{ -(\beta_r+\beta_i)u + 2v \} A_{r,j-1}^{(k-1)} - u\frac{\partial A_{r,j}^{(k-1)}}{\partial v} \right], \\ Q_{r,i,j}^{(k-1)} &= \frac{1}{4(\alpha_i\beta_i-1)} \left[\{ 2u - (\alpha_r+\alpha_i)v \} A_{r,j-1}^{(k-1)} - v\frac{\partial A_{r,j}^{(k-1)}}{\partial u} \right], \\ &(j = 1, 2, 3, \dots, 2k-1), \end{aligned}\right\} \quad (32)$$

in the application of which formulæ we must remember to put

$$A_{r,0}^{(k-1)} = A_{2k-1}^{(k-1)} = 0.$$

Clearly, $P_{r,i,j}^{(k-1)}$ and $Q_{r,i,j}^{(k-1)}$ are homogeneous polynomials in u and v, of degree j. From our general theory we know that the expression under the integral sign in (31) is an exact differential. We must, therefore, have

$$\left.\begin{aligned} (\beta_r-\beta_i) P_{r,i,2k-1}^{(k-1)} &= (\alpha_r-\alpha_i) Q_{r,i,2k-1}^{(k-1)}, \\ (\beta_r-\beta_i) P_{r,i,2k-2}^{(k-1)} + \frac{\partial P_{r,i,2k-1}^{(k-1)}}{\partial v} &= (\alpha_r-\alpha_i) Q_{r,i,2k-2}^{(k-1)} + \frac{\partial Q_{r,i,2k-1}^{(k-1)}}{\partial u}, \\ \dots\dots\dots\dots & \dots\dots\dots\dots, \\ (\beta_r-\beta_i) P_{r,i,1}^{(k-1)} + \frac{\partial P_{r,i,2}^{(k-1)}}{\partial v} &= (\alpha_r-\alpha_i) Q_{r,i,1}^{(k-1)} + \frac{\partial Q_{r,i,2}^{(k-1)}}{\partial u}, \\ \frac{\partial P_{r,i,1}^{(k-1)}}{\partial v} &= \frac{\partial Q_{r,i,1}^{(k-1)}}{\partial u}. \end{aligned}\right\} \quad (33)$$

Consider first the case $i \neq r$. We may write

$$l_{r,i}^{(k)} = [e^{(\alpha_r-\alpha_i)u+(\beta_r-\beta_i)v} \{ R_{r,i,0}^{(k)} + R_{r,i,1}^{(k)} + \dots + R_{r,i,2k-1}^{(k)} \}]_{a_i^{(k)},\, b_i^{(k)}}^{u,\,v},$$

where $R_{r,i,j}^{(k)}$ is a homogeneous polynomial of degree j in u and v. In fact, in order that this may be so, it is necessary and sufficient to satisfy the conditions

$$\frac{\partial R^{(k)}_{r,i,1}}{\partial u}+(\alpha_r-\alpha_i)R^{(k)}_{r,i,0}=0, \qquad \frac{\partial R^{(k)}_{r,i,1}}{\partial v}+(\beta_r-\beta_i)R^{(k)}_{r,i,0}=0,$$

$$\frac{\partial R^{(k)}_{r,i,2}}{\partial u}+(\alpha_r-\alpha_i)R^{(k)}_{r,i,1}=P^{(k-1)}_{r,i,1}, \qquad \frac{\partial R^{(k)}_{r,i,2}}{\partial v}+(\beta_r-\beta_i)R^{(k)}_{r,i,1}=Q^{(k-1)}_{r,i,1},$$

$$\cdots\cdots\cdots\cdots, \qquad \cdots\cdots\cdots\cdots,$$

$$\frac{\partial R^{(k)}_{r,i,2k-1}}{\partial u}+(\alpha_r-\alpha_i)R^{(k)}_{r,i,2k-2}=P^{(k-1)}_{r,i,2k-2}, \qquad \frac{\partial R^{(k)}_{r,i,2k-1}}{\partial v}+(\beta_r-\beta_i)R^{(k)}_{r,i,2k-2}=Q^{(k-1)}_{r,i,2k-2},$$

$$(\alpha_r-\alpha_i)R^{(k)}_{r,i,2k-1}=P^{(k-1)}_{r,i,2k-1}, \qquad (\beta_r-\beta_i)R^{(k)}_{r,i,2k-1}=Q^{(k-1)}_{r,i,2k-1},$$

which are consistent on account of (33) and give

$$R^{(k)}_{r,i,2k-1}=\frac{P^{(k-1)}_{r,i,2k-1}}{\alpha_r-\alpha_i}=\frac{Q^{(k-1)}_{r,i,2k-1}}{\beta_r-\beta_i},$$

$$R^{(k)}_{r,i,2k-2}=\frac{P^{(k-1)}_{r,i,2k-2}}{\alpha_r-\alpha_i}-\frac{\dfrac{\partial P^{(k-1)}_{r,i,2k-1}}{\partial u}}{(\alpha_r-\alpha_i)^2}=\frac{Q^{(k-1)}_{r,i,2k-2}}{\beta_r-\beta_i}-\frac{\dfrac{\partial Q^{(k-1)}_{r,i,2k-1}}{\partial v}}{(\beta_r-\beta_i)^2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$R^{(k)}_{r,i,2k-\lambda}=\frac{P^{(k-1)}_{r,i,2k-\lambda}}{\alpha_r-\alpha_i}-\frac{\dfrac{\partial P^{(k-1)}_{r,i,2k-\lambda+1}}{\partial u}}{(\alpha_r-\alpha_i)^2}+\cdots+(-1)^{\lambda-1}\frac{\dfrac{\partial^{\lambda-1}P^{(k-1)}_{r,i,2k-1}}{\partial u^{\lambda-1}}}{(\alpha_r-\alpha_i)^{\lambda}}$$

$$=\frac{Q^{(k-1)}_{r,i,2k-\lambda}}{\beta_r-\beta_i}-\frac{\dfrac{\partial Q^{(k-1)}_{r,i,2k-\lambda+1}}{\partial v}}{(\beta_r-\beta_i)^2}+\cdots+(-1)^{\lambda-1}\frac{\dfrac{\partial^{\lambda-1}Q^{(k-1)}_{r,i,2k-1}}{\partial v^{\lambda-1}}}{(\beta_r-\beta_i)^{\lambda}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

If we write $2k-\lambda=j$, we obtain the general formula

$$R^{(k)}_{r,i,j}=\sum_{\lambda=0}^{2k-j-1}\frac{(-1)^{\lambda}}{(\alpha_r-\alpha_i)^{\lambda+1}}\frac{\partial^{\lambda}P^{(k-1)}_{r,i,j+\lambda}}{\partial u^{\lambda}}=\sum_{\lambda=0}^{2k-j-1}\frac{(-1)^{\lambda}}{(\beta_r-\beta_i)^{\lambda+1}}\frac{\partial^{\lambda}Q^{(k-1)}_{r,i,j+\lambda}}{\partial v^{\lambda}}, \tag{34}$$

$$(j=0,1,2,\ldots,2k-1), \quad (i=1,2,3,4;\ i\neq r),$$

which gives, upon substituting the values (32), the following two equivalent expressions:

$$\left.\begin{aligned}
R^{(k)}_{r,i,j}=\sum_{\lambda=0}^{2k-j-1}&\frac{(-1)^{\lambda}}{4(\alpha_r-\alpha_i)^{\lambda+1}(\alpha_i\beta_i-1)}\Big[\{-\beta_r+\beta_i)u+2v\}\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda-1}}{\partial u^{\lambda}}\\
&-\lambda(\beta_r+\beta_i)\frac{\partial^{\lambda-1}A^{(k-1)}_{r,j+\lambda-1}}{\partial u^{\lambda-1}}-u\frac{\partial^{\lambda+1}A^{(k-1)}_{r,j+\lambda}}{\partial u^{\lambda}\partial v}-\lambda\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda}}{\partial u^{\lambda-1}\partial v}\Big],\\
R^{(k)}_{r,i,j}=\sum_{\lambda=0}^{2k-j-1}&\frac{(-1)^{\lambda}}{4(\beta_r-\beta_i)^{\lambda+1}(\alpha_i\beta_i-1)}\Big[\{2u-(\alpha_r+\alpha_i)v\}\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda-1}}{\partial v^{\lambda}}\\
&-\lambda(\alpha_r+\alpha_i)\frac{\partial^{\lambda-1}A^{(k-1)}_{r,j+\lambda-1}}{\partial v^{\lambda-1}}-v\frac{\partial^{\lambda+1}A^{(k-1)}_{r,j+\lambda}}{\partial u\,\partial v^{\lambda}}-\lambda\frac{\partial^{\lambda}A^{(k-1)}_{r,j+\lambda}}{\partial u\,\partial v^{\lambda-1}}\Big],\\
(j=0,1,2,&\ldots,2k-1), \quad (i=1,2,3,4;\ i\neq r).
\end{aligned}\right\} \tag{35}$$

For $i = r$, we find

$$l^{(k)}_{r,r} = \int_{(a_r^{(k)},\, b_r^{(k)})}^{(u,\,v)} [\{P^{(k-1)}_{r,r,1} + \dots + P^{(k-1)}_{r,r,2k-1}\} du + \{Q^{(k-1)}_{r,r,1} + \dots + Q^{(k-1)}_{r,r,2k-1}\} dv],$$

and the conditions (33) reduce to

$$\frac{\partial P^{(k-1)}_{r,r,j}}{\partial v} = \frac{\partial Q^{(k-1)}_{r,r,j}}{\partial u}, \qquad (j = 1, 2, \dots, 2k-1).$$

We may write

$$l^{(k)}_{r,r} = [R^{(k)}_{r,r,0} + R^{(k)}_{r,r,1} + \dots + R^{(k)}_{r,r,2k}]^{(u,\,v)}_{(a_r^{(k)},\, b_r^{(k)})},$$

where $R^{(k)}_{r,r,j+1}$ is a homogeneous polynomial of order $j+1$, subject to the compatible conditions

$$\frac{\partial R^{(k)}_{r,r,j+1}}{\partial u} = P^{(k-1)}_{r,r,j} = \frac{1}{4(\alpha_r \beta_r - 1)} \left[\frac{\partial q_r}{\partial u} A^{(k-1)}_{r,j-1} - u \frac{\partial A^{(k-1)}_{r,j}}{\partial v}\right],$$

$$\frac{\partial R^{(k)}_{r,r,j+1}}{\partial v} = Q^{(k-1)}_{r,r,j} = \frac{1}{4(\alpha_r \beta_r - 1)} \left[\frac{\partial q_r}{\partial v} A^{(k-1)}_{r,j-1} - v \frac{\partial A^{(k-1)}_{r,j}}{\partial u}\right],$$

$$(j = 1, 2, 3, \dots, 2k-1),$$

where

$$q_r = -\beta_r u^2 + 2uv - \alpha_r v^2. \tag{36}$$

If we multiply both members of the first equation by u, and those of the second by v, and add, making use of Euler's theorem on homogeneous functions, we find

$$(j+1) R^{(k)}_{r,r,j+1} = \frac{1}{4(\alpha_r \beta_r - 1)} \left[2 q_r A^{(k-1)}_{r,j-1} - \left(v^2 \frac{\partial A^{(k-1)}_{r,j}}{\partial u} + u^2 \frac{\partial A^{(k-1)}_{r,j}}{\partial v}\right)\right], \tag{37}$$
$$(j = 1, 2, \dots, 2k-1),$$

to which we may add

$$R^{(k)}_{r,r,1} = 0. \tag{37a}$$

The value of the arbitrary constant $R^{(k)}_{r,r,0}$ is immaterial. The resulting expression for $y^{(k)}_r$ will be

$$\begin{aligned} y^{(k)}_r = {} & e^{\alpha_r u + \beta_r v} [R^{(k)}_{r,r,2} + \dots + R^{(k)}_{r,r,2k}]^{u,\,v}_{a_r^{(k)},\, b_r^{(k)}} \\ & + e^{\alpha_r u + \beta_r v} \sum_{i=1}^{4}{}' \{R^{(k)}_{r,i,0} + \dots + R^{(k)}_{r,i,2k-1}\} \\ & - \sum_{i=1}^{4}{}' e^{\alpha_i u + \beta_i v + (\alpha_r - \alpha_i) a_i^{(k)} + (\beta_r - \beta_i) b_i^{(k)}} [R^{(k)}_{r,i,0} + \dots + R^{(k)}_{r,i,2k-1}]_{a_i^{(k)},\, b_i^{(k)}}, \end{aligned} \tag{38}$$

and this will have the desired form, if and only if the limits are chosen in such a way that

$$\left.\begin{aligned} & [R^{(k)}_{r,i,0} + \dots + R^{(k)}_{r,i,2k-1}]_{a_i^{(k)},\, b_i^{(k)}} = 0, \\ & [R^{(k)}_{r,r,2} + \dots + R^{(k)}_{r,r,2k}]_{a_r^{(k)},\, b_r^{(k)}} = \sum_{i=1}^{4}{}' R^{(k)}_{r,i,0}. \end{aligned}\right\} (i = 1, 2, 3, 4;\ i \neq r), \tag{39}$$

For every finite value of k this can certainly be done, and we shall then have the following formula for $y_r^{(k)}$:

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} [R^{(k)}_{r,r,2k} + \sum_{i=1}^{4} (R^{(k)}_{r,i,1} + \dots + R^{(k)}_{r,i,2k-1})],$$

if we remember that, according to (37a),

$$R^{(k)}_{r,r,1} = 0.$$

Therefore,

$$A^{(k)}_{r,j} = \sum_{i=1}^{4} R^{(k)}_{r,i,j}, \quad (j = 1, 2, \dots, 2k-1), \quad A^{(k)}_{r,2k} = R^{(k)}_{r,r,2k},$$

whence, making use of (35) and (37),

$$\begin{aligned} A^{(k)}_{r,j} = {} & \frac{1}{4j(\alpha_r\beta_r - 1} \left[2q_r A^{(k-1)}_{r,j-2} - \left(v^2 \frac{\partial A^{(k-1)}_{r,j-1}}{\partial u} + u^2 \frac{\partial A^{(k-1)}_{r,j-1}}{\partial v} \right) \right] \\ & + \frac{1}{4} \sum_{i=1}^{4}{}' \sum_{\lambda=0}^{2k-j-1} \frac{(-1)^\lambda}{(\alpha_r - \alpha_i)^{\lambda+1}(\alpha_i\beta_i - 1)} \left[\{ -(\beta_r + \beta_i)u + 2v \} \frac{\partial^\lambda A^{(k-1)}_{r,j+\lambda-1}}{\partial u^\lambda} \right. \\ & \left. - \lambda(\beta_r + \beta_i) \frac{\partial^{\lambda-1} A^{(k-1)}_{r,j+\lambda-1}}{\partial u^{\lambda-1}} - u \frac{\partial^{\lambda+1} A^{(k-1)}_{r,j+\lambda}}{\partial u^\lambda \partial v} - \lambda \frac{\partial^\lambda A^{(k-1)}_{r,j+\lambda}}{\partial u^{\lambda-1} \partial v} \right], \\ & (j = 1, 2, \dots, 2k-1). \end{aligned}$$

$$A^{(k)}_{r,2k} = \frac{q_r}{4k(\alpha_r\beta_r - 1)} A^{(k-1)}_{r,2k-2}.$$

A second equivalent form for $A^{(k)}_{r,j}$ may be found by using the second of the two equations (35).

From the last formula we find at once

$$A^{(k)}_{2k} = \frac{q_r^k}{4^k k! (\alpha_r\beta_r - 1)^k}, \qquad (\lambda = 1, 2, 3, \dots). \tag{40}$$

Let us put

$$s_{r,\lambda} = \sum_{i=1}^{4}{}' \frac{1}{(\alpha_r - \alpha_i)^{\lambda+1}(\alpha_i\beta_i - 1)}, \quad t_{r,\lambda} = \sum_{i=1}^{4}{}' \frac{\beta_r + \beta_i}{(\alpha_r - \alpha_i)^{\lambda+1}(\alpha_i\beta_i - 1)}. \tag{41}$$

Then we find

$$\begin{aligned} A^{(k)}_{r,j} = {} & \frac{1}{4j(\alpha_r\beta_r - 1)} \left[2q_r A^{(k-1)}_{r,j-2} - \left(v^2 \frac{\partial A^{(k-1)}_{r,j-1}}{\partial u} + u^2 \frac{\partial A^{(k-1)}_{r,j-1}}{\partial v} \right) \right] \\ & + \frac{1}{4} \sum_{\lambda=0}^{2k-j-1} (-1)^\lambda \left\{ (-t_{r,\lambda} u + 2s_{r,\lambda} v) \frac{\partial^\lambda A^{(k-1)}_{r,j+\lambda-1}}{\partial u^\lambda} - \lambda t_{r,\lambda} \frac{\partial^{\lambda-1} A^{(k-1)}_{r,j+\lambda-1}}{\partial u^{\lambda-1}} \right. \\ & \left. - s_{r,\lambda} u \frac{\partial^{\lambda+1} A^{(k-1)}_{r,j+\lambda}}{\partial u^\lambda \partial v} - \lambda s_{r,\lambda} \frac{\partial^\lambda A^{(k-1)}_{r,j+\lambda}}{\partial u^{\lambda-1} \partial v} \right\}, \\ & (j = 1, 2, \dots, 2k-1), \end{aligned} \tag{42}$$

and a second equivalent form corresponding to the second form of equation (35).

Thus we may obtain a solution of (S) in the form

$$y_r = y_r^0 + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \dots + y_r^{(k)} c_0^k + \mathfrak{R},$$

where the further development of $\mathfrak{R}$ would give powers of higher than the k-th order in c_0, k being any finite integer, and each of the functions $y_r^{(i)}$ being of the form

$$y_r^{(i)} = e^{\alpha_r u + \beta_r v} \sum_{j=1}^{2i} A_{r,j}^{(i)}.$$

We may not, however, conclude in this fashion that a solution of (S) exists of the form

$$y_r^{(0)} + y_r^{(1)} c_0 + y_r^{(2)} c_0^2 + \dots \textit{ ad infinitum},$$

since we can not be certain that the limits of the integrals, as determined by equations (39), do not tend to become infinite with k. If this were the case, the convergence theorem which was proved in § 3 would not be applicable.

In order to establish the existence of such solutions, we may proceed as follows: In equation (25) let us put $a = b = 0$. We then find, making use of (30) and (30a),

$$y_r^{(1)} = e^{\alpha_r u + \beta_r v} P_r^{(1)}(u, v) + \sum_{s=1}^{4} c_{r,s}^{(1)} e^{\alpha_s u + \beta_s v},$$

where

$$c_{r,s}^{(1)} = \frac{1}{2(\alpha_s \beta_s - 1)(\alpha_r - \alpha_s)(\beta_r - \beta_s)}, \quad (s \neq r), \quad \sum_{s=1}^{4} c_{r,s}^{(1)} = 0.$$

With this value of $y_r^{(1)}$ we find further

$$y_r^{(2)} = e^{\alpha_r u + \beta_r v} P_r^{(2)}(u, v) + \sum_{s=1}^{4} c_{r,s}^{(1)} \left\{ e^{\alpha_s u + \beta_s v} P_s^{(1)}(u, v) + \sum_{t=1}^{4} c_{s,t}^{(1)} e^{\alpha_t u + \beta_t v} \right\} + \sum_{s=1}^{4} c_{r,s}^{(1,1)} e^{\alpha_s u + \beta_s v},$$

where

$$c_{r,s}^{(1,1)} = - R_{r,i,0}^{(2)}, \quad (s \neq r), \quad \sum_{s=1}^{4} c_{r,s}^{(1,1)} = 0,$$

or

$$y_r^{(2)} = e^{\alpha_r u + \beta_r v} P_r^{(2)}(u, v) + \sum_{s=1}^{4} c_{r,s}^{(1)} e^{\alpha_s u + \beta_s v} P_s^{(1)}(u, v) + \sum_{s=1}^{4} c_{r,s}^{(2)} e^{\alpha_s u + \beta_s v},$$

if we write

$$c_{r,s}^{(2)} = c_{r,s}^{(1,1)} + \sum_{j=1}^{4} c_{r,j}^{(1)} c_{j,s}^{(1)}.$$

We may now prove by induction that

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} P_r^{(k)}(u, v) + \sum_{s=1}^{4} c_{r,s}^{(1)} e^{\alpha_s u + \beta_s v} P_s^{(k-1)}(u, v) + \dots + \sum_{s=1}^{4} c_{r,s}^{(k)} e^{\alpha_s u + \beta_s v},$$

or

$$y_r^{(k)} = e^{\alpha_r u + \beta_r v} P_r^{(k)}(u, v) + \sum_{s=1}^{4} \sum_{j=1}^{k} c_{r,s}^{(j)} e^{\alpha_s u + \beta_s v} P_s^{(k-j)}(u, v), \tag{36}$$

all of the quantities $c_{r,s}^{(j)}$ being constants, and the functions $P_\mu^{(\lambda)}$ being polynomials of degree 2λ.

We know that the series

$$y_r = e^{\alpha_r u + \beta_r v} + \sum_{k=1}^{\infty} y_r^{(k)} c_0^k \tag{37}$$

is absolutely and uniformly convergent for all finite values of u, v and c_0, and that for $u = v = 0$ we shall have

$$y_r = 1, \quad \frac{\partial y_r}{\partial u} = \alpha_r, \quad \frac{\partial y_r}{\partial v} = \beta_r, \quad \frac{\partial^2 y_r}{\partial u \partial v} = \alpha_r \beta_r. \tag{38}$$

From the general theory of linear partial differential equations, we know further that the uniquely determined solution of system (S), which satisfies the initial conditions (38), may be represented as a power-series $\mathfrak{P}(u, v, c_0)$ in the three variables u, v, c_0, absolutely and uniformly convergent for all finite values of these variables. Moreover, for any system of finite values of u, v, c_0, there exists a convergent series D of positive constants which dominates the series of absolute values of the terms of $\mathfrak{P}(u, v, c_0)$.

Since the series (37) and $\mathfrak{P}(u, v, c_0)$ are both absolutely and uniformly convergent, and since both of these series represent, for all finite values of u, v, c_0, the same solution of (S), we shall obtain $\mathfrak{P}(u, v, c_0)$ from (37), if we substitute in (37) the expressions (36) for $y_r^{(k)}$, expand the exponentials into power-series and perform the indicated multiplications.

Now the quantities $|e^{\alpha_s u + \beta_s v}|$ and $|P_r^{(\lambda)}(u, v)|$ are at most equal to the sum of the moduli of the terms of their respective expansions. Let us replace the several terms $y_r^{(k)} c_0^k$, of series (37), by the sums resulting from (36), and let us regard the individual terms of the sums thus obtained as terms of a new series Σ. The series of the absolute values of the terms of Σ will be dominated by a series of positive constants, whose sum is at most equal to the sum of the series D of positive constants, which dominates the series of absolute values of the terms of $\mathfrak{P}(u, v, c_0)$. Consequently, the series Σ itself and any series composed of terms selected from those of Σ, provided that no term of Σ be taken more than once, will be absolutely and uniformly convergent.

Therefore, the series

$$y = e^{\alpha_r u + \beta_r v} \sum_{k=0}^{\infty} P_r^{(k)}(u, v)\, c_0^k,$$

which is known to satisfy system (S) formally, is absolutely and uniformly convergent. If now we differentiate this series term by term, either once or twice, with respect to u or v, and compare the resulting series with the corresponding first- or second-order derivatives of $\mathfrak{P}(u, v, c_0)$, we recognize by a

repetition of the above argument that these series also are absolutely and uniformly convergent and therefore represent $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$, etc.

We finally obtain the following theorem:

If u_0, v_0 is a point for which the exponents $\alpha_1, \ldots, \alpha_4$ are distinct, there exist four linearly independent solutions of system (S), of the form

$$y_r = e^{\alpha_r(u-u_0)+\beta_r(v-v_0)}[1+\sum_{k=1}^{\infty} P_r^{(k)}(u-u_0,\, v-v_0)\, c_0^k], \quad (r=1,2,3,4),$$

where $P_r^{(k)}$, the coefficient of c_0^k, is a polynomial of degree $2k$.

In formulating this theorem we have made use of a terminology suggested by our discussion of the effect of the linear transformation (28a). In the vicinity of the point u_0, v_0, the coefficients of y in the two equations of (S) may be written in the form

$$c_0(u-u_0)+c_0u_0+c_1, \quad c_0(v-v_0)+c_0v_0+c_2,$$

and we may speak of the four pairs of numbers α, β which satisfy the equations

$$\alpha^2+2\beta+c_0u_0+c_1=0, \quad \beta^2+2\alpha+c_0v_0+c_2=0,$$

as the *exponents for the point* u_0, v_0.

We see that, associated with a system of form (S), there is a set of polynomials. We have seen, moreover, how these polynomials may be calculated by means of certain recursion formulæ. These formulæ, however, are rather complicated, and we now proceed to study the polynomials $P_r^{(k)}(u, v)$ by a different method, which gives far simpler results.

Let us substitute, in (S),

$$y = \eta\, e^{\alpha u+\beta v},$$

where α and β are any two numbers which satisfy the *characteristic* equations

$$\alpha^2+2\beta+c_1=0, \quad \beta^2+2\alpha+c_2=0.$$

The system (S) will be transformed into the following system:

$$\eta_{uu}+2\alpha\eta_u+2\eta_v+c_0u\eta=0, \quad \eta_{vv}+2\eta_u+2\beta\eta_v+c_0v\eta=0. \qquad (40)$$

Since (S) has a solution of the form

$$e^{\alpha u+\beta v}[1+\sum_{k=1}^{\infty} P^{(k)}(u,v)\, c_0^k],$$

(40) must have a solution of the form

$$\eta = 1+\sum_{k=1}^{\infty} P^{(k)}(u,v)\, c_0^k, \qquad (41)$$

where

$$P^{(k)}(u,v) = \sum_{i=1}^{2k} A_i^{(k)}(u,v), \qquad (42)$$

$A_i^{(k)}$ being a homogeneous polynomial of degree i.

If we substitute the expression (41) for η into (40), we find

$$\left.\begin{aligned} P_{uu}^{(k)} + 2\alpha P_u^{(k)} + 2P_v^{(k)} + uP^{(k-1)} = 0, \\ P_{vv}^{(k)} + 2P_u^{(k)} + 2\beta P_v^{(k)} + vP^{(k-1)} = 0, \end{aligned}\right\} \quad (k = 1, 2, 3, \ldots.). \tag{43}$$

From (43) and (42) we find, further,

$$\left.\begin{aligned} \sum_{i=1}^{2k}\left(\frac{\partial^2 A_i^{(k)}}{\partial u^2} + 2\alpha\frac{\partial A_i^{(k)}}{\partial u} + 2\frac{\partial A_i^{(k)}}{\partial v}\right) + u\sum_{i=1}^{2k-2} A_i^{(k-1)} = 0, \\ \sum_{i=1}^{2k}\left(\frac{\partial^2 A_i^{(k)}}{\partial v^2} + 2\frac{\partial A_i^{(k)}}{\partial u} + 2\beta\frac{\partial A_i^{(k)}}{\partial v}\right) + v\sum_{i=1}^{2k-2} A_i^{(k-1)} = 0. \end{aligned}\right\} \tag{44}$$

If we equate to zero the terms of degree $2k-1$, we obtain the equations

$$2\alpha\frac{\partial A_{2k}^{(k)}}{\partial u} + 2\frac{\partial A_{2k}^{(k)}}{\partial v} + uA_{2(k-1)}^{(k-1)} = 0, \quad 2\frac{\partial A_{2k}^{(k)}}{\partial u} + 2\beta\frac{\partial A_{2k}^{(k)}}{\partial v} + vA_{2(k-1)}^{(k-1)} = 0,$$

whence

$$2(\alpha\beta - 1)\frac{\partial A_{2k}^{(k)}}{\partial u} = \tfrac{1}{2} q_u A_{2(k-1)}^{(k-1)}, \quad 2(\alpha\beta - 1)\frac{\partial A_{2k}^{(k)}}{\partial v} = \tfrac{1}{2} q_v A_{2(k-1)}^{(k-1)},$$

where

$$q = -\beta u^2 + 2uv - \alpha v^2, \quad q_u = \frac{\partial q}{\partial u}, \quad q_v = \frac{\partial q}{\partial v}, \tag{45}$$

and where $\alpha\beta - 1$ is different from zero if, as we may assume, α is not a repeated root of the characteristic equations. These equations show that the Jacobian of $A_{2k}^{(k)}$ and q is equal to zero, so that $A_{2k}^{(k)}$ must be a function of q. We easily find

$$A_{2k}^{(k)} = \frac{q^k}{4^k(\alpha\beta - 1)^k k!}, \quad (k = 1, 2, 3, \ldots.), \tag{40}$$

a result which we had already obtained by our previous method.

Since $A_0^{(k)} = 0$, the terms of degree zero and unity, in (44), require special attention. We find

$$\frac{\partial^2 A_2^{(k)}}{\partial u^2} + 2\alpha\frac{\partial A_1^{(k)}}{\partial u} + 2\frac{\partial A_1^{(k)}}{\partial v} = 0, \quad \frac{\partial^2 A_2^{(k)}}{\partial v^2} + 2\frac{\partial A_1^{(k)}}{\partial u} + 2\beta\frac{\partial A_1^{(k)}}{\partial v} = 0,$$

whence

$$2(\alpha\beta - 1)\frac{\partial A_1^{(k)}}{\partial u} = -\beta\frac{\partial^2 A_2^{(k)}}{\partial u^2} + \frac{\partial^2 A_2^{(k)}}{\partial v^2},$$

$$2(\alpha\beta - 1)\frac{\partial A_1^{(k)}}{\partial v} = +\frac{\partial^2 A_2^{(k)}}{\partial u^2} - \alpha\frac{\partial^2 A_2^{(k)}}{\partial v^2}.$$

Let us multiply the members of these equations by u and v respectively, and apply Euler's theorem on homogeneous functions. We find

$$A_1^{(k)} = \frac{1}{2(\alpha\beta - 1)}\Omega(A_2^{(k)}), \tag{46}$$

where Ω denotes the operator

$$\Omega = (-\beta u + v)\frac{\partial^2}{\partial u^2} + (u - \alpha v)\frac{\partial^2}{\partial v^2} = \frac{1}{2}\left(q_u \frac{\partial^2}{\partial u^2} + q_v \frac{\partial^2}{\partial v^2}\right), \tag{47}$$

which plays a fundamental rôle in this theory. In the same way we find, from (44), equating to zero the terms of the first degree,

$$A_2^{(k)} = \frac{1}{4(\alpha\beta - 1)}\,\Omega\,(A_3^{(k)}). \tag{48}$$

Let us now equate to zero the terms of degree $2k - i - 1$ in (44), where $1 < 2k - i - 1 < 2k - 1$. A precisely similar treatment gives the following equation:

$$2(2k - i)(\alpha\beta - 1)\,A_{2k-i}^{(k)} = q\,A_{2(k-1)-i}^{(k-1)} + \Omega\,(A_{2k-i+1}^{(k)}).$$

If we put $2k - i = \nu$, we find

$$A_\nu^{(k)} = \frac{\Omega\,(A_{\nu+1}^{(k)}) + q\,A_{\nu-2}^{(k-1)}}{2\nu(\alpha\beta - 1)}, \quad (\nu = 1, 2, 3, \ldots, 2k - 1), \tag{49}$$

for this formula will also be valid for $\nu = 1$ or 2 if we equate to zero $A_i^{(k)}$ whenever i becomes zero or negative.

By means of formulæ (40) and (49) all of the homogeneous polynomials $A_i^{(k)}$ may be calculated. We shall actually write down their values for $k = 1$ and $k = 2$, viz.:

$$\left.\begin{aligned} A_1^{(1)} &= \frac{1}{4(\alpha\beta - 1)^2}\,[(\beta^2 - \alpha)\,u + (\alpha^2 - \beta)\,v], \\ A_1^{(2)} &= \frac{1}{4(\alpha\beta - 1)}\,[-\beta u^2 + 2uv - \alpha v^2] = \frac{q}{4(\alpha\beta - 1)}, \end{aligned}\right\} \tag{50}$$

and

$$\left.\begin{aligned} A_1^{(2)} &= \\ &\frac{(-6\alpha + 11\beta^2 - 9\alpha^2\beta + 5\alpha^4 + 4\alpha\beta^3 - 5\beta^5)\,u + (-6\beta + 11\alpha^2 - 9\alpha\beta^2 + 5\beta^4 + 4\alpha^3\beta - 5\alpha^5)\,v}{32(\alpha\beta - 1)^5}, \\ A_2^{(2)} &= \\ &\frac{(-6\beta + 5\alpha^2 - 4\alpha\beta^2 + 5\beta^4)\,u^2 + 2(2 + 7\alpha\beta - 5\alpha^3 - 5\beta^3 + \alpha^2\beta^2)\,uv + (-6\alpha + 5\beta^2 - 4\alpha^2\beta + 5\alpha^4)\,v^2}{32(\alpha\beta - 1)^4}, \\ A_3^{(2)} &= \\ &\frac{(2 + 3\alpha\beta - 5\beta^3)\,u^3 + 3(-4\alpha + 5\beta^2 - \alpha^2\beta)\,u^2v + 3(-4\beta + 5\alpha^2 - \alpha\beta^2)\,uv^2 + (2 + 3\alpha\beta - 5\alpha^3)\,v^3}{48(\alpha\beta - 1)^3}, \\ A_4^{(2)} &= \\ &\frac{\beta^2u^4 - 4\beta u^3v + 2(\alpha\beta + 2)\,u^2v^2 - 4\alpha uv^3 + \alpha^2v^4}{32(\alpha\beta - 1)^2} = \frac{q^2}{32(\alpha\beta - 1)^2}. \end{aligned}\right\} \tag{51}$$

We observe in all of these cases that

$$A_\nu^{(k)} = \frac{B_\nu^{(k)}}{(\alpha\beta - 1)^{3k-\nu}}, \tag{52}$$

where $B_\nu^{(k)}$ is a homogeneous polynomial of degree ν in u and v, whose coefficients are integral rational functions of α, β of degree $3k-\nu$. Making use of equation (49), it is easy to prove by induction that this law is true in general.

If we make the substitution (52), our recursion formulæ become

$$B_{2k}^{(k)} = \frac{q^k}{4^k k!}, \quad B_\nu^{(k)} = \frac{1}{2\nu}\left[\Omega\left(B_{\nu+1}^{(k)}\right) + q B_{\nu-2}^{(k-1)}\right], \quad (\nu = 1, 2, \ldots, 2k-1), \tag{53}$$

where we must remember that $B_0^{(j)} = B_{-1}^{(j)} = 0$.

By repeated application of these formulæ we find

$$B_{\nu}^{(k)} = \frac{\Omega^{2k-\nu}(q^k)}{2^{4k-\nu}k!\,\nu(\nu+1)\ldots(2k-1)} + \sum_{j=1}^{2k-\nu}\frac{\Omega^{2k-\nu-j}\left(q B_{2(k-1)-j}^{(k-1)}\right)}{2^{2k-\nu-j+1}\nu(\nu+1)\ldots(2k-j)},$$
$$(\nu = 1, 2, \ldots, 2k-1), \tag{54}$$

where the symbol Ω^λ indicates that the operation Ω is to be performed λ times in succession.

We may obtain another formula for $B_\nu^{(k)}$ from the equations

$$B_\nu^{(k)} = \frac{1}{2\nu}\left[q B_{\nu-2}^{(k-1)} + \Omega\left(B_{\nu+1}^{(k)}\right)\right],$$

$$B_{\nu-2}^{(k)} = \frac{1}{2(\nu-2)}\left[q B_{\nu-4}^{(k-2)} + \Omega\left(B_{\nu-1}^{(k-1)}\right)\right],$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

by eliminating the first terms occurring in the right members. In this process we must distinguish two cases, according as ν is even or odd. We find

$$\left.\begin{aligned} B_{2\lambda}^{(k)} &= \sum_{i=0}^{\lambda-1}\frac{q^i\,\Omega\left(B_{2(\lambda-i)+1}^{(k-i)}\right)}{4^{i+1}\lambda(\lambda-1)\ldots(\lambda-i)}, \\ B_{2\lambda-1}^{(k)} &= \sum_{i=0}^{\lambda-1}\frac{q^i\,\Omega\left(B_{2(\lambda-i)}^{(k-i)}\right)}{2^{i+1}(2\lambda-1)(2\lambda-3)\ldots(2\lambda-2i-1)}, \end{aligned}\right\} \tag{55}$$

or, to include both cases,

$$B_\nu^{(k)} = \sum_{i=0}^{\lambda-1}\frac{q^i\,\Omega\left(B_{\nu-2i+1}^{(k-i)}\right)}{2^{i+1}\nu(\nu-2)\ldots(\nu-2i)}, \tag{56}$$

where $\lambda = \frac{\nu}{2}$ if ν is even, and $\lambda = \frac{\nu+1}{2}$ if ν is odd.

In the second equation of (55), let us put $\lambda = k$. We find

$$B_{2k-1}^{(k)} = \sum_{i=0}^{k-1}\frac{q^i\,\Omega(q^{k-i})}{2^{2k-i+1}(2k-1)(2k-3)\ldots(2k-2i-1)(k-i)!}.$$

Moreover,

$$\Omega(q^{k-i}) = 2(k-i)\,q^{k-i-1}l_1 + 4(k-i)(k-i-1)\,q^{k-i-2}l_3, \tag{57}$$

where

$$\left.\begin{aligned} l_1 &= (\beta^2-\alpha)\,u + (\alpha^2-\beta)\,v, \\ l_3 &= (1-\beta^3)\,u^3 + 3\,(\beta^2-\alpha)\,u^2 v + 3\,(\alpha^2-\beta)\,u v^2 + (1-\alpha^3)\,v^3. \end{aligned}\right\} \quad (58)$$

Consequently,

$$B_{2k-1}^{(k)} = \frac{q^{k-2}}{4^k}\,[q\,l_1\,\theta_1(k) + l_3\,\theta_2(k)], \quad (59)$$

where

$$\left.\begin{aligned} \theta_1(k) &= \sum_{i=0}^{k-1} \frac{2^i}{(2k-1)(2k-3)\ldots.(2k-2i-1)(k-i-1)!}, \\ \theta_2(k) &= \sum_{i=0}^{k-1} \frac{2^{i+1}}{(2k-1)(2k-3)\ldots.(2k-2i-1)(k-i-2)!}. \end{aligned}\right\} \quad (60)$$

We may evaluate these sums and thereby very materially simplify equation (59) by the following method: If we equate to zero the terms of degree $2k-2$ in (44), a simple combination of the two resulting equations gives

$$q_v \frac{\partial B_{2k-1}^{(k)}}{\partial u} - q_u \frac{\partial B_{2k-1}^{(k)}}{\partial v} = (\alpha\beta-1)\left(v\frac{\partial^2 B_{2k}^{(k)}}{\partial u^2} - u\frac{\partial^2 B_{2k}^{(k)}}{\partial v^2}\right). \quad (61)$$

In this equation let us substitute the values (53) and (59) for $B_{2k}^{(k)}$ and $B_{2k-1}^{(k)}$. The left member of (61) becomes

$$L = \frac{\theta_1(k)}{4^k}\,q^{k-1}\,[(\beta^2-\alpha)\,q_v - (\alpha^2-\beta)\,q_u] + \frac{\theta_2(k)}{4^k}\,q^{k-2}\left(q_v\frac{\partial l_3}{\partial u} - q_u\frac{\partial l_3}{\partial v}\right),$$

and the right member reduces to

$$R = \frac{\alpha\beta-1}{4^k\,(k-1)!}\,[q^{k-1}\,(v\,q_{uu} - u\,q_{vv}) + (k-1)\,q^{k-2}\,(v\,q_u^2 - u\,q_v^2)].$$

But

$$\begin{aligned} (\beta^2-\alpha)\,q_v - (\alpha^2-\beta)\,q_u &= 2\,(\alpha\beta-1)\,(\alpha u - \beta v), \\ q_v\frac{\partial l_3}{\partial u} - q_u\frac{\partial l_3}{\partial v} &= 6\,(\alpha\beta-1)\,[-u^3 + (2\alpha+\beta^2)\,u^2 v - (2\beta+\alpha^2)\,u v^2 + v^3], \\ v\,q_{uu} - u\,q_{vv} &= 2\,(\alpha u - \beta v), \\ v\,q_u^2 - u\,q_v^2 &= 4\,[-u^3 + (2\alpha+\beta^2)\,u^2 v - (2\beta+\alpha^2)\,u v^2 + v^3], \end{aligned}$$

so that the equation $L = R$ assumes the form

$$\begin{aligned} &q\,(\alpha u - \beta v)\left[\theta_1(k) - \frac{1}{(k-1)!}\right] \\ &\qquad + [-u^3 + (2\alpha+\beta^2)\,u^2 v - (2\beta+\alpha^2)\,u v^2 + v^3]\left[3\,\theta_2(k) - \frac{2}{(k-2)!}\right] = 0. \end{aligned}$$

The two cubic expressions which appear in the left member of this equation are linearly independent, since $\alpha\beta - 1$ is different from zero. Consequently we must have

$$\theta_1(k) = \frac{1}{(k-1)!}, \quad \theta_2(k) = \frac{2}{3(k-2)!}, \tag{62}$$

so that

$$B^{(k)}_{2k-1} = \frac{q^{k-2}[3ql_1 + 2(k-1)l_3]}{3\cdot 4^k (k-1)!}. \tag{63}$$

The explicit expression for $B^{(k)}_{2k-2}$, obtained by an extension of this method, is already rather complicated, viz.:

$$\begin{aligned} B^{(k)}_{2k-2} = \frac{2q^{k-4}}{3^2\cdot 4^{k+1}(k-2)!}&[18\{2(k-2)-3(\alpha\beta-1)\}q^3 \\ &+9\{5l_1^2+4(k-2)(\alpha\beta-1)uv-8(\alpha\beta-1)^2uv\}q^2 \\ &+6\{(k-2)(2l_1l_3-3l_4)+3(k-2)(\alpha\beta-1)^2u^2v^2\}q \\ &+4(k-2)(k-3)l_3^2], \end{aligned} \tag{64}$$

where

$$l_4 = \beta(-\beta u + v)^4 + \alpha(u - \alpha v)^4. \tag{65}$$

The transformation (28a) enables us to simplify these formulæ very considerably. Let us assume that such a transformation has been made so as to reduce c_1 and c_2 to zero. The characteristic equations become

$$\alpha^2 + 2\beta = \beta^2 + 2\alpha = 0,$$

and one pair of solutions is $\alpha = \beta = 0$. The operator Ω reduces to

$$\Omega = v\frac{\partial^2}{\partial u^2} + u\frac{\partial^2}{\partial v^2},$$

and it becomes an easy matter to calculate as many of the polynomials $P^{(k)}(u,v)$ as may be desired, although, even with this simplification, the general expression seems to be rather complicated. We find

$$\left.\begin{aligned} P^{(0)} &= 1, \\ P^{(1)} &= -\frac{1}{2}uv, \\ P^{(2)} &= -\frac{1}{8}uv - \frac{1}{24}(u^3+v^3) + \frac{1}{8}u^2v^2, \\ P^{(3)} &= -\frac{3}{32}uv + \frac{1}{32}(u^3+v^3) - \frac{3}{32}u^2v^2 + \frac{1}{48}uv(u^3+v^3) - \frac{1}{48}u^3v^3, \\ P^{(4)} &= \frac{7}{2^6}uv - \frac{7}{2^6\cdot 3}(u^3+v^3) + \frac{7}{2^6}u^2v^2 - \frac{11}{2^7\cdot 3}uv(u^3+v^3) \\ &\quad + \frac{1}{2^7\cdot 3^2}[(u^3+v^3)^2 + 36u^3v^3] - \frac{1}{2^6\cdot 3}u^2v^2(u^3+v^3) + \frac{u^4v^4}{2^4\cdot 4!}. \end{aligned}\right\} \tag{66}$$

We shall close this discussion with one further remark. The expression (41) for η may be written

$$\eta = e^{\frac{c_0 q}{4(\alpha\beta-1)}} + \sum_{k=1}^{\infty} Q^{(k)}(u,v)\,c_0^k, \tag{67}$$

where the functions $Q^{(k)}(u,v)$ are polynomials of degree $2k-1$, differing from $P^{(k)}(u,v)$ merely by the absence of the terms of degree $2k$. Thus, by introduction of the first exponential term of (67), the polynomials $P^{(k)}$ have been deprived of their terms of highest degree. By means of equation (63) this process may be carried out one step further, depriving the polynomials $Q^{(k)}$, in their turn, of their terms of highest degree. However, for the present, we shall refrain from any further developments in this direction.

§ 5. *Integral Equations Satisfied by the Solutions of (S).*

If Y is any solution of (S), it may also be regarded as a solution of the non-homogeneous system

$$y_{uu}+2y_v+c_1y+c_0uY=0, \qquad y_{vv}+2y_u+c_2y+c_0vY=0.$$

The method of § 2 for integrating such a system is applicable, since the integrability conditions are all satisfied. Consequently we obtain the following relation:

$$(J) \quad Y=c_0\sum_{i=1}^{4}e^{\alpha_iu+\beta_iv}\int_{a,b}^{u,v}\frac{e^{-\alpha_iu-\beta_iv}}{4(\alpha_i\beta_i-1)}\left[\left\{(-\beta_iu+2v)Y-u\frac{\partial Y}{\partial v}\right\}du+\left\{(2u-\alpha_iv)Y-v\frac{\partial Y}{\partial u}\right\}dv\right]+\sum_{i=1}^{4}a_ie^{\alpha_iu+\beta_iv},$$

where $a_1,\ldots,a_4$ are constants. The integrals on the right member of this equation are independent of the path of integration, as a consequence of the differential equations satisfied by Y, and have definite finite values since all of the solutions of (S) are integral transcendental functions of u and v.

Moreover, if these integrals be denoted by l_i, they have been determined, in accordance with the method of § 2, in such a way that, if we write

$$\eta=\sum_{i=1}^{4}l_ie^{\alpha_iu+\beta_iv},$$

we shall have

$$\frac{\partial\eta}{\partial u}=\sum_{i=1}^{4}\alpha_il_ie^{\alpha_iu+\beta_iv}, \quad \frac{\partial\eta}{\partial v}=\sum_{i=1}^{4}\beta_il_ie^{\alpha_iu+\beta_iv}, \quad \frac{\partial^2\eta}{\partial u\,\partial v}=\sum_{i=1}^{4}\alpha_i\beta_il_ie^{\alpha_iu+\beta_iv},$$

just as though $l_1,\ldots,l_4$ were constants. Consequently, the solution of the integral equation (J) will satisfy the same initial conditions for $u=a$, $v=b$ as the function

$$\sum_{i=1}^{4}a_ie^{\alpha_iu+\beta_iv},$$

so that the equation of form (J) satisfied by y_r will be obtained if we replace this sum by $e^{\alpha_r u+\beta_r v}$.

Let us change the notation for the variables of integration. Then, the integral equation satisfied by y_r may be written as follows:

$$(J_r)\quad Y(u,v)=e^{\alpha_r u+\beta_r v}$$
$$+c_0\int_{a,b}^{u,v}\sum_{i=1}^{4}\frac{e^{\alpha_i(u-p)+\beta_i(v-q)}}{4(\alpha_i\beta_i-1)}\left[\left\{(-\beta_i p+2q)\,Y(p,q)-p\frac{\partial Y(p,q)}{\partial q}\right\}dp\right.$$
$$\left.+\left\{(2p-\alpha_i q)\,Y(p,q)-q\frac{\partial Y(p,q)}{\partial p}\right\}dq\right].$$

Evidently the Liouville-Neumann method of successive substitution, as applied to this integral equation, will lead us back to the solution already obtained in § 3.

It is easy to see, conversely, that if α_i, β_i are the four pairs of solutions (supposed distinct) of the equations

$$\alpha^2+2\beta+c_1=0,\quad \beta^2+2\alpha+c_2=0,$$

any continuous solution of (J_r), with continuous first and second derivatives, will be a solution of system (S), satisfying the given initial conditions for $u=a$, $v=b$, thus establishing, by a second method, the uniqueness of the corresponding solution of the integral equation (J_r).

In fact, we find from (J_r) by direct differentiation

$$Y_{uu}+2Y_v+c_1Y=c_0\sum_{i=1}^{4}\frac{1}{4(\alpha_i\beta_i-1)}\left[(-\beta_i+4u-\alpha_i\beta_i u)\,Y+\beta_i u\frac{\partial Y}{\partial u}\right.$$
$$\left.-(1+\alpha_i u)\frac{\partial Y}{\partial v}-u\frac{\partial^2 Y}{\partial u\,\partial v}\right]. \tag{68}$$

But we have from (24)

$$\frac{1}{\alpha_i\beta_i-1}=\frac{2k_i}{\Delta},$$

and $\frac{1}{2}k_i$ is the co-factor of $\alpha_i\,\beta_i$ in the determinant Δ defined by (22). Consequently,

$$\sum_{i=1}^{4}\frac{1}{\alpha_i\beta_i-1}=\sum_{i=1}^{4}\frac{\alpha_i}{\alpha_i\beta_i-1}=\sum_{i=1}^{4}\frac{\beta_i}{\alpha_i\beta_i-1}=0,\quad \sum_{i=1}^{4}\frac{\alpha_i\beta_i}{\alpha_i\beta_i-1}=4,$$

so that the right member of (68) reduces to $-c_0uY$. In the same way we may prove that Y must also satisfy the second equation of system (S).

The integral equations (J) and (J_r) are simple examples of a type of integral equations which apparently has not yet been considered. The characteristic property of equations of this type is that they contain open line integrals subject to the condition of being independent of the path. We shall leave the general discussion of integral equations of this kind for a future occasion.

We shall close this section by pointing out a remarkable formula for the partial derivative, with respect to c_0, of any solution of system (S). Let

$$\frac{\partial y}{\partial c_0} = \eta.$$

Then we see that η is a solution of the non-homogeneous system

$$(S') \qquad \begin{aligned} \eta_{uu} + 2\,\eta_v + (c_0 u + c_1)\,\eta + u\,y = 0, \\ \eta_{vv} + 2\,\eta_u + (c_0 v + c_1)\,\eta + v\,y = 0. \end{aligned}$$

The corresponding homogeneous system is (S). If we apply the method of § 2 to the integration of system (S'), we obtain $\frac{\partial y}{\partial c_0}$ expressed as a sum of integrals of exact differentials depending upon the solutions $y_1, \ldots, y_4$ of (S). Of course, $\frac{\partial y}{\partial c_0}$ may also be regarded as a solution of an integral equation obtained from (J) by differentiation with respect to c_0.

THE UNIVERSITY OF CHICAGO, *December* 26, 1912.

On the Connection of an Abstract Set, with Applications to the Theory of Functions of a General Variable.

By Arthur D. Pitcher.

In that portion of the thesis of Fréchet* which relates to the theory of functions† of a general variable there appear theorems which secure for a continuous function of such a general variable the more fundamental properties possessed by continuous functions of an ordinary real variable. For instance, given a system $(\mathfrak{Q}; L)$, *i. e.*, a set $\mathfrak{Q}$ of elements q on which a limit L satisfying certain postulates is defined, a sufficient condition that every continuous function on $\mathfrak{Q}$ (1) be bounded, (2) attain its bounds, is that $\mathfrak{Q}$ be extremal ‡ (compact and closed). A sufficient condition that every continuous function on $\mathfrak{Q}$ (3) assume every value between each pair of its values is that $\mathfrak{Q}$ be "continuous," where the term "continuous" is defined in terms of a certain postulated correspondence between subsets of $\mathfrak{Q}$ and the linear interval (0 1).§ Later, for a system $(\mathfrak{Q}; \delta)$ (*i. e.*, a set $\mathfrak{Q}$ of elements on which a distance function δ satisfying certain postulates is defined) it is shown that a necessary and sufficient condition for (1) and (2) is that $\mathfrak{Q}$ be extremal.|| Fréchet does not consider (3) further. It will be recognized that (3) is a property not less important than (2) and (1).

In his dissertation entitled "A Contribution to the Foundations of Fréchet's Calcul Fonctionnel,"¶ T. H. Hildebrandt gives an excellent analysis of the Fréchet theory, securing many of the Fréchet theorems under hypotheses milder than those used by Fréchet. Hilderbrandt considers especially systems $(\mathfrak{Q}; K)$, replacing the δ of Fréchet by a relation K between the pairs of elements of $\mathfrak{Q}$ and the single elements of the class $[m]$ of positive and negative integers m.**

* "Sur Quelques Points du Calcul Fonctionnel," Paris, reprinted in *Rendiconti del Circolo Matematico di Palermo*, Vol. XXII, pp. 1–64.

† The term "function" is used here and in the sequel in the sense of real-valued, single-valued function.

‡ Fréchet, *loc. cit.*, § 11.

§ Fréchet, *loc. cit.*, § 12.

|| Fréchet, *loc. cit.*, § 51.

¶ American Journal of Mathematics, Vol. XXXIV, pp. 237–290. We refer to this paper in the sequel as Hildebrandt.

** It is very convenient and precise to speak of this relation as a relation on $\mathfrak{Q}\mathfrak{Q}\mathfrak{J}$, where $\mathfrak{J}$ is the class of positive and negative integers. Cf. § 69 of the memoir by E. H. Moore entitled "Introduction to a Form of General Analysis," *The New Haven Mathematical Colloquium*, New Haven, 1910. The relation K_2 introduced by Moore is essentially the K relation used by Hildebrandt.

He conditions the K by postulates sufficient to secure the theorems of Fréchet and points out the precise relation between the K and the δ.* In particular Hildebrandt indicates that for a system $(\mathfrak{Q}; K)$, where K is properly conditioned, a necessary and sufficient condition for (1) and (2) above is that $\mathfrak{Q}$ be extremal. He does not consider (3).

It is the purpose of the present paper to give, in terms of the K relation of Hildebrandt, a definition of the notion *connection*† as applied to a general range; to give some simple but general theorems in the theory of functions of a general variable; to give for systems $(\mathfrak{Q}; K)$, where K is properly conditioned and $\mathfrak{Q}$ is extremal, a *necessary* and *sufficient* condition that every continuous function on $\mathfrak{Q}$ possess the property (3) above; and finally to give a set of conditions *necessary* and *sufficient* that every continuous function on $\mathfrak{Q}$ possess the properties (1), (2), (3). The theorems are applicable to the Fréchet systems $(\mathfrak{Q}; \delta)$.

Consider a class $\mathfrak{Q}\equiv[q]$ of elements q, unconditioned except by the postulates of the sequel, and the class $[m]$ of all integers, positive and negative. Consider also a relation K between pairs of elements q_1q_2 of $\mathfrak{Q}$ and single integers m. Denote the fact that q_1q_2 and m are in the relation K by the symbol Kq_1q_2m. The relation K may have a variety of properties. (Cf. Hildebrandt, § 4.) For instance, the relation K may be such that the following condition is satisfied:

$$\text{If } m_0 \leq m_1, \text{ then } Kq_1q_2m_1 \text{ implies } Kq_1q_2m_0. \tag{1}$$

In the sequel a K relation for which (1) holds is denoted by K^1, while K denotes the general K relation.

In a system $(\mathfrak{Q}; K)$ the sequence $\{q_n\}$ of elements of $\mathfrak{Q}$ is said to have the element q as a limit if, for every m, there is an n_m such that $n \geq n_m$ implies Kq_nqm.‡

A function μ is said to be continuous on the range $\mathfrak{Q}\equiv[q]$ of a system $(\mathfrak{Q}; K)$ if, for every q and positive number e, there is an m_{qe} such that Kq_1qm_{qe} implies $|\mu(q_1)-\mu(q)|\leq e$.

The function μ is said to be uniformly continuous if, for every e, there is an m_e such that $Kq_1q_2m_e$ implies $|\mu(q_1)-\mu(q_2)|\leq e$.

Two elements $\hat{q}$ and $\tilde{q}$ of the range $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K)$ are *connected for a given m* if there is a finite set $q_1, q_2, \ldots, q_k$ such that $\hat{q}=q_1$ and $\tilde{q}=q_k$,

* Hildebrandt, § 6.

† The idea of the connection of an abstract range was first brought to my attention by Professor E. H. Moore at the time when I had the privilege of attending his lectures on "General Analysis" at the University of Chicago.

F. Riesz, in his article "Stetigkeitsbegriff und Abstracte Mengenlehre," *Atti del IV. Congresso Internazionale dei Matematici*, Rome, 1908, discusses the connection of subsets of an abstract point set.

‡ Hildebrandt, § 7.

and such that, for each pair q_iq_{i+1} of elements of the set, we have either $Kq_iq_{i+1}m$ or else $Kq_{i+1}q_im$.

The elements $\hat{q}\tilde{q}$ are *connected* if they are connected for every m.

The range $\mathfrak{Q}$ is said to be *connected* when each pair q_1q_2 of elements of $\mathfrak{Q}$ is connected.

The set $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K)$ is *extremal** if every infinitude of elements of $\mathfrak{Q}$ gives rise to at least one limit element of $\mathfrak{Q}$. That is, if $\mathfrak{Q}_1$ is a subclass of $\mathfrak{Q}$ containing an infinite number of elements, then there is a sequence $\{q_{1n}\}$ of $\mathfrak{Q}_1$ and an element q of $\mathfrak{Q}$ such that $\underset{n}{L}q_n=q$.

The following propositions relative to systems $(\mathfrak{Q}; K)$ are easily proved.

If q_1q_2 are connected for m_0 and q_2q_3 are connected for m_0, then q_1q_3 are connected for m_0.

If q_1q_2 are connected and q_2q_3 are connected, then q_1q_3 are connected.

If $\mathfrak{R}$ is a connected subclass of $\mathfrak{Q}$, then $\mathfrak{R}+\mathfrak{R}'$ is connected. ($\mathfrak{R}'$ is the derived set of $\mathfrak{R}$.)

If $\mathfrak{Q}_1$ is the class of all elements connected with a given q_1, then $\mathfrak{Q}_1$ is closed. (A subset $\mathfrak{Q}_1$ of $\mathfrak{Q}$ is closed in case all the elements of $\mathfrak{Q}$ which are limits of sequences of $\mathfrak{Q}_1$ are also elements of $\mathfrak{Q}_1$.)

If $\mathfrak{Q}_1$ is the class of elements composed of q_1 and all elements q connected with q_1, and $\mathfrak{Q}_2$ the class of elements composed of q_2 and all elements connected with q_2, then the necessary and sufficient condition that $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have a common element, is that q_1 and q_2 be connected. In fact, if $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ have a single element in common, they are identical.

The following proposition is for systems $(\mathfrak{Q}; K^1)$,

If q_1 and q_2 are connected for m, they are connected for every $m_0 \leq m$.

Denote by $\mathfrak{A}_\mu$ the set of real.numbers composed of the functional values of μ, and by $(\mathfrak{A}_\mu)_1$ the set $\mathfrak{A}_\mu$ plus its derived set. The following theorem is relative to a system $(\mathfrak{Q}; K)$.

THEOREM I. (a) *If q_1 and q_2 are connected and μ is a uniformly continuous function on $\mathfrak{Q}$ such that $\mu(q_1) < h < \mu(q_2)$, then h belongs to $(\mathfrak{A}_\mu)_1$.* (b) *If $\mathfrak{Q}$ is connected and k is any number between two values of a uniformly continuous function μ on $\mathfrak{Q}$, then k belongs to $(\mathfrak{A}_\mu)_1$.*

The second part of the theorem is an immediate corollary of the first part. To prove the first part it is sufficient to prove that if $\mu(q_1) < 0 < \mu(q_2)$, then 0 belongs to $(\mathfrak{A}_\mu)_1$. The method of proof is standard. (Cf. Goursat-Hedrick, "Mathematical Analysis," p. 145.) Denote by $\hat{\mathfrak{Q}} \equiv [\hat{q}]$ the class of all elements $\hat{q}$ such that $\mu(\hat{q}) > 0$, and by $\bar{\mathfrak{Q}} \equiv [\bar{q}]$ the class of all elements $\bar{q}$ such

*Cf. Hildebrandt, § 10.

that $\mu(\bar{q})<0$. Denote by $\hat{\mathfrak{A}}_\mu$ the set of all positive functional values of μ, and by $\bar{\mathfrak{A}}_\mu$ the set of all negative functional values of μ. Then the lower limit of $\hat{\mathfrak{A}}_\mu$ is either 0 or $\varepsilon>0$. Suppose the latter to be true. Then $|\mu(\hat{q})-\mu(\bar{q})|\geq\varepsilon$ for every $\hat{q}\bar{q}$. But since q_1q_2 are connected, there is for every m a pair $\hat{q}_m\bar{q}_m$ such that either $K\hat{q}_m\bar{q}_m m$ or $K\bar{q}_m\hat{q}_m m$. Since μ is uniformly continuous, there is for $\varepsilon/2$ an $m_{\varepsilon/2}$ such that $Kq_1q_2m_{\varepsilon/2}$ implies $|\mu(q_1)-\mu(q_2)|<\varepsilon/2$. From above there is a pair $\hat{q}_{m_{\varepsilon/2}}\bar{q}_{m_{\varepsilon/2}}$ such that $K\hat{q}_{m_{\varepsilon/2}}\bar{q}_{m_{\varepsilon/2}}m_{\varepsilon/2}$, and such that $|\mu(\hat{q}_{m_{\varepsilon/2}})-\mu(\bar{q}_{m_{\varepsilon/2}})|>\varepsilon/2$. Thus we have a contradiction.

THEOREM II. (a) *If the class $\mathfrak{Q}$ of a system $(\mathfrak{Q};K)$ is extremal and q_1, q_2 are two connected elements of $\mathfrak{Q}$, and μ is a uniformly continuous function on $\mathfrak{Q}$ such that $\mu(q_1)<h<\mu(q_2)$, then h belongs to $\mathfrak{A}_\mu$.* (b) *If $\mathfrak{Q}$ is extremal and connected, then every uniformly continuous function μ on $\mathfrak{Q}$ assumes every value between each pair of its values.*

The second part of the theorem follows from the first. The first part may be seen readily as follows:* By the previous theorem h belongs to $(\mathfrak{A}_\mu)_1$. Then either h belongs to $\mathfrak{A}_\mu$ or else there is a sequence $\{h_n\}$ of numbers of $\mathfrak{A}_\mu$ such that $\underset{n}{L}h_n=h$. This, combined with the fact that $\mathfrak{Q}$ is extremal, implies that there is a sequence $\{q_n\}$ of $\mathfrak{Q}$ and an element q_0 of $\mathfrak{Q}$ such that $\underset{n}{L}q_n=q_0$ and $\underset{n}{L}\mu(q_n)=h$. It follows readily from this and the continuity of μ that $\mu(q_0)=h$.

If $\mathfrak{Q}$ is extremal, then every continuous function on $\mathfrak{Q}$ is bounded and attains its least upper and greatest lower bounds. (Cf. Hildebrandt, § 19.) From this and the previous theorem we have:

THEOREM III. *If a class $\mathfrak{Q}$, of a system $(\mathfrak{Q};K)$, is extremal and connected, then every uniformly continuous function μ on $\mathfrak{Q}$ is bounded and assumes its least upper and greatest lower bounds and every value between these bounds.*

THEOREM IV. (a) *A neccessary and sufficient condition that every function μ uniformly continuous on $\mathfrak{Q}$, of a system $(\mathfrak{Q};K^1)$, be such that if $\mu(q_1)<h<\mu(q_2)$ then h belongs to $(\mathfrak{A}_\mu)_1$, is that q_1 and q_2 be connected.* (b) *A necessary and sufficient condition that every function μ uniformly continuous on $\mathfrak{Q}$, of a system $(\mathfrak{Q}, K^1)$, be such that every value between any two of its values belongs to $(\mathfrak{A}_\mu)_1$, is that $\mathfrak{Q}$ be connected.*

The second part of the theorem is a consequence of the first part. We have shown in Theorem I that the condition in the first part of the theorem is sufficient. To show the necessity of the condition, we show that if q_1 and q_2 are not connected, then there is a function μ, uniformly continuous on $\mathfrak{Q}$, such that $\mu(q_1)=1$, $\mu(q_2)=-1$, and such that 0 does not belong to $(\mathfrak{A}_\mu)_1$.

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* Cf. Hildebrandt, § 19.

In a system $(\mathfrak{Q}; K^1)$, if q_1 and q_2 are not connected there is an $\overline{m}$ such that q_1q_2 are connected for no $m > \overline{m}$. Consider the division of $\mathfrak{Q}$ into subclasses as follows: $\mathfrak{Q}_1'$ is composed of q_1 and all elements q each of which is for some $m > \overline{m}$ connected with q_1. $\mathfrak{Q}_2'$ is composed of q_2 and all elements q each of which is for some $m > \overline{m}$ connected with q_2. $\mathfrak{Q}_3'$ is composed of all elements which are in neither $\mathfrak{Q}_1'$ nor $\mathfrak{Q}_2'$. $\mathfrak{Q}_1'$ and $\mathfrak{Q}_2'$ each contain at least one element. $\mathfrak{Q}_3'$ may contain no elements.

$\mathfrak{Q}_1'$ and $\mathfrak{Q}_2'$ have no common elements. It is clear that neither q_1 nor q_2 can be common to $\mathfrak{Q}_1'$ and $\mathfrak{Q}_2'$. Suppose there is an element q', neither q_1 nor q_2, which is common to $\mathfrak{Q}_1'$ and $\mathfrak{Q}_2'$. q' is connected with q_1 for a certain $m_1 > \overline{m}$. q' is connected with q_2 for a certain $m_2 > \overline{m}$. Call m_0 the smaller of m_1 and m_2. Then for m_0, which exceeds $\overline{m}$, q_1 and q_2 are connected. This contradicts the fact that q_1 and q_2 are connected for no $m > \overline{m}$. From the way $\mathfrak{Q}_3'$ is defined, it can have no elements in common with either $\mathfrak{Q}_1'$ or $\mathfrak{Q}_2'$.

There is no $m > \overline{m}$ for which a q_1' of $\mathfrak{Q}_1'$ and a q_2' of $\mathfrak{Q}_2'$ are connected. For suppose there were such an m and such a pair of elements $q_1'q_2'$. q_1' is connected with q_1 for $m_1 > \overline{m}$. q_2' is connected with q_2 for $m_2 > \overline{m}$. Consider m_0 the smaller of m, m_1, m_2. For m_0, q_1 is connected with q_1', q_1' with q_2', and q_2' with q_2. Therefore q_1 is connected with q_2 for an $m_0 > \overline{m}$. This is a contradiction. Similarly there is no $m > \overline{m}$ for which a q_1' of $\mathfrak{Q}_1'$ and a q_3' of $\mathfrak{Q}_3'$ are connected. Neither is there an $m > \overline{m}$ for which a q_3' of $\mathfrak{Q}_3'$ and a q_2' of $\mathfrak{Q}_2'$ are connected. Thus, if there is an $m > \overline{m}$ such that the relation $K\hat{q}\tilde{q}m$ holds, then $\hat{q}$ and $\tilde{q}$ must both belong to the same one of the classes $\mathfrak{Q}_1', \mathfrak{Q}_2', \mathfrak{Q}_3'$.

Now consider a function μ such that $\mu(q) = 1$ if q belongs to $\mathfrak{Q}_1'$, and such that $\mu(q) = -1$ if q does not belong to $\mathfrak{Q}_1'$. μ is uniformly continuous on $\mathfrak{Q}$, and the totality of its functional values consists of the two values 1 and -1.

From the previous theorem and the example just given we have the following theorem:

Theorem V. (a) *A necessary and sufficient condition that every function μ, uniformly continuous on an extremal set $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K^1)$, be such that if $\mu(q_1) < h < \mu(q_2)$ then h belongs to $\mathfrak{A}_\mu$, is that q_1 and q_2 be connected.* (b) *A necessary and sufficient condition that every function μ, uniformly continuous on the extremal set $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K^1)$, assume every value between each of its values, is that $\mathfrak{Q}$ be connected.*

For systems $(\mathfrak{Q}; K)$ which are such that every continuous function on $\mathfrak{Q}$ is likewise uniformly continuous, the preceding theorems hold if in the hypotheses uniform continuity be replaced by continuity. Hildebrandt shows* that in

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* Hildebrandt, § 22.

the case of systems $(\mathfrak{Q}; K^{1367})$, where 3, 6, 7 are properties of the relation K defined in a note below* and $\mathfrak{Q}$ is extremal, the continuity of a function μ on $\mathfrak{Q}$ implies its uniform continuity. In view of this, Theorems II, III, V hold for systems $(\mathfrak{Q}; K^{1367})$ if continuity be substituted for uniform continuity in the hypotheses. In particular we have:

THEOREM VI. *A necessary and sufficient condition that every function μ continuous on an extremal set $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K^{1367})$*† *assume every value between each pair of its values, is that $\mathfrak{Q}$ be connected.*

Hildebrandt also indicates‡ that a necessary and sufficient condition that every function μ continuous on $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K^{1367})$ (1) be bounded, (2) attain its bounds, is that $\mathfrak{Q}$ be extremal. Using the results of Hildebrandt and Theorem VI, we have the following theorem:

THEOREM VII. *A necessary and sufficient condition that every function μ continuous on $\mathfrak{Q}$ of a system $(\mathfrak{Q}; K^{1367})$ shall at the same time be bounded and attain its least upper and greatest lower bounds and every value between these bounds, is that $\mathfrak{Q}$ be extremal and connected.*

Among the systems $(\mathfrak{Q}; K^{1367})$ of Hildebrandt are the systems $(\mathfrak{Q}; \delta)$ as used by Fréchet.§

Various independence considerations might arise here, but we content ourselves by giving two simple examples of systems $(\mathfrak{Q}; K^{12345678})$‖ where $\mathfrak{Q}$ is extremal. In the first of these examples $\mathfrak{Q}$ is connected, and in the second $\mathfrak{Q}$ is not connected.

(a) $\mathfrak{Q}$ is the class of all real numbers q such that $0 \leq q \leq 1$. q_1 and q_2 are in the relation Kq_1q_2m if $|q_1-q_2| \leq \frac{1}{2^m}$.

(b) $\mathfrak{Q}$ is the class of all real numbers q such that $0 \leq q \leq 1$ or such that $3 \leq q \leq 4$. q_1 and q_2 are in the relation Kq_1q_2m if $|q_1-q_2| \leq \frac{1}{2^m}$.

DARTMOUTH COLLEGE, *March*, 1913.

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* Hildebrandt, § 4.

† K has the property 3 if the relation Kq_1q_2m holds for every m only in the case when q_1 and q_2 are identical.

K has the property 5 if there is an integral-valued function, say ϕ_m, which increases indefinitely with m, such that, if we have the relations Kq_1q_2m Kq_2q_3m, then we have the relation $Kq_1q_3\phi_m$.

K has the property 6 if there is an integral-valued function ϕ_m which increases indefinitely with m, such that, if we have Kq_2q_1m Kq_2q_3m, then we have $Kq_1q_3\phi_m$. K^5 and K^6 are equivalent if K is symmetrical. Cf. Hildebrandt, § 4.

‡ Hildebrandt, § 22.

§ Cf. Hildebrandt, p. 266, foot-note.

‖ Hildebrandt, § 4.

*On Series of Iterated Linear Fractional Functions.**

By R. D. Carmichael.

Introduction.

The two classes of power series (ascending and descending) are the most important series known to mathematical analysis. Recent investigations have brought to prominent notice another class of series which are also of prime importance; namely, the factorial series

$$a_0 + \frac{a_1}{x} + \sum_{n=1}^{\infty} \frac{a_{n+1} n!}{x(x+1)(x+2)\dots(x+n)}.$$

The simplicity and elegance of the general theory associated with these series is seen from the development of their fundamental properties by Landau.† Their importance throughout a large range of modern mathematical analysis is apparent from the demonstration by Watson‡ that most of the ordinary functions of analysis (which possess asymptotic expansions) actually are capable of being expressed in the form of convergent factorial series. An earlier result, of a character similar to this, is due to Horn,§ who has shown that the divergent Thomé normal series, which satisfy a linear differential equation with rational coefficients, may be transformed into convergent factorial series. That these series may be of great value in special problems is effectively illustrated in their recent use by Nörlund,|| in his elegant paper on the integration of linear difference equations by means of factorial series.

With factorial series are to be associated the so-called binomial coefficient series

$$a_0 + \sum_{n=1}^{\infty} a_n \frac{(x-1)(x-2)\dots(x-n)}{n!},$$

the general theory of which has recently been developed by Landau¶ and others.

* Presented to the American Mathematical Society, September, 1913.

† *Sitzungsber. d. Math.-Phys. Klasse d. Kgl. Bayer. Akad. d. Wiss.*, XXXVI (1906), pp. 151–218. See also the references in this paper.

‡ In a memoir crowned by the Danish Royal Academy of Science. Published in *Rendiconti del Circolo Matematico di Palermo*, XXXIV (1912), pp. 41–88. See also the references in this paper.

§ *Mathematische Annalen*, LXXI (1912), pp. 510–532.

|| *Rendiconti del Circolo Matematico di Palermo*, XXXV (1913), pp. 177–216.

¶ Landau, *loc. cit.*, pp. 192–197. See also the literature cited by Landau here and on p. 154 of the same memoir.

Now, power series (both ascending and descending), factorial series and binomial coefficient series are all included as special cases in the following two types of series of iterated linear fractional functions:

$$\alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots.S_n(x)}, \tag{1}$$

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n S_1(x)S_2(x)\ldots.S_n(x), \tag{2}$$

where

$$S_1(x) = \frac{ax+b}{cx+d}, \quad ad-bc \neq 0; \qquad S_k(x) = \frac{aS_{k-1}(x)+b}{cS_{k-1}(x)+d}, \quad k>1,$$

a, b, c, d being constants. We shall refer to these two series as of type I and type II respectively.

If $S_1(x) = ax$, it is clear that (1) is a descending power series and that (2) is an ascending power series. If $S_1(x) = x+1$, then (1) is a factorial series of the form above; while, if $S_1(x) = x-1$, (2) is a binomial coefficient series of the form above.

The object of the present paper is to develop the fundamental elements of a general theory of both the above types of series of iterated linear fractional functions.

In § 1 I introduce some preliminary notations and definitions and state some lemmas which are of frequent use in the convergence proofs.

In § 2 I determine the character of the regions of convergence, of absolute convergence and of conditional convergence of series of both the types I and II. An upper bound to the magnitude of the region of conditional convergence is obtained. Thus we have generalizations of the corresponding theories for the case of power series, factorial series and binomial coefficient series. The methods of Landau for series of the two latter kinds are in the main employed. The general plan of treatment is improved in one respect by the use of the criterion of Gauss for the convergence of series; thus it is no longer necessary to employ properties of the gamma function for factorial series and binomial coefficient series, or of corresponding functions for the other cases (treated for the first time in the present paper). It should be noted here that the general results are in some respects in marked contrast to the simpler ones for the special cases which have been investigated heretofore.

In § 3 the regions of uniform convergence of the series are determined and some immediate consequences are stated.

In § 4 I determine the boundaries of the regions of convergence and absolute convergence of the series in terms of their coefficients.

§ 1. *Preliminary Definitions, Notations and Lemmas.*

In relation to the matter of convergence of series (1) or series (2), it is obvious that an exceptional rôle is played by a point x for which $S_k(x)$ is either zero or infinity for some value of k. In the case of series (1), the point $x=0$ is also exceptional. Later, it will be seen that an exceptional rôle is played by a point x_0 such that $S_1(x_0)=x_0$, whence $S_k(x_0)=x_0$ for every k. Hence we shall employ the following definition:

For series (2) a point x_0 such that $S_k(x_0)=0$, x_0 or ∞, for some k, is called an *exceptional* point. For series (1) these points and the point $x=0$ are said to be *exceptional.* In either case, the remaining points are called *non-exceptional* points.

By the *region of convergence* of series (1) [series (2)] we shall mean that portion of the plane which is made up of those non-exceptional points x at which the series converges and those exceptional points which have the property that there is some neighborhood of each of them such that all points in these neighborhoods (except possibly the exceptional points themselves) are points of convergence of the series. We also define similarly the *region of absolute convergence* of each series.

We shall say that the substitution $x'=S_1(x)$ is the substitution *corresponding* to series (1) or to series (2); also, that these series correspond to the given substitution.

Throughout the paper we shall require to have at hand explicit formulæ for $S_n(x)$ in terms of x and n. In the statement of these formulæ it is convenient to distinguish four cases as follows:*

CASE A. The substitution $x'=S_1(x)$ has two double points in the finite plane. Denote these by α and β. Then for the substitution itself and for the value of $S_n(x)$ we have respectively:

$$\frac{x'-\alpha}{x'-\beta}=k\frac{x-\alpha}{x-\beta},\quad k\neq 0,1;\qquad S_n(x)\equiv\frac{k^n\beta(x-\alpha)-\alpha(x-\beta)}{k^n(x-\alpha)-(x-\beta)}.$$

CASE B. The substitution $x'=S_1(x)$ has two double points, one of them being at infinity. If the other is at α, then for the substitution and for $S_n(x)$ we have respectively:

$$x'-\alpha=k(x-\alpha),\quad k\neq 0,1;\qquad S_n(x)\equiv k^n(x-\alpha)+\alpha.$$

CASE C. The substitution $x'=S_1(x)$ has only one double point, and this point lies in the finite plane. If it is at α, then for the substitution and for $S_n(x)$ we have respectively:

*As a matter of convenience, the identical substitution is excluded throughout. This involves no loss of generality, since the series corresponding to the substitution $x'=x$, in each type, is of the same form as that corresponding to the more general substitution $x'=ax$, where a is any non-zero constant whatever.

$$\frac{1}{x'-\alpha}=\frac{1}{x-\alpha}+t,\quad t\neq 0;\qquad S_n(x)\equiv\frac{\alpha nt(x-\alpha)+x}{nt(x-\alpha)+1}.$$

CASE D. The substitution $x'=S_1(x)$ has a single double point, and this point is at infinity. Then for the substitution and for $S_n(x)$ we have respectively:

$$x'=x+t,\quad t\neq 0;\qquad S_n(x)\equiv x+nt.$$

It is convenient to note here the behavior of $S_n(x)$ for n approaching infinity. This is simplest in the cases C and D. Let $\bar{x}$ be any finite point which is not a double point of the substitution $x'=S_1(x)$. Then, in cases C and D, it is clear that $\lim_{n=\infty} S_n(\bar{x})$ exists and is the double point of the substitution.

In case A [B], if $\bar{x}$ is any finite point which is not a double point of $x'=S_1(x)$, then $\lim_{n=\infty} S_n(\bar{x})$ exists or not, according as $|k|\neq 1$ or $|k|=1$. In case $|k|>1$, the limit is the double point $\beta[\infty]$. In case $|k|<1$, the limit is the double point $\alpha[\alpha]$. In case $|k|=1$, in which case the limit does not exist, it is clear that $|S_n(\bar{x})|$ is less than some fixed constant for all n greater than some N.

It is convenient to state here the following lemmas which will be useful in several convergence proofs:

LEMMA I. *Let $b_0, b_1, b_2, \ldots.$ and $c_0, c_1, c_2, \ldots.$ be two sequences of complex numbers such that both of the series*

$$\sum_{n=0}^{\infty} b_n,\qquad \sum_{n=0}^{\infty} |c_n-c_{n+1}|$$

are convergent. Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

also converges.

LEMMA II. *Let $b_0, b_1, b_2, \ldots.$ and $c_0, c_1, c_2, \ldots.$ be two sequences of complex numbers such that*

1) *there exists a number B such that $|\sum_{n=0}^{t} b_n|<B$ for every t;*

2) $\lim_{n=\infty} c_n$ *exists and is zero;*

3) *the series $\sum_{n=0}^{\infty} |c_n-c_{n+1}|$ converges.*

Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

also converges.

LEMMA III. *Let $b_0, b_1, b_2, \ldots.$ be a sequence of complex constants and $c_0, c_1, c_2, \ldots.$ be a sequence of functions of the complex variable x which are*

regular throughout a given domain D (including the boundary). Suppose that

1) *there exists a number B such that* $|\sum_{n=0}^{t} b_n| < B$ *for every* t;

2) *as n approaches infinity* c_n *converges to* 0 *uniformly in D;*

3) *the series* $\sum_{n=0}^{\infty} |c_n - c_{n+1}|$ *converges uniformly in D.*

Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

converges uniformly in D.

LEMMA IV. *Let* $b_0, b_1, b_2, \ldots$ *be a sequence of complex constants such that the series* $b_0+b_1+b_2+\ldots$ *is convergent. Let* $c_0, c_1, c_2, \ldots$ *be a sequence of functions of the complex variable* x *which are regular throughout a given domain D (including the boundary) and are such that the series*

$$\sum_{n=0}^{\infty} |c_n - c_{n+1}|$$

converges uniformly in D. Then the series

$$\sum_{n=0}^{\infty} b_n c_n$$

is uniformly convergent in D.

Elegant elementary proofs of the first three of these lemmas are given by Landau (*loc. cit.*, pp. 155–157, 160–161); he also gives references to their earlier use. The fourth lemma appears to have been first employed by Nielsen;* but Nielsen's statement of it is not entirely accurate, as Landau† has already pointed out.

§ 2. *Character of the Regions of Convergence and Absolute Convergence.*

By means of lemma I we shall now determine the character of the region of convergence of the series

$$\Omega(x) = \alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots S_n(x)}.$$

Let x_0 and x_1 be two non-exceptional values of x for the series. We shall assume that the series above converges for $x=x_0$, and shall express this briefly by saying that $\Omega(x_0)$ converges. We shall determine relations between x_0 and x_1 which are sufficient to ensure that $\Omega(x_1)$ also converges.

Let us put

$$b_n = \frac{\alpha_{n+1}}{x_0S_1(x_0)S_2(x_0)\ldots S_n(x_0)}, \qquad c_n = \frac{x_0S_1(x_0)S_2(x_0)\ldots S_n(x_0)}{x_1S_1(x_1)S_2(x_1)\ldots S_n(x_1)}.$$

* *Annali di matematica*, (3) XV (1908), pp. 275–282.

† *Sitzungsber. d. Kgl. Bayer. Akad. d. Wiss., Math.-Phys.*, 1909.

Then the series $b_1+b_2+b_3+\ldots.$ converges, by hypothesis. From lemma I it follows that the series $b_1c_1+b_2c_2+b_3c_3+\ldots.$, and hence the series $\Omega(x_1)$, converges provided that

$$\sum_{n=1}^{\infty}|c_n-c_{n+1}| \tag{3}$$

is convergent. Here we have

$$c_n-c_{n+1}=\frac{x_0S_1(x_0)\ldots.S_n(x_0)}{x_1S_1(x_1)\ldots.S_n(x_1)}\left(1-\frac{S_{n+1}(x_0)}{S_{n+1}(x_1)}\right).$$

For determining the relation between x_0 and x_1 so that (3) shall converge, we shall have use for the following formulæ, which are easily verified:

$$\left.\begin{aligned}Q_n(x_0, x_1)&\equiv\frac{S_n(x_0)}{S_n(x_1)}\\ &=1+\frac{k^n(\alpha-\beta)^2(x_0-x_1)}{\{k^n(x_0-\alpha)-(x_0-\beta)\}\{k^n\beta(x_1-\alpha)-\alpha(x_1-\beta)\}},\ \text{in case A;*}\\ &=1+\frac{k^n(x_0-x_1)}{k^n(x_1-\alpha)+\alpha},\ \text{in case B;}\\ &=1+\frac{x_0-x_1}{\{nt(x_0-\alpha)+1\}\{\alpha nt(x_1-\alpha)+x_1\}},\ \text{in case C;}\\ &=1+\frac{x_0-x_1}{x_1+nt},\ \text{in case D.}\end{aligned}\right\} \tag{4}$$

The ratio $r_n(x_1)$ of the $(n+1)$-th term to the n-th term of series (3) is

$$r_n(x_1)=|Q_{n+1}(x_0, x_1)|\cdot\frac{|1-Q_{n+2}(x_0, x_1)|}{|1-Q_{n+1}(x_0, x_1)|}.$$

Making use of the above values of the function $Q_n(x_0, x_1)$, we have by obvious reductions the following formulæ:

$$\begin{aligned}r_n(x_1)&=|Q_{n+1}(x_0, x_1)|\cdot\left|k\frac{\{k^{n+1}(x_0-\alpha)-(x_0-\beta)\}\{k^{n+1}\beta(x_1-\alpha)-\alpha(x_1-\beta)\}}{\{k^{n+2}(x_0-\alpha)-(x_0-\beta)\}\{k^{n+2}\beta(x_1-\alpha)-\alpha(x_1-\beta)\}}\right|,\\ &\qquad\text{in case A;}\\ &=|Q_{n+1}(x_0, x_1)|\cdot\left|k\frac{k^{n+1}(x_1-\alpha)+\alpha}{k^{n+2}(x_1-\alpha)+\alpha}\right|,\ \text{in case B;}\\ &=|Q_{n+1}(x_0, x_1)|\cdot\left|\frac{\{(n+1)t(x_0-\alpha)+1\}\{(n+1)\alpha t(x_1-\alpha)+x_1\}}{\{(n+2)t(x_0-\alpha)+1\}\{(n+2)\alpha t(x_1-\alpha)+x_1\}}\right|,\\ &\qquad\text{in case C;}\\ &=|Q_{n+1}(x_0, x_1)|\cdot\left|\frac{x_1+(n+1)t}{x_1+(n+2)t}\right|,\ \text{in case D.}\end{aligned}$$

In the discussion immediately following we treat separately the cases A, B, C, D.

* The definition of the several cases is given in § 1.

In case A, several possibilities arise:

1) If $|k|<1$ and $\alpha\neq 0$, then $\lim\limits_{n=\infty}|Q_n(x_0, x_1)|=1$, and we have

$$\lim_{n=\infty} r_n=|k|.$$

2) If $|k|>1$ and $\beta\neq 0$, then $\lim\limits_{n=\infty}|Q_n(x_0, x_1)|=1$, and we have

$$\lim_{n=\infty} r_n=\left|\frac{1}{k}\right|.$$

3) If $|k|<1$ and $\alpha=0$, we have by easy reductions

$$\lim_{n=\infty} r_n=\lim_{n=\infty}\left|\frac{x_0}{x_1}\right|\cdot\left|\frac{k^n x_1-(x_1-\beta)}{k^n x_0-(x_0-\beta)}\right|\cdot\left|\frac{k^{n+1}x_0-(x_0-\beta)}{k^{n+2}x_0-(x_0-\beta)}\right|=\left|\frac{x_0}{x_1}\right|\cdot\left|\frac{x_1-\beta}{x_0-\beta}\right|.$$

4) If $|k|>1$ and $\beta=0$, we have similarly

$$\lim_{n=\infty} r_n=\left|\frac{x_0}{x_1}\right|\cdot\left|\frac{x_1-\alpha}{x_0-\alpha}\right|.$$

5) If $|k|=1$, we have

$$r_n=\left|\frac{k^{n+2}\beta(x_0-\alpha)-\alpha k(x_0-\beta)}{k^{n+2}(x_0-\alpha)-(x_0-\beta)}\right|\cdot\left|\frac{k^{n+2}(x_1-\alpha)-k(x_1-\beta)}{k^{n+2}\beta(x_1-\alpha)-\alpha(x_1-\beta)}\right|.$$

From these results we conclude that, in cases 1) and 2), $\Omega(x_1)$ converges for every x_1 (subject to the initial conditions specified above). In case 3), $\Omega(x_1)$ converges if

$$\left|\frac{x_1-\beta}{x_1}\right|<\left|\frac{x_0-\beta}{x_0}\right|.$$

In case 4), $\Omega(x_1)$ converges if

$$\left|\frac{x_1-\alpha}{x_1}\right|<\left|\frac{x_0-\alpha}{x_0}\right|.$$

In case 5), $\Omega(x_1)$ converges if $\limsup\limits_{n=\infty} r_n<1$. In several respects this case is exceptional. A corresponding exception arises under case B below; in connection with the latter, an example is given to indicate the nature of the irregularity.

In case B, we have readily

$$r_n=\left|\frac{k^{n+2}(x_0-\alpha)+k\alpha}{k^{n+2}(x_1-\alpha)+\alpha}\right|.$$

Hence, if $\alpha=0$ or if $|k|>1$, we have

$$\lim_{n=\infty} r_n=\left|\frac{x_0-\alpha}{x_1-\alpha}\right|.$$

In this case, series (3), and hence $\Omega(x_1)$, converges if $|x_1-\alpha|>|x_0-\alpha|$. If $\alpha\neq 0$ and $|k|<1$, we have

$$\lim_{n=\infty} r_n=|k|;$$

and therefore, in this case, $\Omega(x_1)$ converges for every x_1 (subject to the initial conditions specified above). If $\alpha\neq 0$ and $|k|=1$, $\Omega(x_1)$ will converge if

$$\limsup_{n=\infty} r_n<1.$$

This last case is quite exceptional in its character. In our discussion below, we shall point out that the region of convergence (when it is not the entire plane) is always bounded by a circle (or a straight line, considered as a limiting case of a circle) except for cases A and B in which $|k|=1$. By means of examples we shall here show that in this exceptional case the region of convergence may be bounded by a curve of the fourth or even higher degree. Suppose that the substitution is $x'=-x+2$. Then $S_{2m}(x)\equiv x$ and $S_{2m+1}(x)\equiv -x+2$. Consider the series

$$\Omega(x)=1+\frac{1}{x}+\frac{1}{x(x-2)}+\frac{1}{x^2(x-2)}+\frac{1}{x^2(x-2)^2}+\ldots.$$

If $\bar{x}$ is any non-exceptional value of x, it is easy to show that this series converges if $|\bar{x}(\bar{x}-2)|<1$, and diverges if $|\bar{x}(\bar{x}-2)|>1$. Hence the boundary of its region of convergence is the curve $|x(x-2)|=1$. This is obviously a curve of the fourth degree. By taking a periodic substitution of period greater than 2, we should similarly obtain series whose regions of convergence are bounded by curves of higher degree. Similar examples are readily constructed for the case A, when $|k|=1$. On account of the exceptional character of the cases when $|k|=1$, they will be excluded from further consideration.

We turn now to a further consideration of case C. We have readily

$$r_n=\left|\frac{(\{n+1)\alpha t(x_0-\alpha)+x_0\}\{(n+1)t(x_1-\alpha)+1\}}{(\{n+2)\alpha t(x_1-\alpha)+x_1\}\{(n+2)t(x_0-\alpha)+1\}}\right|.$$

If $\alpha\neq 0$, we have

$$r_n=\left|1-\frac{2}{n}+\ldots\right|,$$

where the terms omitted involve higher powers of $1/n$. Hence

$$r_n=1-\frac{2}{n}+\ldots.$$

Applying the criterion of Gauss,* we see that series (3), and hence $\Omega(x_1)$, is convergent. If $\alpha=0$, we have

$$r_n=\left|\frac{x_0}{x_1}\cdot\frac{(n+1)tx_1+1}{(n+2)tx_0+1}\right|=\left|1-\frac{1+\frac{1}{t}\left(\frac{1}{x_0}-\frac{1}{x_1}\right)}{n}+\ldots\right|;$$

hence, if we use the notation $R(z)$ for the real part of z, we have

$$r_n=1-\frac{1+R\left(\frac{1}{tx_0}-\frac{1}{tx_1}\right)}{n}+\ldots.$$

* See "Encyclopédie des sciences mathématiques," I_1, p. 216.

From this, by aid of the criterion of Gauss, we conclude that series (3), and hence $\Omega(x_1)$, converges if

$$R\left(\frac{1}{tx_1}\right) < R\left(\frac{1}{tx_0}\right).$$

Finally, let us consider case D. Here we have

$$r_n = \left|\frac{x_0+(n+1)t}{x_1+(n+2)t}\right| = 1 - \frac{1+R\left(\frac{x_1-x_0}{t}\right)}{n} + \ldots\ldots$$

Using again the criterion of Gauss, we conclude that series (3), and hence $\Omega(x_1)$, converges if

$$R\left(\frac{x_1}{t}\right) > R\left(\frac{x_0}{t}\right).$$

In the discussion above we have noted that cases A and B are exceptional if $|k|=1$. The results for the other cases may be stated in the following theorem:

THEOREM I_1.* *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$\Omega(x) = \alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots.S_n(x)},$$

and suppose that the series converges for $x=x_0$. Then the series converges in the following cases:

CASE A.† *If $|k|<1$ and $\alpha \neq 0$, or if $|k|>1$ and $\beta \neq 0$, then $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|<1$ and $\alpha=0$, $\Omega(x_1)$ converges if*

$$\left|\frac{x_1-\beta}{x_1}\right| < \left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $\Omega(x_1)$ converges if

$$\left|\frac{x_1-\alpha}{x_1}\right| < \left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|<1$ and $\alpha \neq 0$, $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|>1$, or if $\alpha=0$, then $\Omega(x_1)$ converges if*

$$|x_1-\alpha| > |x_0-\alpha|.$$

CASE C. *If $\alpha \neq 0$, $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $\alpha=0$, $\Omega(x_1)$ converges if*

$$R\left(\frac{1}{tx_1}\right) < R\left(\frac{1}{tx_0}\right).$$

CASE D. *In this case, $\Omega(x_1)$ converges if*

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*The subscript 1 [2] attached to the number of a theorem indicates that the theorem refers to a series of type I [II]. See the Introduction for definition of types.

†See the definition of cases A, B, C, D in § 1.

$$R\left(\frac{x_1}{t}\right) > R\left(\frac{x_0}{t}\right).$$

If one employs lemma II instead of lemma I, the above theorem can in some cases be slightly strengthened by requiring, in the hypothesis, that the sum of t terms of $\Omega(x_0)$ shall be bounded in absolute value, instead of making the stronger assumption that this series converges. To prove this, it is sufficient to show further that c_n approaches the limit zero as n increases indefinitely. This is equivalent to showing that the infinite product

$$\prod_{n=1}^{\infty} \frac{S_n(x_0)}{S_n(x_1)}$$

diverges to zero when x_0 and x_1 are connected by the relations given in the theorem. If we make use of equations (4) we see readily that this product is zero in case A if $|k|<1$ and $\alpha=0$ or if $|k|>1$ and $\beta=0$, in case B if $|k|>1$ or if $\alpha=0$, in case C if $\alpha=0$, in case D for all t. Hence, in these cases the theorem may be strengthened as indicated.

It is not difficult to construct examples in which $\Omega(x)$ converges for no non-exceptional value of x whatever. For the moment, we exclude this case from consideration. Then the region of convergence* of $\Omega(x)$ is the entire plane in case A if $|k|>1$ and $\beta \neq 0$, in cases A and B if $|k|<1$ and $\alpha \neq 0$, in case C if $\alpha \neq 0$. In all other cases the region of convergence may or may not be the entire plane, the fact in a particular case depending on the coefficients α of the series. We shall now take up these remaining cases separately and determine the exact nature of the region of convergence when it is not the entire plane.

The results are simple in case D, and hence we shall treat this first. Consider the straight line $(0t)$ through the points 0 and t. If $\Omega(x)$ converges at any non-exceptional point $\bar{x}$ on this line, it converges at every non-exceptional point to the right of a line through $\bar{x}$ and perpendicular to $(0t)$, the directions right and left being determined by saying that t is to the right of 0. From this it follows readily that there exists a straight line l perpendicular to $(0t)$ such that $\Omega(x)$ converges for every non-exceptional x to the right of l and diverges for every non-exceptional x to the left of l. On l, its character as to convergence or divergence varies as in the case of a power series in relation to its circle of convergence, as one might show by examples. The line l is called the *line of convergence* of the series. We may look upon l as a circle through the double point ∞ of the substitution $x'=x+t$ corresponding to the series $\Omega(x)$.

* See the definition of region of convergence in § 1.

In case C, when $\alpha=0$, the results are analogous to those in case D, as we shall now show. Here the double point of the substitution corresponding to $\Omega(x)$ is 0. Consider the system S of circles C which pass through the point 0 and have their centers on the straight line $(0t)$ through the points 0 and t. We shall say that t is to the right of 0. We make the following conventions concerning the *interior* and *exterior* of these circles C: If C lies to the right of the straight line l through 0 and perpendicular to $(0t)$, we shall say that the *interior* of the circle is that part of the plane bounded by C and containing its center; if C coincides with l, its interior is to the right of l; if C is to the left of l, then its interior is that part of the plane bounded by C and not containing its center. Now if $\Omega(x)$ converges for any non-exceptional point on a given circle C of the system S, it follows from the above theorem that it converges at every non-exceptional point *exterior* to C. From this it is readily seen that there exists a circle $\bar{C}$ of this system S such that $\Omega(x)$ converges for every non-exceptional point *exterior* to $\bar{C}$ and diverges for every non-exceptional point *interior* to $\bar{C}$. At a point on $\bar{C}$ it may either converge or diverge, as one might show by examples. The circle $\bar{C}$ will be referred to as the *circle of convergence* of $\Omega(x)$.

Under B we have two cases to examine; namely, that when $\alpha=0$ and that when $|k|>1$. In either case the above theorem leads readily to the conclusion that there exists a circle $\bar{C}$ about α as a center such that $\Omega(x)$ converges for every non-exceptional point x *exterior* to $\bar{C}$ and diverges for every non-exceptional point *interior* to $\bar{C}$, the words exterior and interior being now employed in their usual (elementary) sense. The circle $\bar{C}$ is called the *circle of convergence* of $\Omega(x)$.

Finally, under A we have also two cases to consider; namely, that when $|k|<1$ and $\alpha=0$ and that when $|k|>1$ and $\beta=0$. On account of the similarity of these two cases, it is sufficient to treat in detail one of them alone; we take that when $|k|<1$ and $\alpha=0$. Consider the systems S of circles C such that a circle C of the system is defined by the property that the distances of a point P on C from the points 0 and β is a constant for all positions of P. This system contains as a particular case the straight line which bisects perpendicularly the straight line segment joining 0 and β. For every circle C the points 0 and β are on opposite sides of C. The part of the plane bounded by C and containing $0[\beta]$ will be called the *interior* [*exterior*] of the circle. It is now easy to show, by aid of the above theorem, that there exists a circle $\bar{C}$ of the system S such that $\Omega(x)$ converges at every non-exceptional point *exterior* to $\bar{C}$ and diverges at every non-exceptional point *interior* to $\bar{C}$. The circle $\bar{C}$ is called the *circle of convergence* of $\Omega(x)$.

Corresponding to theorem I_1 for series of type I, we have the following theorem I_2 for series of type II:

THEOREM I_2. *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$W(x)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n S_1(x)S_2(x)\ldots.S_n(x),$$

and suppose that the series converges for $x=x_0$. Then the series $W(x_1)$ converges in the following cases:

CASE A. *If $|k|<1$ and $\alpha\neq 0$, or if $|k|>1$ and $\beta\neq 0$, then $W(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|<1$ and $\alpha=0$, $W(x_1)$ converges if*

$$\left|\frac{x_1-\beta}{x_1}\right|>\left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $W(x_1)$ converges if

$$\left|\frac{x_1-\alpha}{x_1}\right|>\left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|<1$ and $\alpha\neq 0$, $\Omega(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $|k|>1$, or if $\alpha=0$, then $W(x_1)$ converges if*

$$|x_1-\alpha|<|x_0-\alpha|.$$

CASE C. *If $\alpha\neq 0$, $W(x_1)$ converges for every value of x_1 (subject to the initial conditions specified above); if $\alpha=0$, $W(x_1)$ converges if*

$$R\left(\frac{1}{tx_1}\right)>R\left(\frac{1}{tx_0}\right).$$

CASE D. *In this case, $W(x_1)$ converges if*

$$R\left(\frac{x_1}{t}\right)<R\left(\frac{x_0}{t}\right).$$

For the proof of this theorem also we employ lemma I. We write

$$b_n=\alpha_n S_1(x_0)S_2(x_0)\ldots.S_n(x_0),\qquad c_n=\frac{S_1(x_1)S_2(x_1)\ldots.S_n(x_1)}{S_1(x_0)S_2(x_0)\ldots.S_n(x_0)}.$$

Then the series $b_1+b_2+b_3+\ldots.$ converges, by hypothesis. From lemma I it follows that the series $b_1c_1+b_2c_2+b_3c_3+\ldots.$, and hence $W(x_1)$, converges, provided that

$$\sum_{n=1}^{\infty}|c_n-c_{n+1}|$$

is convergent. Now this series may be obtained from (3) by multiplying each term of series (3) by $|x_1/x_0|$ and in the result exchanging x_0 and x_1. Hence, we have only to repeat the argument for theorem I_1, interchanging the rôles of x_0 and x_1, in order to complete the proof of theorem I_2.

By the use of lemma II instead of lemma I, it is possible to strengthen this theorem by a slight weakening of the hypothesis. The discussion is analogous to that for theorem I_1; it is omitted here.

It is clear that the present theorem enables us to determine completely the character of the region of convergence for $W(x)$. It may consist of the entire plane or it may be non-existent, in special cases. Laying aside these possibilities, it is easy to see that the boundary of the region of convergence, in every case, must be of the same character for $W(x)$ as for $\Omega(x)$. In each case, however, the region of convergence for $W(x)$ is on the opposite side of the boundary from that for $\Omega(x)$. This is analogous to the corresponding facts for ascending and descending power series. This is natural, since $W(x)$ contains ascending power series as a special case, while $\Omega(x)$ contains descending power series as a special case.

We turn now to a consideration of the character of the region of absolute convergence of $\Omega(x)$. Let us suppose that $\Omega(x_0)$ is absolutely convergent, and ascertain conditions which are sufficient to ensure that $\Omega(x_1)$ is absolutely convergent. One such condition is that the ratio

$$c_n = \frac{x_0 S_1(x_0) \dots . S_n(x_0)}{x_1 S_1(x_1) \dots . S_n(x_1)} \tag{5}$$

of corresponding terms of $\Omega(x_1)$ and $\Omega(x_0)$ is bounded in absolute value. Clearly this ratio is bounded if the infinite product

$$\prod_{n=1}^{\infty} \left| \frac{S_n(x_0)}{S_n(x_1)} \right|$$

is convergent or if it diverges to zero. For the study of this matter we may make the same separation into cases as in theorem I_1. By taking up each case separately, making use of equations (4), and applying elementary tests of convergence, or divergence to zero, of an infinite product, one may easily show that $|c_n|$ is bounded, in each case, provided that x_0 and x_1 satisfy the relations specified in theorem I_1 for such case. Hence we have the following result:

THEOREM II_1. *If throughout theorem* I_1 *we replace the word "converges" by the words "converges absolutely," we obtain a new theorem which is valid.*

By a precisely similar argument one may also prove the following theorem:

THEOREM II_2. *If throughout theorem* I_2 *we replace the word "converges" by the words "converges absolutely," we obtain a new theorem which is valid.*

By a discussion in all respects similar to that which follows theorem I_1, it is now possible to determine completely the character of the regions of absolute convergence of both $\Omega(x)$ and $W(x)$. It is clear that in each case this region is of the same character as the region of convergence for the corresponding case. Consequently, it is unnecessary to go into the treatment in detail.

In view of the preceding discussion, it is an easy matter to construct series having the following interesting property: They converge for every non-exceptional value of x; they converge absolutely for no non-exceptional value of x. We shall illustrate this remark by a single example.

In case A, put $\alpha=0$, $\beta=1$, $k=2$. Then we have

$$S_n(x)=\frac{2^n x}{(2^n-1)x+1}.$$

Consider the series

$$\Omega(x)=\sum_{n=1}^{\infty}\frac{a_{n+1}}{xS_1(x)\ldots S_n(x)},$$

where

$$a_{n+1}=\frac{(-1)^n}{n}\prod_{k=1}^{n}\frac{2^{k+1}}{2^{k+1}-1}.$$

For the non-exceptional value $x=2$, this series converges, but it does not converge absolutely. Hence, from theorems I_1 and II_1 (and an examination of the series for exceptional values of x) it follows that $\Omega(x)$ converges for every x different from zero; it converges absolutely, however, only for the exceptional values

$$x=-\frac{1}{2^n-1},\qquad n=1,2,3,\ldots;$$

and then trivially, since for each of these values it has only a finite number of terms different from zero.

This example raises the general question as to the character of the region of conditional convergence of the series $\Omega(x)$ and $W(x)$. It is easy to see, in the light of the preceding theorems, and for the case of a series of each type, that the region of conditional convergence may be non-existent (through the series either diverging everywhere or converging absolutely throughout its region of convergence) or may be the entire region of convergence, in the following cases:

Case A: $|k|<1$ and $\alpha\neq 0$; $|k|>1$ and $\beta\neq 0$.

Case B: $|k|<1$ and $\alpha\neq 0$.

Case C: $\alpha\neq 0$.

To investigate the remaining cases we proceed as follows: Let x_0 and x_1 be non-exceptional values for the series $\Omega(x)$. Suppose that there exists a constant A such that the n-th term of $\Omega(x_0)$ is in absolute value less than A for every n. Denote by c_n the ratio (given in (5)) between corresponding terms of $\Omega(x_0)$ and $\Omega(x_1)$. Then $\Omega(x_1)$ converges absolutely if

$$|c_1|+|c_2|+|c_3|+\ldots$$

converges. Now the ratio $|c_n|/|c_{n-1}|$ of two consecutive terms of this series

is $|Q_n(x_0, x_1)|$. If, now, in investigating the convergence of this series, we employ equations (4) and make use of the simple ratio test and the Gauss criterion for convergence of series, we are led to the following theorem:

THEOREM III$_1$. *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$\Omega(x)=\alpha_0+\frac{\alpha_1}{x}+\sum_{n=1}^{\infty}\frac{\alpha_{n+1}}{xS_1(x)S_2(x)\ldots.S_n(x)},$$

and suppose that there exists a constant A such that the n-th term of $\Omega(x_0)$ is in absolute value less than A for every n. Then $\Omega(x_1)$ converges absolutely in the following cases:

CASE A. *If $|k|<1$ and $\alpha=0$, $\Omega(x_1)$ converges absolutely if*

$$\left|\frac{x_1-\beta}{x_1}\right|<\left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $\Omega(x_1)$ converges absolutely if

$$\left|\frac{x_1-\alpha}{x_1}\right|<\left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|>1$, or if $\alpha=0$, then $\Omega(x_1)$ converges absolutely if*

$$|x_1-\alpha|>|x_0-\alpha|.$$

CASE C. *If $\alpha=0$, $\Omega(x_1)$ converges absolutely if*

$$R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx_1}\right)>1.$$

CASE D. *In this case, $\Omega(x_1)$ converges absolutely if*

$$R\left(\frac{x_1}{t}\right)-R\left(\frac{x_0}{t}\right)>1.$$

In a similar way one readily proves the following theorem:

THEOREM III$_2$. *Let x_0 and x_1 be two non-exceptional values of x for the series*

$$W(x)=\alpha_0+\sum_{n=1}^{\infty}\alpha_nS_1(x)S_2(x)\ldots.S_n(x),$$

and suppose that there exists a constant A such that the n-th term of $W(x_0)$ is in absolute value less than A for every n. Then $W(x_1)$ converges absolutely in the following cases:

CASE A. *If $|k|<1$ and $\alpha=0$, $W(x_1)$ converges absolutely if*

$$\left|\frac{x_1-\beta}{x_1}\right|>\left|\frac{x_0-\beta}{x_0}\right|;$$

if $|k|>1$ and $\beta=0$, $W(x_1)$ converges absolutely if

$$\left|\frac{x_1-\alpha}{x_1}\right|>\left|\frac{x_0-\alpha}{x_0}\right|.$$

CASE B. *If $|k|>1$, or if $\alpha=0$, then $W(x_1)$ converges absolutely if*

$$|x_1-\alpha|<|x_0-\alpha|.$$

CASE C. *If $\alpha=0$, $W(x_1)$ converges absolutely if*

$$R\left(\frac{1}{tx_1}\right)-R\left(\frac{1}{tx_0}\right)>1.$$

CASE D. *In this case, $W(x_1)$ converges absolutely if*

$$R\left(\frac{x_0}{t}\right)-R\left(\frac{x_1}{t}\right)>1.$$

From the last two theorems it follows that $\Omega(x)$ and $W(x)$ both converge absolutely at every non-exceptional point in the interior of their regions of convergence, in the following cases: Case A, when $|k|>1$ and $\beta=0$; cases A and B, when $|k|<1$ and $\alpha=0$. Here the general theory is analogous to that of power series, as indeed it should be, since in these cases the series are direct generalizations of power series.

In case D, the region of conditional convergence is a strip bounded by the line of convergence and the line of absolute convergence, the width of this strip being at most $|t|$. It is possible to construct examples (cf. Landau, *loc. cit.*) to show that each of the logical possibilities may arise; namely, that the strip of conditional convergence is non-existent, that it exists and is of width less than $|t|$, that it is of width $|t|$.

In case C, when $\alpha=0$, it is easy to see that there may again exist a region of conditional convergence. It is bounded by two circles, each passing through the point 0 and having its center on the straight line $(0t)$ through the points 0 and t. The maximum distance between the centers of these two circles—the circle of convergence and the circle of absolute convergence—is determined in the case of $\Omega(x)$ $[W(x)]$ by means of the relation

$$R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx_1}\right)\leqq 1 \qquad \left[R\left(\frac{1}{tx_1}\right)-R\left(\frac{1}{tx_0}\right)\leqq 1\right],$$

where x_0 is on the circle of convergence and x_1 is on the circle of absolute convergence. The greatest possible width of the region of conditional convergence is thus a function of the radius of convergence of the series.

Examples might be constructed to show (for this case as well as for the preceding) that each of the three logical possibilities may actually arise. This may also be seen indirectly by observing that $\Omega(x)$ $[W(x)]$, for case C when $\alpha=0$, transforms into $W(x-t)$ $[\Omega(x+t)]$ for case D by replacing x by $1/x$.

§ 3. *Uniform Convergence of the Series. Nature of the Functions Defined by Them.*

We shall now prove the following theorem:

THEOREM IV. *The series* $\Omega(x)$ $[W(x)]$ converges uniformly throughout any closed domain D lying within its region of convergence and containing no point x which is exceptional for $\Omega(x)$ $[W(x)]$.*

*The exceptional subdivisions of cases A and B in which $|k|$ is unity are naturally excluded here.

Let us consider first the case of the series $\Omega(x)$. Suppose that x_0 is a non-exceptional value of x for which $\Omega(x)$ is convergent; later we shall subject x_0 to such further conditions as will serve our convenience. Write

$$b_n = \frac{a_{n+1}}{x_0 S_1(x_0) \ldots . S_n(x_0)}, \qquad c_n = \frac{x_0 S_1(x_0) \ldots . S_n(x_0)}{x_1 S_1(x_1) \ldots . S_n(x_1)}.$$

Then the series $b_1 + b_2 + b_3 + \ldots .$ converges, by hypothesis. From lemma IV it follows that the series $b_1 c_1 + b_2 c_2 + b_3 c_3 + \ldots .$, and hence the series $\Omega(x)$, converges uniformly throughout D, provided that

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}| \tag{6}$$

converges uniformly in D.

Now series (6) is what series (3) becomes on replacing x_1 by x; and therefore we may employ the results of the reckoning in connection with (3) in the proof of the uniform convergence of (6). Thus we see that, when n increases indefinitely, the ratio $r_n(x)$ of two consecutive terms of (6) approaches a limit which is less than unity (whatever x_0 may be) in each of the following cases: Case A, when $|k| < 1$ and $\alpha \neq 0$ or when $|k| > 1$ and $\beta \neq 0$; case B, when $|k| < 1$ and $\alpha \neq 0$. Hence it is easy to construct a comparison series of constant terms such that series (6) is term by term less than this comparison series. Hence, in these cases (6) is uniformly convergent. In case C, when $\alpha \neq 0$, it is also unnecessary to place any restriction on x_0, since in this case

$$r_n(x) = 1 - \frac{2}{n} + \ldots .$$

For, as a comparison series, one may employ a series of the form

$$\sum_{n=1}^{\infty} \frac{A}{n^{3/2}},$$

where A is a properly chosen constant.

In each of the other cases it is necessary to place further restrictions on x_0.

In case A, when $|k| < 1$ and $\alpha = 0$, we have

$$\lim_{n=\infty} r_n(x) = \left| \frac{x_0}{x} \right| \cdot \left| \frac{x - \beta}{x_0 - \beta} \right|.$$

Since x is in D and D lies within the region of convergence for $\Omega(x)$, it is clear that x_0 can be chosen so that the above limit is less than a properly chosen positive constant ρ ($\rho < 1$) for every x in D. Hence, by means of a comparison series of constant terms of the form

$$A + A\rho + A\rho^2 + \ldots .,$$

it may be shown in this case also that (6) is uniformly convergent in D. Sim-

ilarly, one may deal with each of the following: Case A, when $|k|>1$ and $\beta=0$; case B, when $\alpha=0$ or $|k|>1$.

In case C, if $\alpha=0$ we have

$$r_n(x)=1-\frac{1+R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx}\right)}{n}+\ldots\ldots$$

Since x is in D and D lies within the region of convergence of $\Omega(x)$, it is easy to see that there exists a positive constant 2ε such that

$$R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx}\right)\geqq 2\varepsilon$$

for every x in D. Therefore, for the proof in this case that (6) converges uniformly in D, it is sufficient to construct a comparison series of the form

$$\sum_{n=1}^{\infty}\frac{A}{n^{1+\epsilon}},$$

where A is a properly chosen positive constant. Similarly, one may deal with case D.

This completes the examination of all the cases for the series $\Omega(x)$; and consequently the theorem is established for this series.

It is unnecessary to give in detail the argument for $W(x)$, since it is so far similar to that for $\Omega(x)$. It is sufficient to point out that one employs lemma IV, using for b_n and c_n the quantities

$$b_n=\alpha_n S_1(x_0)S_2(x_0)\ldots.S_n(x_0),\qquad c_n=\frac{S_1(x)S_2(x)\ldots.S_n(x)}{S_1(x_0)S_2(x_0)\ldots.S_n(x_0)}.$$

For our case D (the case of factorial series and binomial coefficient series), Landau (*loc. cit.*) effects the proof of theorem IV by the use of lemma III. This lemma is obviously not sufficient for the more general case treated here, since it is not always true that c_n approaches zero as n increases indefinitely.

Now, if we make use of the well-known theorem of Weierstrass relative to the analytic character of the function represented as a uniformly convergent series of analytic functions, we are led (in view of theorem IV) to the following theorem:

THEOREM V. *The series* $\Omega(x)$ $[W(x)]$ (*at least if* $|k|$ *is different from unity in cases* A *and* B) *represents a function which is analytic at every non-exceptional point* x *lying within its region of convergence.*

Certain exceptional points for the series are always regular points of the functions represented by them. Thus, a point which is exceptional for one of the series only through causing every term past a certain one in that series to vanish, is clearly a regular point for the function represented by the series, provided that it lies within the region of convergence of the series.

Furthermore, it is easy to show that a point (within the region of convergence of the series) which is exceptional only through causing every term of the series past a certain one to have a pole of the first order at the exceptional point, is either a regular point or a pole of the first order for the function represented by the series (compare Landau, *loc. cit.*, p. 164, where a discussion of this matter for factorial series is given).

§ 4. *Dependence of the Region of Convergence on the Coefficients of the Series.*

In the preceding discussion of the nature of the regions of convergence of the series $\Omega(x)$ and $W(x)$ we have employed no properties of the (constant) coefficients except what is involved in the assumption of convergence or of absolute convergence for some value x_0 of x. We turn now to the question as to how the *magnitude* of the region of convergence depends on the actual coefficients of a given series.

In certain cases this question is trivial; namely, in those cases in which it is true that the series always converges [converges absolutely] for every non-exceptional x as soon as it converges [converges absolutely] for a single non-exceptional x. All that is necessary, in such a case, for a complete answer to the question is to determine the divergence or the nature of the convergence of the series for a single value of x, and then apply the general theorems of § 2.

Our general question here has already been answered by Landau (*loc. cit.*) for the case of factorial series and binomial coefficient series. This obviously affords also the answer to the question for our cases D, since these obviously go over into the simpler cases (treated by Landau) by multiplicative transformations on x.

We have observed above that the simple transformation of replacing x by $1/x$ carries our $\Omega(x)$ $[W(x)]$ for case C when $\alpha=0$ over into our $W(x-t)$ $[\Omega(x+t)]$ for case D. Hence Landau's theory affords an immediate means for the resolution of the present problem for case C when $\alpha=0$.

For case B when $\alpha=0$ the series $\Omega(x)$ and $W(x)$ are both power series; and hence the problem has been solved for this case.

There remains for further consideration essentially two cases: Namely, case A when $|k|<1$ and $\alpha=0$ (this being equivalent to case A when $|k|>1$ and $\beta=0$) and case B when $|k|>1$. These we shall now take up in turn. Since in each of them the boundary of the region of convergence coincides with that of the region of absolute convergence it is sufficient to treat only the former.

Let us first consider case A when $|k|<1$ and $\alpha=0$. We have

$$S_n(x)\equiv\frac{k^n\beta x}{(k^n-1)x+\beta}, \qquad \beta\neq 0, \quad k\neq 0, \quad |k|<1. \tag{7}$$

ilarly, one may deal with each of the following: Case A, when $|k|>1$ and $\beta=0$; case B, when $\alpha=0$ or $|k|>1$.

In case C, if $\alpha=0$ we have

$$r_n(x)=1-\frac{1+R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx}\right)}{n}+\ldots\ldots$$

Since x is in D and D lies within the region of convergence of $\Omega(x)$, it is easy to see that there exists a positive constant 2ε such that

$$R\left(\frac{1}{tx_0}\right)-R\left(\frac{1}{tx}\right)\geqq 2\varepsilon$$

for every x in D. Therefore, for the proof in this case that (6) converges uniformly in D, it is sufficient to construct a comparison series of the form

$$\sum_{n=1}^{\infty}\frac{A}{n^{1+\epsilon}},$$

where A is a properly chosen positive constant. Similarly, one may deal with case D.

This completes the examination of all the cases for the series $\Omega(x)$; and consequently the theorem is established for this series.

It is unnecessary to give in detail the argument for $W(x)$, since it is so far similar to that for $\Omega(x)$. It is sufficient to point out that one employs lemma IV, using for b_n and c_n the quantities

$$b_n=a_nS_1(x_0)S_2(x_0)\ldots.S_n(x_0),\qquad c_n=\frac{S_1(x)S_2(x)\ldots.S_n(x)}{S_1(x_0)S_2(x_0)\ldots.S_n(x_0)}.$$

For our case D (the case of factorial series and binomial coefficient series), Landau (*loc. cit.*) effects the proof of theorem IV by the use of lemma III. This lemma is obviously not sufficient for the more general case treated here, since it is not always true that c_n approaches zero as n increases indefinitely.

Now, if we make use of the well-known theorem of Weierstrass relative to the analytic character of the function represented as a uniformly convergent series of analytic functions, we are led (in view of theorem IV) to the following theorem:

THEOREM V. *The series* $\Omega(x)$ $[W(x)]$ (*at least if* $|k|$ *is different from unity in cases* A *and* B) *represents a function which is analytic at every non-exceptional point* x *lying within its region of convergence.*

Certain exceptional points for the series are always regular points of the functions represented by them. Thus, a point which is exceptional for one of the series only through causing every term past a certain one in that series to vanish, is clearly a regular point for the function represented by the series, provided that it lies within the region of convergence of the series.

Furthermore, it is easy to show that a point (within the region of convergence of the series) which is exceptional only through causing every term of the series past a certain one to have a pole of the first order at the exceptional point, is either a regular point or a pole of the first order for the function represented by the series (compare Landau, *loc. cit.*, p. 164, where a discussion of this matter for factorial series is given).

§ 4. *Dependence of the Region of Convergence on the Coefficients of the Series.*

In the preceding discussion of the nature of the regions of convergence of the series $\Omega(x)$ and $W(x)$ we have employed no properties of the (constant) coefficients except what is involved in the assumption of convergence or of absolute convergence for some value x_0 of x. We turn now to the question as to how the *magnitude* of the region of convergence depends on the actual coefficients of a given series.

In certain cases this question is trivial; namely, in those cases in which it is true that the series always converges [converges absolutely] for every non-exceptional x as soon as it converges [converges absolutely] for a single non-exceptional x. All that is necessary, in such a case, for a complete answer to the question is to determine the divergence or the nature of the convergence of the series for a single value of x, and then apply the general theorems of § 2.

Our general question here has already been answered by Landau (*loc. cit.*) for the case of factorial series and binomial coefficient series. This obviously affords also the answer to the question for our cases D, since these obviously go over into the simpler cases (treated by Landau) by multiplicative transformations on x.

We have observed above that the simple transformation of replacing x by $1/x$ carries our $\Omega(x)$ $[W(x)]$ for case C when $\alpha=0$ over into our $W(x-t)$ $[\Omega(x+t)]$ for case D. Hence Landau's theory affords an immediate means for the resolution of the present problem for case C when $\alpha=0$.

For case B when $\alpha=0$ the series $\Omega(x)$ and $W(x)$ are both power series; and hence the problem has been solved for this case.

There remains for further consideration essentially two cases: Namely, case A when $|k|<1$ and $\alpha=0$ (this being equivalent to case A when $|k|>1$ and $\beta=0$) and case B when $|k|>1$. These we shall now take up in turn. Since in each of them the boundary of the region of convergence coincides with that of the region of absolute convergence it is sufficient to treat only the former.

Let us first consider case A when $|k|<1$ and $\alpha=0$. We have

$$S_n(x)\equiv\frac{k^n\beta x}{(k^n-1)x+\beta}, \qquad \beta\neq 0, \quad k\neq 0, \quad |k|<1. \tag{7}$$

The boundary of the region of convergence is a circle whose equation is of the form

$$\left|\frac{x-\beta}{x}\right|=\rho, \tag{8}$$

where ρ is a constant depending on the coefficients $a_1, a_2, \ldots.$ of the series. We have to determine ρ. We first prove the following theorem:

THEOREM VI$'_1$. *The two series**

$$\Omega(x)=a+\frac{a_0}{x}+\sum_{n=1}^{\infty}\frac{a_n k^{1+2+\cdots+n}(-\beta)^n}{xS_1(x)S_2(x)\ldots S_n(x)}, \qquad \psi(x)=\sum_{n=1}^{\infty}a_n\left(\frac{x-\beta}{x}\right)^n,$$

in which $S_n(x)$ has the value given in (7), *both converge or both diverge for any given value of x which is non-exceptional for $\Omega(x)$.*

The proof falls into two parts.

1. Suppose that $\psi(x)$ converges for a value x_0 of x which is non-exceptional for $\Omega(x)$. We shall prove that $\Omega(x_0)$ is convergent. Write

$$b_n=a_n\left(\frac{x_0-\beta}{x_0}\right)^n, \qquad c_n=\frac{k^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}\left(\frac{x_0}{x_0-\beta}\right)^n.$$

The series $b_1+b_2+\ldots.$ converges, by hypothesis. Hence, from lemma I, it follows that $\Omega(x_0)$ converges, provided that

$$\sum_{n=1}^{\infty}|c_n-c_{n+1}| \tag{9}$$

is convergent.

We have

$$\begin{aligned} c_n-c_{n+1}&=\frac{k^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}\left(\frac{x_0}{x_0-\beta}\right)^n\left(1-\frac{k^{n+1}(-\beta)}{S_{n+1}(x_0)}\cdot\frac{x_0}{x_0-\beta}\right)\\ &=\frac{k^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}\left(\frac{x_0}{x_0-\beta}\right)^n\left(\frac{k^{n+1}x_0}{x_0-\beta}\right).\end{aligned}$$

Thence the ratio r_n of the $(n+1)$-th term to the n-th term of (9) is easily reduced to the form

$$r_n=\left|\frac{k\{(1-k^{n+1})x_0-\beta\}}{x_0-\beta}\right|.$$

Therefore, $\lim\limits_{n=\infty} r_n=|k|$; this being less than unity, it follows that (9), and hence $\Omega(x_0)$, converges.

2. Suppose that $\Omega(x)$ converges for a non-exceptional value x_0 of x. In order to prove that $\psi(x_0)$ converges, it is sufficient to write

$$b_n=\frac{a_nk^{1+2+\cdots+n}(-\beta)^n}{x_0S_1(x_0)\ldots S_n(x_0)}, \qquad c_n=\frac{x_0S_1(x_0)\ldots S_n(x_0)}{k^{1+2+\cdots+n}(-\beta)^n}\left(\frac{x_0-\beta}{x_0}\right)^n,$$

and apply lemma I. It is not necessary to give the argument in detail.

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*It is convenient here to write the (constant) coefficients in $\Omega(x)$ in a new form. Obviously, there is no loss of generality in this.

From the theorem just demonstrated it follows that the regions of convergence of $\Omega(x)$ and $\psi(x)$ coincide. But $\psi(x)$ is a power series in z, $z=(x-\beta)/x$, and therefore its region of convergence in terms of the coefficients a_n is known. It is the circle (8), where

$$\frac{1}{\rho}=\limsup_{n=\infty}\sqrt[n]{|a_n|}.$$

Hence:

THEOREM VII$'_1$. *The series $\Omega(x)$ of theorem VI$'_1$ has for the boundary of its region of convergence the circle*

$$\left|\frac{x-\beta}{x}\right|=\rho,$$

where

$$\frac{1}{\rho}=\limsup_{n=\infty}\sqrt[n]{|a_n|}.$$

In a similar manner one may readily prove the following two theorems:

THEOREM VI$'_2$. *The two series*

$$W(x)=a_0+\sum_{n=1}^{\infty}a_n\frac{S_1(x)S_2(x)\dots .S_n(x)}{k^{1+2+\dots .+n}(-\beta)^n},\qquad \phi(x)=\sum_{n=1}^{\infty}a_n\left(\frac{x}{x-\beta}\right)^n,$$

in which $S_n(x)$ has the value given in (7), both converge or both diverge for any given value of x which is non-exceptional for $W(x)$.

THEOREM VII$'_2$. *The series $W(x)$ of theorem VI$'_2$ has for the boundary of its region of convergence the circle*

$$\left|\frac{x}{x-\beta}\right|=\rho,$$

where

$$\frac{1}{\rho}=\limsup_{n=\infty}\sqrt[n]{|a_n|}.$$

We shall next treat the case B when $|k|>1$. We have

$$S_n(x)=k^n(x-\alpha)+\alpha,\qquad |k|>1.$$

We shall first prove the following theorem:

THEOREM VI$''_1$. *The two series*

$$\Omega(x)=a+\frac{a_0}{x}+\sum_{n=1}^{\infty}\frac{a_n}{x}\prod_{t=1}^{n}\frac{k^t}{k^t(x-\alpha)+\alpha},\quad |k|>1;\qquad \psi(x)=\sum_{n=1}^{\infty}\frac{a_n}{(x-\alpha)^n}$$

both converge or both diverge for any given value of x which is non-exceptional for $\Omega(x)$.

The proof falls into two parts.

1. Suppose that $\psi(x)$ converges for a value x_0 of x which is non-exceptional for $\Omega(x)$. We shall prove that $\Omega(x_0)$ is also convergent. We put

$$b_n=\frac{a_n}{(x_0-\alpha)^n},\qquad c_n=\frac{1}{x_0}\prod_{t=1}^{n}\frac{k^t}{k^t(x_0-\alpha)+\alpha}\cdot(x_0-\alpha)^n$$

and apply lemma I. In order to show that $\Omega(x_0)$ converges, it is sufficient to prove the convergence of

$$\sum_{n=1}^{\infty} |c_n - c_{n+1}|. \tag{10}$$

We have

$$c_n - c_{n+1} = \frac{1}{x_0} \prod_{t=1}^{n} \frac{k^t}{k^t(x_0-\alpha)+\alpha} \cdot (x_0-\alpha)^n \left(1 - \frac{k^{n+1}(x_0-\alpha)}{k^{n+1}(x_0-\alpha)+\alpha}\right)$$

$$= \frac{1}{x_0} \prod_{t=1}^{n} \frac{k^t}{k^t(x_0-\alpha)+\alpha} \cdot (x_0-\alpha)^n \left(\frac{\alpha}{k^{n+1}(x_0-\alpha)+\alpha}\right).$$

Hence, for the ratio r_n of the $(n+1)$-th term to the n-th term of (10), we have readily

$$r_n = \left| \frac{k^{n+1}(x_0-\alpha)}{k^{n+2}(x_0-\alpha)+\alpha} \right|.$$

Therefore, $\lim\limits_{n=\infty} r_n = |1/k|$; this being less than unity, it follows that (10), and hence $\Omega(x_0)$, converges.

2. Suppose that $\Omega(x_0)$ converges, x_0 being a non-exceptional value of x for $\Omega(x)$. In order to show that $\psi(x_0)$ also converges, it is sufficient to put

$$b_n = \frac{a_n}{x_0} \prod_{t=1}^{n} \frac{k^t}{k^t(x_0-\alpha)+\alpha}, \qquad c_n = x_0 \prod_{t=1}^{n} \frac{k^t(x_0-\alpha)+\alpha}{k^t} \cdot \frac{1}{(x_0-\alpha)^n}$$

and apply lemma I. It is unnecessary to give the argument in detail.

From the preceding theorem we have at once the following:

THEOREM VII$_1''$. *The series $\Omega(x)$ of theorem VI$_1''$ has for the boundary of its region of convergence a circle about α as center with radius ρ, where*

$$\rho = \limsup_{n=\infty} \sqrt[n]{|a_n|}.$$

In a similar manner we may readily prove the following two theorems:

THEOREM VI$_2''$. *The two series*

$$W(x) = a_0 + \sum_{n=1}^{\infty} a_n \prod_{t=1}^{n} \frac{k^t(x-\alpha)+\alpha}{k^t}, \quad |k|>1; \qquad \phi(x) = \sum_{n=1}^{\infty} a_n (x-\alpha)^n$$

both converge or both diverge for any given value of x which is non-exceptional for $W(x)$.

THEOREM VII$_2''$. *The series $W(x)$ of theorem VI$_2''$ has for the boundary of its region of convergence a circle about α as center with radius ρ, where*

$$\frac{1}{\rho} = \limsup_{n=\infty} \sqrt[n]{|a_n|}.$$

INDIANA UNIVERSITY, *July*, 1913.

The Derivative of a Function of a Surface.[*]

By Charles A. Fischer.

Introduction.

In a paper published in the *Atti della R. Academia dei Lincei*,[†] Volterra has defined the derivative of a function of a curve, and has proved that if the derivative is continuous and approached uniformly, the first variation of the function is equal to the integral of the product of this derivative and the first variation of the dependent variable y, taken between limits equal to the values of the independent variable x at the end-points of the given curve. In a paper entitled "A Generalization of Volterra's Derivative of a Function of a Curve,"[‡] I have modified the definition of this derivative to make it applicable to a restricted set of curves defined by means of some ordinary differential equations, and applied some resulting theorems to the Lagrange problem of the calculus of variations.

In the present paper I have given a definition of the derivative of a function of a surface analogous to Volterra's derivative of a function of a curve, and have proved substantially that if the derivative is continuous and approached uniformly with any finite order, then the first variation of the given function is equal to the double integral of the product of the derivative of the function and the first variation of the dependent variable z, the integration taking place over the projection of the given surface on the x, y-plane. I have also generalized this definition and theorem to make them applicable to a restricted set of surfaces, in a manner somewhat similar to that employed in my paper already referred to.

The first section contains the definition of the derivative of a function of a surface, the theorem mentioned above, and a proof that the derivative must vanish at every point of a surface which minimizes the given function. In § 2

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* Presented to the American Mathematical Society, April 26, 1913.

† Volterra, *Atti della R. Academia dei Lincei*, Ser. IV, Rend. III_2, p. 97.

‡ Fischer, American Journal of Mathematics, Vol. XXXV (1913), No. 4, p. 369.

the given function is taken as a double integral. Its derivative is found, and an application is made to the theory of maxima and minima of double integrals. In the remainder of the paper only those surfaces which give previously assigned values to a set of functions $M_1(S), M_2(S), \ldots, M_m(S)$ are considered admissible. In § 3 it is proved that in the neighborhood of any admissible surface for which a certain determinant does not vanish, there are other admissible surfaces of the kind needed in finding the derivative of another function $L(S)$ relative to this restricted set of surfaces. In § 4 this derivative is defined and the theorems of the first section are proved to hold for the special set of surfaces also. In the last section all of the functions of surfaces considered are taken to be double integrals, and the results obtained in § 4 are applied to the isoperimetric problem of the theory of maxima and minima of double integrals.

§ 1. *Definition of the Derivative and Volterra's Theorem.*

All of the functions of x and y considered in this paper will be defined over a region R in the x, y-plane, and will be assumed to be of class $C^{(r)}$,* where r is an arbitrary positive integer. In the first part of this section R will be taken as a rectangle defined by the inequalities $a \leqq x \leqq b$ and $c \leqq y \leqq d$, but later this restriction will be removed and R will be an arbitrary closed region bounded by a curve of finite length.

The surfaces S and S_ϵ will be frequently mentioned. They will be defined by the equations

$$\begin{aligned} S: &\quad z=z(x, y), \\ S_\epsilon: &\quad z=z(x, y)+\eta(x, y), \end{aligned}$$

where the function $\eta(x, y)$ is assumed to have the following properties: It vanishes for all values of x and y outside of the region defined by the inequalities $x_0-\epsilon<x<x_0+\epsilon$ and $y_0-\epsilon<y<y_0+\epsilon$, or as much of that region as is inside of R if the point (x_0, y_0) happens to be on the boundary of R. Inside this region it has a permanent sign and is not identically zero. The absolute value of each of its partial derivatives up to and including those of order r is everywhere less than ϵ, the function itself being considered as the partial derivative of order zero.

The derivative of the function $L(S)$ of the surface S can now be defined as follows: *If the limit*

$$L'(S; x_0, y_0) = \lim_{\epsilon=0} \frac{L(S_\epsilon)-L(S)}{\sigma}$$

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* Bolza, "Vorlesungen über Variationsrechnung," p. 14.

exists uniquely, where σ is defined by the equation

$$\sigma=\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\eta(x, y)\,dy\,dx, \tag{1}$$

then this limit is said to be the derivative of the function $L(S)$ at the point (x_0, y_0), and it is said to be approached with order r.

It will be assumed that if $\bar{S}$ is any surface in a fixed neighborhood of order r of the surface S, the function $L(S)$ has the following property: For every $\delta>0$ there exists an $\epsilon>0$ independent of $\bar{S}$ and (x_0, y_0), such that the inequality

$$\left|\frac{L(\bar{S}_\epsilon)-L(\bar{S})}{\sigma}-L'(\bar{S}; x_0, y_0)\right|<\delta$$

is always satisfied. That is to say, the derivative is approached uniformly with order r in the neighborhood of the surface S.

It will be assumed that for every $\delta>0$ there exists an $\epsilon>0$ independent of (x_0, y_0) such that if $\bar{S}$ is any surface in the neighborhood $(\epsilon)_r$ of S, then the inequality

$$|L'(\bar{S}; x_0, y_0)-L'(S; x_0, y_0)|<\delta$$

is satisfied. That is to say, $L'(S; x, y)$ is continuous with order r in the argument S. It will also be assumed that $L'(S; x, y)$ is a continuous function of the variables x and y, and since the region R is closed, it is uniformly continuous.

A family of surfaces will now be considered which is defined by the equation

$$S_\alpha: \quad z=z(x, y)+\omega(x, y, \alpha),$$

where the function $\omega(x, y, \alpha)$ has the following properties: All of its partial derivatives with respect to x and y vanish identically in α at every point of the boundary of R. They also approach zero uniformly in x and y when α approaches zero, and vanish when $\alpha=0$. If a numerically small constant value is given to α, the function $\omega(x, y, \alpha)$ has a permanent sign. The derivative $\omega_\alpha(x, y, 0)$ exists, is continuous and approached uniformly.

The extension of Volterra's theorem, mentioned in the introduction, to functions of surfaces will be stated as follows: *If the function $L(S)$ and the surface S_α satisfy the assumptions given above, then the derivative of $L(S_\alpha)$ with respect to the parameter α is given by the equation*

$$\left.\frac{dL(S_\alpha)}{d\alpha}\right|_{\alpha=0}=\int\int_R L'(S; x, y)\,\omega_\alpha(x, y, 0)\,dy\,dx.$$

Select the constant ϵ arbitrarily. Then divide the region R into n^2 equal rectangles, calling their vertices (x_i, y_j), where i and j have the range 0, 1, 2,

. . . ., n, and taking n so large that the dimensions of each rectangle are less than ε. Evidently $x_0=a$, $x_n=b$, $y_0=c$, and $y_n=d$. Now choose another positive constant h satisfying the equation $nh(b-a+d-c)=\varepsilon$. A set of functions $\theta_1(x)$, $\theta_2(x)$, ..., $\theta_n(x)$ will then be selected, satisfying the conditions

$$\begin{aligned} \theta_i(x)&=0, \quad (a \leqq x_i \leqq x_{i-1} \text{ and } x_i \leqq x \leqq b), \qquad (i=1, 2, \ldots, n),\\ 0<\theta_i(x)&<1, \quad (x_{i-1}<x<x_{i-1}+h \text{ and } x_i-h<x<x_i),\\ \theta_i(x)&=1, \quad (x_{i-1}+h \leqq x \leqq x_i-h). \end{aligned}$$

Another set of functions $\phi_1(y)$, $\phi_2(y)$, ..., $\phi_n(y)$ will also be chosen which satisfy the analogous conditions. The surfaces S_{ij} will then be defined by the equations

$$S_{ij}: \qquad z=z(x, y)+\left[\sum_{k=1}^{i-1}\sum_{l=1}^{n}\theta_k(x)\phi_l(y)+\sum_{l=1}^{j}\theta_i(x)\phi_l(y)\right]\omega(x, y, \alpha),$$
$$(i, j=1, 2, \ldots, n).$$

For convenience in notation $S_{i-1\,n}$ will sometimes be written S_{i0}.

After ε and n have been chosen, it is possible to find a positive constant α_0 such that if $|\alpha|<\alpha_0$ the surfaces S_{ij} will all be in the neighborhood $(\varepsilon)_r$ of S. Then S_{ij} is the kind of varied surface used in forming the derivative

$$L'(S_{i\,j-1};\, x_i,\, y_j),$$

and since it is assumed to be approached uniformly, the following equation is satisfied:

$$\frac{L(S_{nn})-L(S)}{\alpha}=$$
$$\sum_{i,j=1}^{n}[L'(S_{i\,j-1};\, x_i,\, y_j)+\xi_{ij}]\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\theta_i(x)\phi_j(y)\,\frac{\omega(x, y, \alpha)}{\alpha}\,dydx, \qquad (2)$$

where ξ, the upper bound of $|\xi_{ij}|$, approaches zero with ε. The derivatives $L'(S_{i\,j-1};\, x_i,\, y_j)$ may be taken at any points of the proper rectangles instead of at the corners, if desired. Since it is assumed to be continuous in the argument S, $L'(S_{i\,j-1};\, x_i\, y_j)$ can be replaced by $L'(S;\, x_i,\, y_j)$. This changes the values of the quantities ξ_{ij} but does not affect the property of their upper bound mentioned above. The double integral in the right member of equation (2) can be written

$$\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\frac{\omega(x, y, \alpha)}{\alpha}\,dydx-\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}[1-\theta_i(x)\phi_j(y)]\,\frac{\omega(x, y, \alpha)}{\alpha}\,dydx$$
$$=\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\frac{\omega(x, y, \alpha)}{\alpha}\,dydx+2\lambda hN\,\frac{b-a+d-c}{n}, \qquad (3)$$

where $-1<\lambda<1$, and

$$N > \left| \frac{\omega(x, y, \alpha)}{\alpha} \right|.$$

The uniform approach to $\omega_\alpha(x, y, 0)$ implies the existence of such a constant N for α_0 sufficiently small. If a constant M is chosen, satisfying the inequalities

$$M > |L'(S; x_i, y_j) + \xi_{ij}|, \qquad (i, j = 1, 2, \ldots, n),$$

the absolute value of the contribution of the last term of the right member of equation (3) to the value of the right member of equation (2) will be less than the quantity $2nhMN(b-a+d-c) = 2MN\epsilon$, which approaches zero with ϵ. The absolute value of the sum of the terms involving ξ_{ij} is less than $\xi N(b-a)(d-c)$, which also approaches zero with ϵ. The mean-value theorem implies that the equation

$$\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \frac{\omega(x, y, \alpha)}{\alpha}\, dy dx = \frac{\omega(x'_i, y'_j, \alpha)}{\alpha} (x_i - x_{i-1})(y_j - y_{j-1})$$

is satisfied, where (x'_i, y'_j) is a point in the rectangle over which the integration takes place, and in accordance with the remark following equation (2) $L'(S; x'_i, y'_j)$ can be substituted for $L'(S; x_i, y_j)$ in that equation. It follows directly that the equation

$$\lim_{\epsilon,\, \alpha=0} \frac{L(S_{nn}) - L(S)}{\alpha} = \int_a^b \int_c^d L'(S; x, y)\, \omega_\alpha(x, y, 0)\, dy dx \tag{4}$$

is satisfied.

It remains to be proved that

$$\lim_{\epsilon,\, \alpha=0} \frac{L(S_\alpha) - L(S_{nn})}{\alpha} = 0. \tag{5}$$

By definition S_{nn} coincides with S_α, excepting over $n+1$ rectangles in the x, y-plane whose dimensions are $b-a$ and either h or $2h$, and $n+1$ other rectangles whose dimensions are either h or $2h$ and $d-c$; and wherever the surfaces meet, the contact is of order r. The overlapping of the two sets of rectangles has no effect on the work here. The following lemma is needed to establish equation (5):

If the surface S_h is given by the equation

$$S_h: \qquad z = z(x, y) + \eta(x, y, \alpha),$$

where the function $\eta(x, y, \alpha)$ vanishes identically over all of the region R excepting the rectangle $(a < x < b;\, y_0 - h < y < y_0 + h)$ where it has a permanent sign when α is kept constant, if there is a finite constant N satisfying the inequality

$$\left| \frac{\eta(x, y, \alpha)}{\alpha} \right| < N,$$

and all of its partial derivatives with respect to x and y approach zero uniformly with α, then, for sufficiently small values of h and $|\alpha|$, the inequality

$$\left|\frac{L(S_h)-L(S)}{\alpha}\right| < 4hMN(b-a) \tag{6}$$

is satisfied, where M is any constant greater than the upper bound of $|L'(S; x, y)|$.

This can be proved as follows: Divide the interval $[a, b]$ into n equal parts, calling the points of division $x_1, x_2, \ldots, x_{n-1}$. For convenience in notation let $x_0=a$ and $x_m=b$. Then a set of functions $\psi_1(x), \psi_2(x), \ldots, \psi_{n-1}(x)$ can be found which satisfy the conditions

$$\begin{aligned} \psi_i(x)&=1, \quad (a \leqq x \leqq x_i), \qquad (i=1, 2, \ldots, n-1), \\ 0<\psi_i(x)&<1, \quad (x_i<x<x_{i+1}), \\ \psi_i(x)&=0, \quad (x_{i+1} \leqq x \leqq b). \end{aligned}$$

Then the surfaces $S_1, S_2, \ldots, S_{n-1}$ will be defined by the equations

$$S_i: \qquad z=z(x, y)+\psi_i(x)\eta(x, y, \alpha), \qquad (i=1, 2, \ldots, n-1),$$

and for convenience in notation let $S_0=S$ and $S_n=S_h$. It follows from the definition of the derivative of a surface that

$$L'(S_{i-1}; x_i, y_0) = \lim_{\substack{h,\,\alpha=0 \\ n=\infty}} \frac{L(S_i)-L(S_{i-1})}{\int_{x_{i-1}}^{x_{i+1}}\int_{y_0-h}^{y_0+h}[\psi_i(x)-\psi_{i-1}(x)]\eta(x, y, \alpha)\,dydx}.$$

Since this derivative is continuous and approached uniformly, the equation

$$\frac{L(S_h)-L(S)}{\alpha} = \sum_{i=1}^{n}[L'(S; x_i, y_0)+\xi_i]\int_{x_{i-1}}^{x_{i+1}}\int_{y_0-h}^{y_0+h}[\psi_i(x)-\psi_{i-1}(x)]\,\frac{\eta}{\alpha}\,dydx$$

is satisfied, where the upper bound of $|\xi_i|$ approaches zero with h and α. By definition $|\psi_i(x)-\psi_{i-1}(x)| \leqq 1$, and for sufficiently small values of h and α

$$|L'(S; x_i, y_0)+\xi_i| < M, \qquad (i=1, 2, \ldots, n),$$

and

$$\left|\frac{\eta(x, y, \alpha)}{\alpha}\right| < N.$$

The sum of the areas over which the integration takes place is $4h(b-a)$. If their upper bounds are substituted for the respective functions in the last equation, it establishes the inequality (6).

If the equation of S_h is now taken to be

$$z=z(x, y)+\sum_{i=1}^{n}\theta_i(x)\omega(x, y, \alpha),$$

and the surface S_{nn} is taken instead of S, the function $\eta(x, y, \alpha)$ becomes

$$\eta(x, y, \alpha) = \left[1 - \sum_{j=1}^{n} \phi_j(y)\right] \sum_{i=1}^{n} \theta_i(x) \omega(x, y, \alpha),$$

which vanishes excepting in the $n+1$ rectangles $(a<x<b;\ y_j-h<y<y_j+h)$, $(j=0, 1, \ldots, n)$. Since the rest of the hypothesis of the lemma is satisfied, if it is applied $n+1$ times it will give the inequality

$$\left| \frac{L(S_h)-L(S_{nn})}{\alpha} \right| < 4h(n+1)MN(b-a). \tag{7}$$

If the surface S_α is taken instead of S, the function $\eta(x, y, \alpha)$ will become

$$\left[\sum_{i=1}^{n} \theta_i(x) - 1\right] \omega(x, y, \alpha),$$

which vanishes excepting in the rectangles $(x_i-h<x<x_i+h;\ c<y<d)$, $(i=0, 1, 2, \ldots, n)$. Since the proof of the lemma is not affected by interchanging x and y and replacing a and b by c and d, the inequality

$$\left| \frac{L(S_h)-L(S_\alpha)}{\alpha} \right| < 4h(n+1)MN(d-c) \tag{8}$$

is also satisfied for small values of h and $|\alpha|$. Inequalities (7) and (8) can be combined, giving the inequality

$$\left| \frac{L(S_\alpha)-L(S_{nn})}{\alpha} \right| < 4h(n+1)MN(b-a+d-c) < 4MN\varepsilon + 4hMN(b-a+d-c).$$

Since the right member of the last inequality approaches zero with ε, equation (4) may be replaced by the equation

$$\left. \frac{dL(S_\alpha)}{d\alpha} \right|_{\alpha=0} = \iint_R L'(S;\, x, y) \omega_\alpha(x, y, 0)\, dy\, dx. \tag{9}$$

The assumption that the region R be a rectangle is not essential. If it is any region bounded by a curve of finite length, a suitable rectangle R' can be circumscribed about it, and the function $z(x, y)$ defined arbitrarily and $\omega(x, y, \alpha)$ taken as identically zero over the part of R' not included in R. If the function $L(S)$ is defined in such a way that it depends for its value only on that part of S whose projection on the x, y-plane is R, the derivative $L'(S;\, x, y)$ will vanish everywhere outside of R. The fact that it may be discontinuous at the boundary of R will have no effect on the proof of the above theorem. The theorem states that equation (9) would hold with R' substituted for R, and therefore it holds as it stands.

It will now be proved that if the surface S and the function $L(S)$ satisfy the hypothesis of the last theorem and the surface S minimizes the function $L(S)$, then the derivative $L'(S;\, x, y)$ must vanish at every point of the region

R. Suppose there is a point (x_0, y_0) where the derivative is positive. Then, since it is continuous, there is a finite region including the point (x_0, y_0) where it is also positive. The function $\omega(x, y, \alpha,)$ can be chosen in such a way that its partial derivative $\omega_\alpha(x, y, 0)$ will be positive in this region and zero everywhere else. Then equation (9) will imply that

$$\left.\frac{dL(S_\alpha)}{d\alpha}\right|_{\alpha=0} > 0,$$

and the surface S will not minimize the function $L(S)$. If there is a point where the derivative is negative, the argument is essentially the same.

§ 2. *The Derivative of a Double Integral.*

The function $L(S)$ will now be taken as a double integral, and the first necessary condition for a minimum as proved by Lagrange* will be found to be a special case of the last theorem of the preceding section. The arbitrary integer r which designates the order of the approach to the derivative will be taken as 2 in this section.

Let

$$L(S) = \int\int_R f(x, y, z, p, q)\,dydx,$$

where $f(x, y, z, p, q)$ is of class $C^{(2)}$ in the neighborhood of values of the arguments determined by the surface S. The letters p and q represent the partial derivatives of z with respect to x and y. Choose a function $\eta(x, y)$ as in § 1. Then, by definition,

$$L(S_\epsilon) - L(S) = \int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}[f(x, y, z+\eta, p+\eta_x, q+\eta_y) - f(x, y, z, p, q)]\,dydx. \quad (10)$$

If the integrand of this expression is expanded by Taylor's formula, it becomes

$$f_z(x, y, z+\theta\eta, p+\theta\eta_x, q+\theta\eta_y)\eta + f_p\eta_x + f_q\eta_y,$$

where θ is a constant between 0 and 1. It follows from Green's theorem,† and the vanishing of $\eta(x, y)$ along L, the boundary of the region over which the integral is taken, that the equation

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\frac{\partial(f_p\eta)}{\partial x}\,dydx = \int_L f_p\eta\,dy = 0$$

is satisfied, and consequently the equation

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon} f_p\eta_x\,dydx = -\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\frac{\partial f_p}{\partial x}\eta\,dydx. \quad (11)$$

Similarly

* Bolza, *loc. cit.*, p. 655.

† Bolza, *loc. cit.*, p. 654.

$$\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon} f_q\eta_y dy dx = -\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon} \frac{\partial f_q}{\partial y}\eta dy dx. \tag{12}$$

If equations (11) and (12) are substituted in the expanded form of equation (10), it becomes

$$L(S_\epsilon)-L(S)=\int_{x_0-\epsilon}^{x_0+\epsilon}\int_{y_0-\epsilon}^{y_0+\epsilon}\left[f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}\right]\eta dy dx.$$

Since $\eta(x, y)$ has a permanent sign, the mean-value theorem can be applied to the right member of this equation, reducing it to the form

$$L(S_\epsilon)-L(S)=\left[f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}\right]\sigma,$$

where the arguments of the partial derivatives of f are $x_0+\lambda\varepsilon$, $y_0+\mu\varepsilon$, $z(x_0+\lambda\varepsilon, y_0+\mu\varepsilon)+\theta\eta(x_0+\lambda\varepsilon, y_0+\mu\varepsilon)$, $p+\theta\eta_x$, and $q+\theta\eta_y$, and σ is defined by equation (1). The value of the derivative is then seen to be

$$L'(S; x_0, y_0)=f_z(x_0, y_0, z(x_0, y_0), p(x_0, y_0), q(x_0, y_0))-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}.$$

It follows from the assumed continuity of the functions $z(x, y)$ and $f(x, y, z, p, q)$ that $L'(S; x, y)$ is continuous and approached uniformly with order 2. Therefore the Lagrange equation,

$$f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}=0,$$

is equivalent to the equation

$$L'(S; x, y)=0.$$

§ 3. *The Set of Surfaces K.*

In this section there will be given m functions $M_1(S), M_2(S), \ldots, M_m(S)$ whose derivatives are continuous and approached uniformly with order r in a neighborhood of order r of the surface S. It will be assumed that there are some points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ in the region R which satisfy the inequality

$$\begin{vmatrix} M'_j(S; x_i, y_i) \\ (i, j=1, 2, \ldots, m) \end{vmatrix} \neq 0. \tag{13}$$

A surface $\bar{S}$ will be said to belong to the set K if it satisfies the equations

$$M_j(\bar{S})-M_j(S)=0, \qquad (j=1, 2, \ldots, m).$$

It will now be proved that for every $\varepsilon>0$ there exists a $\delta>0$, which will always be taken as less than or equal to ε, for which the following statement is true: If $\eta(x, y)$ is any function all of whose partial derivatives up to and

including those of order r are everywhere less than δ in absolute value, and vanish along the boundary of R, then there is a set of functions $\eta_1(x, y)$, $\eta_2(x, y), \ldots, \eta_m(x, y)$ all of whose partial derivatives, up to and including those of order r, are everywhere less than ε in absolute value, such that $\eta_i(x, y)$ vanishes identically excepting in the region $(x_i-\varepsilon<x<x_i+\varepsilon;\ y_i-\varepsilon<y<y_i+\varepsilon)$, where it has a permanent sign, and the surface

$$S_m: \qquad z=z(x, y)+\eta(x, y)+\sum_{i=1}^{m}\eta_i(x, y) \tag{14}$$

belongs to the set K.

A set of functions $\eta_i(x, y)$ can readily be found which vanish excepting in the specified regions where they have a permanent sign, and whose derivatives are numerically less than ε. Then it must merely be proved that there is a set of multipliers $\alpha_1, \alpha_2, \ldots, \alpha_m$, each numerically less than unity, such that the surface

$$S'_m: \qquad z=z(x, y)+\sum_{i=1}^{m}\alpha_i\eta_i(x, y) \tag{15}$$

belongs to the set K. Then the multipliers α_i can be absorbed into the functions $\eta_i(x, y)$ and the surface S'_m written S_m.

The determinant

$$\left|\begin{matrix} \int_{x_i-\varepsilon}^{x_i+\varepsilon}\int_{y_i-\varepsilon}^{y_i+\varepsilon} M'_j(S;\ x, y)\eta_i(x, y)\,dy\,dx \\ (i, j=1, 2, \ldots, m) \end{matrix}\right| \tag{16}$$

must be considered. If the mean-value theorem is applied to each element, it can be reduced to the form

$$\prod_{i=1}^{m}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\int_{y_i-\varepsilon}^{y_i+\varepsilon}\eta_i(x, y)\,dy\,dx \left|\begin{matrix} M'_j(S;\ x_{ij}, y_{ij}) \\ (i, j=1, 2, \ldots, m) \end{matrix}\right|,$$

where (x_{ij}, y_{ij}) is some point in the proper region. It follows from inequality (13) and the assumed continuity of the derivatives $M'_j(S;\ x, y)$ that for sufficiently small values of ε this determinant can not vanish, and only such values of ε need be considered.

If ε and the functions $\eta_i(x, y)$ are considered as fixed, the expressions $M_j(S'_m)-M_j(S)$ are completely determined by the function $\eta(x, y)$ and the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$, and may be represented by the equations

$$M_j(S'_m)-M_j(S)=\phi_j(H;\ \alpha_1, \alpha_2, \ldots, \alpha_m), \qquad (j=1, 2, \ldots, m),$$

where H represents the surface in the x, y, η-space whose equation is

$\eta=\eta(x, y)$. The surface whose equation is $\eta=0$ will be called H_0. The following equations come directly from the principal theorem of § 1:

$$\frac{\partial\phi_j(H; \alpha_1, \alpha_2, \ldots, \alpha_m)}{\partial\alpha_i}=\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}M'_j(S'_m; x, y)\eta_i(x, y)\,dydx, \qquad (i, j=1, 2, \ldots, m); \tag{17}$$

and the determinant

$$\left|\begin{matrix}\dfrac{\partial\phi_j(H_0; 0, 0, \ldots, 0)}{\partial\alpha_i}\\ (i, j=1, 2, \ldots, m)\end{matrix}\right|$$

is equal to the determinant (16), which can not vanish. The functions $\phi_j(H, \alpha_1, \alpha_2, \ldots, \alpha_m)$ and their partial derivatives given by equations (17) are seen to be continuous in all arguments in the neighborhood of $(H_0; 0, 0, \ldots, 0)$. Thus the hypothesis of an existence theorem in the paper entitled "A Generalization of Volterra's Derivative of a Function of a Curve,"* mentioned in the introduction, is satisfied. The fact that H now represents a surface instead of a curve has no effect on the proof of this theorem. The conclusion of this existence theorem is that for every $\epsilon>0$ sufficiently small there is a $\delta>0$ such that to every admissible function $\eta(x, y)$ there corresponds a unique set of constants $\alpha_1, \alpha_2, \ldots, \alpha_m$, each numerically less than unity, which satisfy the equations

$$\phi_j(H; \alpha_1, \alpha_2, \ldots, \alpha_m)=0, \qquad (j=1, 2, \ldots, m).$$

The surface S'_m then belongs to the set K. It is evident that if a δ is effective for a particular value of ϵ, as ϵ_0, it is also effective for every value of ϵ greater than ϵ_0, since the functions $\eta_i(x, y)$ can be taken identically zero over as much of the regions $(x_i-\epsilon<x<x_i+\epsilon;\ y_i-\epsilon<y<y_i+\epsilon)$ as is desired, provided they are different from zero in a part of each.

§ 4. *The Derivative Relative to the Set K.*

The derivative $L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)$ of the function $L(S)$ relative to the set of surfaces K can now be defined, and evaluated in terms of the derivatives $L'(S; x, y)$, $M'_1(S; x, y)$, $\ldots$, $M'_m(S; x, y)$, if they are continuous and approached unformly with any finite order r.

The function $\eta(x, y)$ discussed in the preceding section will be restricted by the additional condition that it vanish identically excepting in the region $(x_0-\epsilon<x<x_0+\epsilon;\ y_0-\epsilon<y<y_0+\epsilon)$, where it has a permanent sign and is not identically zero. *Then the limit*

* Fischer, *loc. cit.*, p. 373.

$$L'(S;\, x_0,\, y_0;\, x_1,\, y_1,\, \ldots,\, x_m,\, y_m) = \lim_{\epsilon=0} \frac{L(S_m)-L(S)}{\sigma}$$

will be called the derivative of the function $L(S)$ at the point (x_0, y_0), relative to the set of surfaces K and the points (x_1, y_1), (x_2, y_2), ..., (x_m, y_m). The integral σ and the surface S_m have already been defined by equations (1) and (14).

This derivative can be evaluated as follows: Define the surfaces S_0, S_1, ..., S_{m-1} by the equations

$$S_0: \qquad z = z(x, y) + \eta(x, y),$$

$$S_i: \qquad z = z(x, y) + \eta(x, y) + \sum_{k=1}^{i} \eta_k(x, y), \qquad (i = 1, 2, \ldots, m-1);$$

and for convenience in the notation let

$$\sigma_i = \int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon} \eta_i(x, y)\, dy\, dx, \qquad (i = 1, 2, \ldots, m).$$

Then the derivative may be written in the form

$$L'(S;\, x_0,\, y_0;\, x_1,\, y_1,\, \ldots,\, x_m,\, y_m) = \lim_{\epsilon=0}\left[\frac{L(S_0)-L(S)}{\sigma} + \sum_{i=1}^{m} \frac{L(S_i)-L(S_{i-1})}{\sigma_i}\cdot\frac{\sigma_i}{\sigma}\right],$$

which is easily reduced to the equation

$$L'(S;\, x_0,\, y_0;\, x_1,\, y_1,\, \ldots,\, x_m,\, y_m) = L'(S;\, x_0,\, y_0) + \sum_{i=1}^{m} L'(S;\, x_i,\, y_i) \lim_{\epsilon=0} \frac{\sigma_i}{\sigma}. \tag{18}$$

It is now necessary to evaluate the expressions

$$\lim_{\epsilon=0} \frac{\sigma_i}{\sigma}.$$

Since S_m belongs to the set K, the equations

$$\frac{M_j(S_m)-M_j(S)}{\sigma} = 0, \qquad (j = 1, 2, \ldots, m),$$

are always satisfied. If ϵ is made to approach zero, they become

$$M'_j(S;\, x_0,\, y_0) + \sum_{i=1}^{m} M'_j(S;\, x_i,\, y_i) \lim_{\epsilon=0} \frac{\sigma_i}{\sigma} = 0, \qquad (j = 1, 2, \ldots, m).$$

The determinant of the coefficients of $\lim \sigma_i/\sigma$ is not zero, and therefore the solution is unique. Solving for these limits and substituting their values in equation (18), it becomes

$$L'(S; x_0, y_0; x_1, y_1, \ldots, x_m, y_m) = L'(S; x_0, y_0) - \sum_{k=1}^{m} L'(S; x_k, y_k) \frac{\begin{vmatrix} M'_1(S; x_1, y_1) \,.\,.\, M'_1(S; x_{k-1}, y_{k-1}) M'_1(S; x_0, y_0) \,.\,.\, M'_1(S; x_m, y_m) \\ M'_2(S; x_1, y_1) \,.\,.\, M'_2(S; x_{k-1}, y_{k-1}) M'_2(S; x_0, y_0) \,.\,.\, M'_2(S; x_m, y_m) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M'_m(S; x_1, y_1) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots M'_m(S; x_m, y_m) \end{vmatrix}}{\begin{vmatrix} M'_j(S; x_i, y_i) \\ (i, j = 1, 2, \ldots, m) \end{vmatrix}}.$$

This may be written either in the form

$$L'(S; x_0, y_0; x_1, y_1, \ldots, x_m, y_m) = L'(S; x_0, y_0) + \sum_{k=1}^{m} \lambda_k M'_k(S; x_0, y_0), \tag{19}$$

where the quantities $\lambda_1, \lambda_2, \ldots, \lambda_m$ are defined by the equations

$$\lambda_k = (-1)^k \frac{\begin{vmatrix} L'(S; x_1, y_1) & L'(S; x_2, y_2) \ldots & L'(S; x_m, y_m) \\ M'_1(S; x_1, y_1) & M'_1(S; x_2, y_2) \ldots & M'_1(S; x_m, y_m) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ M'_{k-1}(S; x_1, y_1) & M'_{k-1}(S; x_2, y_2) \ldots & M'_{k-1}(S; x_m, y_m) \\ M'_{k+1}(S; x_1, y_1) & M'_{k+1}(S; x_2, y_2) \ldots & M'_{k+1}(S; x_m, y_m) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ M'_m(S; x_1, y_1) & M'_m(S; x_2, y_2) \ldots & M'_m(S; x_m, y_m) \end{vmatrix}}{\begin{vmatrix} M'_j(S; x_i, y_i) \\ (i, j = 1, 2, \ldots, m) \end{vmatrix}}, \tag{20}$$

or in the form

$$\begin{vmatrix} M'_j(S; x_i, y_i) \\ (i, j = 1, 2, \ldots, m) \end{vmatrix} L'(S; x_0, y_0; x_1, y_1, \ldots, x_m, y_m) = \begin{vmatrix} L'(S; x_0, y_0) & L'(S; x_1, y_1) \ldots & L'(S; x_m, y_m) \\ M'_1(S; x_0, y_0) & M'_1(S; x_1, y_1) \ldots & M'_1(S; x_m, y_m) \\ M'_2(S; x_0, y_0) & M'_2(S; x_1, y_1) \ldots & M'_2(S; x_m, y_m) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ M'_m(S; x_0, y_0) & M'_m(S; x_1, y_1) \ldots & M'_m(S; x_m, y_m) \end{vmatrix}. \tag{21}$$

Since $L'(S; x, y), M'_1(S; x, y), \ldots, M'_m(S; x, y)$ are continuous and approached uniformly with order r, $L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)$ is also.

The following equations, which will be used later, follow immediately from equation (21):

$$L'(S; x_i, y_i; x_1, y_1, \ldots, x_m, y_m) = 0, \qquad (i = 1, 2, \ldots, m).$$

The extension of Volterra's theorem proved in § 1 can readily be proved for the case where only surfaces of the set K are considered. Suppose that the one-parameter family of surfaces

$$S_\alpha: \qquad z=z(x, y)+\omega(x, y, \alpha)$$

discussed in § 1 belongs to the set K. Then it follows from equation (9) that

$$\left.\frac{dL(S_\alpha)}{d\alpha}\right|_{\alpha=0}=\int\int_R L'(S;\, x, y)\omega_\alpha(x, y, 0)\,dydx, \tag{22}$$

and similarly

$$\left.\frac{dM_j(S_\alpha)}{d\alpha}\right|_{\alpha=0}=\int\int_R M'_j(S;\, x, y)\omega_\alpha(x, y, 0)\,dydx, \qquad (j=1, 2, \ldots, m). \tag{23}$$

The left members of equation (23) must vanish since S_α belongs to the set K. If the right members are multiplied by $\lambda_1, \lambda_2, \ldots, \lambda m$ respectively, equated to zero, and added to the right member of equation (22), it becomes

$$\left.\frac{dL(S_\alpha)}{d\alpha}\right|_{\alpha=0}=\int\int_R L'(S;\, x, y;\, x_1, y_1, \ldots, x_m, y_m)\omega_\alpha(x, y, 0)\,dydx. \tag{24}$$

It will now be proved that if the surface S minimizes the function $L(S)$ relative to the set of surfaces K, then the derivative $L'(S;\, x, y;\, x_1, y_1, \ldots, x_m, y_m)$ must vanish for all values of x and y in the region R.

Suppose that there is a point (x_0, y_0) where it does not vanish, and to fix ideas, suppose that it is positive at (x_0, y_0). Then, since the derivative is continuous, two positive constants h and k can be found such that the inequality

$$L'(S;\, x, y;\, x_1, y_1, \ldots, x_m, y_m)>k \tag{25}$$

is satisfied at every point in the region $(x_0-h\leqq x\leqq x_0+h;\, y_0-h\leqq y\leqq y_0+h)$. Since the derivative vanishes at each of the points (x_1, y_1), $(x_2, y_2,)$, $\ldots$, (x_m, y_m), for every $\delta>0$ there exists an $\varepsilon>0$ such that the inequality

$$|L'(S;\, x, y;\, x_1, y_1, \ldots, x_m, y_m)|<\delta \tag{26}$$

is satisfied at every point in each of the regions

$$(x_i-\varepsilon\leqq x\leqq x_i+\varepsilon;\, y_i-\varepsilon\leqq y\leqq y_i+\varepsilon), \qquad (i=1, 2, \ldots, m).$$

Select a function $\eta(x, y)$ which vanishes at all points outside of the region $(x_0-h<x<x_0+h;\; y_0-h<y<y_0+h)$, where it is positive, and m functions $\eta_i(x, y)$ which vanish at all points outside of the respective regions $(x_i-\varepsilon<x<x_i+\varepsilon;\, y_i-\varepsilon<y<y_i+\varepsilon)$, where they are positive. Then it follows from the proof of the existence theorem of § 3, that if ε is taken sufficiently small, for every value of α numerically less than a fixed positive number, there exists a unique set of numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, each numerically less than another fixed number, such that the surface

$$S_\alpha: \qquad z=z(x, y)+\alpha\eta(x, y)+\sum_{i=1}^{m}\alpha_i\eta_i(x, y)$$

belongs to the set K. The function $\omega(x, y, \alpha)$ will now be defined by the equation

$$\omega(x, y, \alpha) = \alpha\eta(x, y) + \sum_{i=1}^{m} \alpha_i \eta_i(x, y). \tag{27}$$

As the existence of the partial derivative $\omega_\alpha(x, y, 0)$ can not be easily proved, the assumption that it exists will be replaced by the less exacting assumption that there is a constant N, such that the inequality

$$\left| \frac{\omega(x, y, \alpha)}{\alpha} \right| < N \tag{28}$$

is satisfied, for all numerically small values of α. This necessitates some changes in the statement and proof of the extension of Volterra's theorem. Equation (4) must be replaced by the equation

$$\lim_{\epsilon,\, \alpha=0} \left[\frac{L(S_{nn}) - L(S)}{\alpha} - \int_a^b \int_c^d L'(S; x, y) \frac{\omega(x, y, \alpha)}{\alpha} \, dy dx \right] = 0,$$

and equation (9) by the equation

$$\lim_{\alpha=0} \left[\frac{L(S_\alpha) - L(S)}{\alpha} - \iint_R L'(S; x, y) \frac{\omega(x, y, \alpha)}{\alpha} \, dy dx \right] = 0.$$

The proof of the theorem will then be valid. In the present section equations (22) and (23) must be replaced by the equations

$$\lim_{\alpha=0} \left[\frac{L(S_\alpha) - L(S)}{\alpha} - \iint_R L'(S; x, y) \frac{\omega(x, y, \alpha)}{\alpha} \, dy dx \right] = 0, \tag{29}$$

and

$$\lim_{\alpha=0} \left[\frac{M_j(S_\alpha) - M_j(S)}{\alpha} - \iint_R M'_j(S; x, y) \frac{\omega(x, y, \alpha)}{\alpha} \, dy dx \right] = 0, \tag{30}$$

$$(j = 1, 2, \ldots, m).$$

Since the surface S_α belongs to the set K, equations (30) may be written

$$\lim_{\alpha=0} \iint_R M'_j(S; x, y) \frac{\omega(x, y, \alpha)}{\alpha} \, dy dx = 0, \qquad (j = 1, 2, \ldots, m). \tag{31}$$

If equations (31) are multiplied by $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively, and subtracted from equation (29), it becomes by virtue of equation (19)

$$\lim_{\alpha=0} \left[\frac{L(S_\alpha) - L(S)}{\alpha} \right.$$
$$\left. - \iint_R L'(S; x, y; x_1, y_1, \ldots, x_m, y_m) \frac{\omega(x, y, \alpha)}{\alpha} \, dy dx \right] = 0, \tag{32}$$

which is the desired generalization of equation (24).

It will next be provėd that if the function $\omega(x, y, \alpha)$ is defined by equation (27) a constant N must exist, satisfying inequality (28). Let

$$S_0: \qquad z=z(x, y)+\alpha\eta(x, y),$$
$$S_i: \qquad z=z(x, y)+\alpha\eta(x, y)+\sum_{k=1}^{i}\alpha_k\eta_k(x, y), \qquad (i=1, 2, \ldots, m).$$

Then, since S_m, which is the same as S_α, belongs to the set K, the equations

$$\frac{M_j(S_m)-M_j(S)}{\alpha}=\frac{M_j(S_0)-M_j(S)}{\alpha}+\sum_{i=1}^{m}\frac{M_j(S_i)-M_j(S_{i-1})}{\sigma_i}\cdot\frac{\sigma_i}{\alpha}=0$$

are satisfied, where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are defined by the equations

$$\sigma_i=\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}\alpha_i\eta_i(x, y)\,dydx.$$

When ϵ and α both approach zero, the determinant of the coefficients of σ_i/α approaches the determinant (13) uniformly, and therefore it can not vanish if $|\alpha|$ and ϵ are sufficiently small. The terms $1/\alpha[M_j(S_0)-M_j(S)]$ approach finite limits as α approaches zero. Thus a constant N', independent of ϵ and α, can be found, such that, if ϵ is given any value less than a fixed positive constant and then α is taken less numerically than another constant, then the inequalities $|\sigma_i/\alpha|<N'$ will be satisfied. Since the expressions

$$\frac{\sigma_i}{\alpha_i}=\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}\eta_i(x, y)\,dydx$$

are positive and independent of α, the ratios α_i/α do not become infinite as α approaches zero; and if $\omega(x, y, \alpha)$ is defined by equation (27), a constant N can be found satisfying equation (28). Since N is not independent of ϵ, it will be better to use N' in what follows.

Equation (32) may now be written

$$\frac{L(S_\alpha)-L(S)}{\alpha}=\int_{x_0-h}^{x_0+h}\int_{y_0-h}^{y_0+h}L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)\eta\,dydx$$
$$+\sum_{i=1}^{m}\int_{x_i-\epsilon}^{x_i+\epsilon}\int_{y_i-\epsilon}^{y_i+\epsilon}L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)\frac{\alpha_i\eta_i}{\alpha}\,dydx+\xi(\alpha),$$

where $\xi(\alpha)$ approaches zero with α. It follows from this equation and inequalities (25) and (26) that the inequality

$$\frac{L(S_\alpha)-L(S)}{\alpha}>k\int_{x_0-h}^{x_0+h}\int_{y_0-h}^{y_0+h}\eta\,dydx-mN'\delta-|\xi(\alpha)|$$

is satisfied. Since the first term of the right member is positive and independent of δ, ε, and α, if δ is taken sufficiently small the inequality

$$\frac{L(S_\alpha)-L(S)}{\alpha}>0$$

is satisfied for small values of α both positive and negative. Therefore the surface S does not minimize the function $L(S)$. If $L'(S; x_0, y_0; x_1, y_1, \ldots, x_m, y_m)$ were negative, the argument would be essentially the same.

The quantities $\lambda_1, \lambda_2, \ldots, \lambda_m$ defined by equations (20) are functions of the $2m$ arguments $x_1, y_1, \ldots, x_m, y_m$; but in the special case where the derivative vanishes identically, they are constants, as can be proved as follows: Let $x'_1, y'_1, \ldots, x'_m, y'_m$ be a fixed set of values for $x_1, y_1, \ldots, x_m, y_m$, and let those letters without accents be considered as variables. Then the equations

$$L'(S; x'_i, y'_i; x_1, y_1, \ldots, x_m, y_m)=$$
$$L'(S; x'_i, y'_i)+\sum_{j=1}^{m}\lambda_j(x_1, y_1, \ldots, x_m, y_m)M'_j(S; x'_i, y'_i)=0,$$
$$(i=1, 2, \ldots, m),$$

are satisfied identically. Comparing the solutions of these equations with equations (20), it is evident that

$$\lambda_j(x_1, y_1, \ldots, x_m, y_m)=\lambda_j(x'_1, y'_1, \ldots, x'_m, y'_m), \qquad (j=1, 2, \ldots, m).$$

That is, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are constants.

§ 5. *The Derivative of a Double Integral Relative to the Set K.*

If the functions $L(S), M_1(S), \ldots, M_m(S)$ are defined by the equations

$$L(S)=\iint_R f(x, y, z, p, q)\,dydx, \tag{33}$$

$$M_j(S)=\iint_R g_j(x, y, z, p, q)\,dydx, \qquad (j=1, 2, \ldots, m), \tag{34}$$

where $z(x, y)$, $f(x, y, z, p, q)$ and $g_j(x, y, z, p, q)$ are of class $C^{(2)}$, and the determinant (13) is distinct from zero, the derivative of $L(S)$ with respect to the restricted set of surfaces is given by the equation

$$L'(S; x, y; x_1, y_1, \ldots, x_m, y_m)=f_z-\frac{\partial f_p}{\partial x}-\frac{\partial f_q}{\partial y}+\sum_{j=1}^{m}\lambda_j\left[g_{jz}-\frac{\partial g_{jp}}{\partial x}-\frac{\partial g_{jq}}{\partial y}\right], \tag{35}$$

where the arguments of the partial derivatives of f and g_j are x, y, $z(x, y)$, $p(x, y)$, $q(x, y)$, and the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ are defined by the equations

$$\lambda_k=(-1)^k\frac{\begin{vmatrix} \left[f_z-\dfrac{\partial f_p}{\partial x}-\dfrac{\partial f_q}{\partial y}\right]_{x_1,\,y_1} & \cdots\cdots\cdots & \left[f_z-\dfrac{\partial f_p}{\partial x}-\dfrac{\partial f_q}{\partial y}\right]_{x_m,\,y_m} \\ \left[g_{1z}-\dfrac{\partial g_{1p}}{\partial x}-\dfrac{\partial g_{1q}}{\partial y}\right]_{x_1,\,y_1} & \cdots\cdots\cdots & \left[g_{1z}-\dfrac{\partial g_{1p}}{\partial x}-\dfrac{\partial g_{1q}}{\partial y}\right]_{x_m,\,y_m} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \left[g_{k-1z}-\dfrac{\partial g_{k-1p}}{\partial x}-\dfrac{\partial g_{k-1q}}{\partial y}\right]_{x_1,\,y_1} & \cdots & \left[g_{k-1z}-\dfrac{\partial g_{k-1p}}{\partial x}-\dfrac{\partial g_{k-1q}}{\partial y}\right]_{x_m,\,y_m} \\ \left[g_{k+1z}-\dfrac{\partial g_{k+1p}}{\partial x}-\dfrac{\partial g_{k+1q}}{\partial y}\right]_{x_1,\,y_1} & \cdots & \left[g_{k+1z}-\dfrac{\partial g_{k+1p}}{\partial x}-\dfrac{\partial g_{k+1q}}{\partial y}\right]_{x_m,\,y_m} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \left[g_{mz}-\dfrac{\partial g_{mp}}{\partial x}-\dfrac{\partial g_{mq}}{\partial y}\right]_{x_1,\,y_1} & \cdots\cdots\cdots & \left[g_{mz}-\dfrac{\partial g_{mp}}{\partial x}-\dfrac{\partial g_{mq}}{\partial y}\right]_{x_m,\,y_m} \end{vmatrix}}{\begin{vmatrix} \left[g_{jz}-\dfrac{\partial g_{jp}}{\partial x}-\dfrac{\partial g_{jq}}{\partial y}\right]_{x_i,\,y_i} \\ (i,\,j=1,\,2,\,\ldots.,\,m) \end{vmatrix}}.$$

The right member of equation (35) equated to zero constitutes the first necessary condition that the surface S minimize the double integral (33) relative to the surfaces which give fixed values to the integrals (34).

COLUMBIA UNIVERSITY, *April*, 1913.

Some Invariants and Covariants of Ternary Collineations.

By Henry Bayard Phillips.

Introduction.

1. The analytical basis of the present paper is the form of Grassmann's Lückenausdruck which Gibbs called a *dyadic*. This, as the sequel shows, is merely a bilinear form from which the variables are omitted. It can then represent a collineation or correlation, and may be manipulated practically like an ordinary symbolical bilinear form.

Using this as a basis, the object of the paper is in the first place to give an interpretation of the multiple products defined by Gibbs, and to obtain some of the properties of the invariants and covariants involved. The field of operation is plane projective geometry, and the products are formed according to the outer multiplication of Grassmann.

Finally, in the third part there is considered a skew symmetric function of any number of collineations which is called an *alternant*. It is a combinant, linear in the coefficients of each collineation, and presenting in some ways, for forms in two sets of variables, properties analogous to those of the expressions resulting from the outer multiplication of linear manifolds.

Part I. Notation.

I. *The Open Product or Dyadic.*

2. In a space of two dimensions a sum of products of similar construction, each containing a factor X, can be written in the form

$$(A_1 X)\, B_1 + (A_2 X)\, B_2 + (A_3 X)\, B_3,$$

where the parentheses are used merely to show the order of multiplication. A_i, B_i and X are geometric quantities, points or lines of the plane, and all products are formed according to the outer multiplication. This can be considered as resulting from the operation of X on the expression

$$A_1 (\ )\, B_1 + A_2 (\ )\, B_2 + A_3 (\ )\, B_3,$$

the operation consisting in placing the variable X in the parentheses. This last expression is an example of what Grassmann called an *open* product.*

* "Ausdehnungslehre" (1878), p. 265.

Gibbs wrote the open product in the form

$$A_1 B_1 + A_2 B_2 + A_3 B_3,$$

and from the nature of its construction called it a *dyadic.** The variable is supposed to operate on the dyadic from the outside, and so give as result

$$(X A_1) B_1 + (X A_2) B_2 + (X A_3) B_3,$$

or

$$A_1 (B_1 X) + A_2 (B_2 X) + A_3 (B_3 X),$$

according as X is used as prefactor or as postfactor.

In the present paper the notation of Gibbs is used, and outer products are represented either by placing the letters in parentheses or by placing a bar over them. It is found convenient to use the parentheses when the product reduces to a scalar, or number, and in all other cases to use the bar. Unless otherwise expressly stated, the variable is supposed to enter as postfactor; *i. e.*, the dyadic operates on the variable. From analogy with the ordinary symbolism for a row product we write

$$AB = A_1 B_1 + A_2 B_2 + A_3 B_3.$$

It is to be observed that A_i and B_i in this expression have definite sizes or intensities. If they are only projectively given, the dyadic has the form

$$AB = \lambda_1 A_1 B_1 + \lambda_2 A_2 B_3 + \lambda_3 A_3 B_3,$$

where the λ's are numbers determined when definite intensities are given to A_i and B_i.

3. As an operator the dyadic gives a linear transformation of quantities contragredient to B_i. For, X being such a quantity, since $(B_i X)$ is a number,

$$A(BX) = \lambda_1 (B_1 X) A_1 + \lambda_2 (B_2 X) A_2 + \lambda_3 (B_3 X) A_3,$$

which is a simple quantity of the same dimension as A_i.

There are two cases of present interest. When A_i and B_i are contragredient, we have a collineation; when cogredient, a correlation.

A dyadic of the form

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3,$$

where the a's are points and the α's lines,† represents a point collineation.‡

† Gibbs's "Vector Analysis" (Wilson), Chap. V.

* We shall use the letters a, b, c, x, y to represent points; $\alpha, \beta, \gamma, \xi, \eta$ to represent lines; and the large capitals A, B, C, X in general discussions to represent either points or lines. The letters λ, μ, ν, ρ will be used to represent abstract numbers.

† In the notation of Clebsch this is of course

$$(a\xi)(ax) = \Sigma \lambda_i (a_i \xi)(a_i x) = 0,$$

where x is given and ξ variable. If ξ were given and x variable, we should write the dyadic $a\alpha$. We thus regard the dyadic not as giving a connex but as determining a definite transformation.

In particular, to the point $\overline{\alpha_2 \alpha_3}$ corresponds the point

$$a(\alpha\alpha_2\alpha_3) = \lambda_1(\alpha_1\alpha_2\alpha_3)\, a_1.$$

To the vertices of the triangle α then correspond the points a_i. Since the lines of the plane can be expressed as linear functions of any three not passing through a point, we can assume the α's to be any three linearly independent lines of the plane, the collineation then determining the corresponding points a_i.

Similarly, the dyadic

$$\alpha\beta = \lambda_1\alpha_1\beta_1 + \lambda_2\alpha_2\beta_2 + \lambda_3\alpha_3\beta_3$$

represents a correlation in which the lines α_i correspond to the points of the triangle β. A similar interpretation can be given to the dual forms αa and ab.

II. *Tetrad and Counter-tetrad.*

4. A collineation or correlation is determined by four sets of corresponding elements. It is sometimes useful to have the dyadic expressed in terms of these corresponding elements. Since the postfactors in the dyadic are complementary in dimension to the variable operated on, we must associate with a set of four points a set of four lines to be used as postfactors. For this purpose we associate with a 4-point the set of four lines obtained by taking the polar of each point with respect to the triangle of the other three. It is well known, then, that conversely from this 4-line the 4-point is obtained by taking the polar of each line with respect to the triangle of the other three. These mutually related systems have been called *tetrad* and *counter-tetrad.**

Four points p_i in the plane satisfy a linear relation

$$(p_2p_3p_4)\,p_1 - (p_1p_3p_4)\,p_2 + (p_1p_2p_4)\,p_3 - (p_1p_2p_3)\,p_4 = 0.$$

Represent the terms $(p_2p_3p_4)\,p_1$, etc., in this equation by the letters a_1, a_2, etc. The equation then becomes

$$a_1 + a_2 + a_3 + a_4 = 0. \tag{1}$$

Operating on this identity with the products $\overline{a_i a_j}$ we find that the products $(a_i a_j a_k)$ are all equal in absolute value. If no three of the points lie on a line (which we shall always assume), we can replace the points a_i by such multiples of themselves that (1) still holds and

$$(a_2a_3a_4) = 4.$$

We now have

$$(a_ia_ja_k) = \pm 4, \qquad (i < j < k), \tag{2}$$

the sign being positive or negative according as i, j, k is complementary to an odd or an even term of the sequence 1, 2, 3, 4.

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* F. Morley, *Trans. Amer. Math. Society*, Vol. IV, p. 291.

Using the letter α_1 for the polar of a_1 with respect to the triangle a_2, a_3, a_4 (second polar of a_1 with respect to the triangle $\overline{a_2 a_3}, \overline{a_3 a_4}, \overline{a_4 a_2}$), and similarly α_i for the polar of a_i with respect to the corresponding triangle, we find by using (1) and (2) and choosing the intensities of the α's properly,

$$\left.\begin{aligned} 4\alpha_1 &= -\overline{a_2 a_3} - \overline{a_3 a_4} - \overline{a_4 a_2}, \\ 4\alpha_2 &= \overline{a_4 a_1} + \overline{a_1 a_3} + \overline{a_3 a_4}, \\ 4\alpha_3 &= -\overline{a_1 a_2} - \overline{a_2 a_4} - \overline{a_4 a_1}, \\ 4\alpha_4 &= \overline{a_1 a_2} + \overline{a_2 a_3} + \overline{a_3 a_1}. \end{aligned}\right\} \tag{3}$$

From these equations by addition we obtain

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0. \tag{4}$$

Multiplying the equations (3) by a_i and making use of (2), it is easily seen that

$$(a_i \alpha_i) = 3, \quad (a_i \alpha_j) = -1, \qquad (i \neq j). \tag{5}$$

These last equations are sixteen in number. From the equations

$$\sum_i (a_i \alpha_j) = \sum_j (a_i \alpha_j) = 0$$

it is seen that seven of them have for effect to make a_i and α_i subject to the conditions (1) and (4). The remaining nine equations of (5), when one of the tetrads is given, determine the eight geometrical constants of the other together with an intensity not fixed by the equations (1) and (4). The equations (5) may then be taken as canonically defining a tetrad and counter-tetrad. Their symmetry in a_i and α_i indicates the mutuality of the relations of the two tetrads.

From (3), by direct multiplication and the use of (2), we get

$$\overline{\alpha_i \alpha_j} = a_l - a_k, \tag{6}$$

where i, j, k, l is any positive permutation of the numbers 1, 2, 3, 4. Multiplying by α_k and using (5), we get

$$(\alpha_i \alpha_j \alpha_k) = \pm 4, \qquad (i < j < k), \tag{7}$$

the sign being positive or negative according as i, j, k is an odd or an even minor of 1, 2, 3, 4. Finally, from (3), by subtracting the equations in pairs and using (1), we get

$$\overline{a_i a_j} = \alpha_l - \alpha_k, \tag{8}$$

the rule of sign being the same as in (6). Equations differing from (3) only in the interchange of a's and α's can easily be proved.

5. The application of the preceding to the study of dyadics in the plane is now simple. Consider the collineation

$$\Sigma\, (a_2 a_3 a_4)\, (\beta_2 \beta_3 \beta_4)\, a_1 \beta_1, \tag{9}$$

no three of the points a_i being on a line, and no three of the lines β_i passing through a point. Taking a_i and β_i subject to the conditions (2) and (7), the products $(a_i a_j a_k)(\beta_i \beta_j \beta_k)$ all become equal and (9) becomes a multiple of

$$a_1 \beta_1 + a_2 \beta_2 + a_3 \beta_3 + a_4 \beta_4 .$$

Operating on this with b_1, a point of the counter-tetrad of β, and making use of (5), we get

$$3a_1 - a_2 - a_3 - a_4 = 4a_1 .$$

The points b_i pass by (9) into the points a_i. The collineation therefore transforms the associated system b, β into the system a, α, and so the dyadic in this form involves a correspondence of tetrads.

In the same way we see that

$$\Sigma\,(\alpha_2 \alpha_3 \alpha_4)(\beta_2 \beta_3 \beta_4)\,\alpha_1 \beta_1 \tag{10}$$

is a correlation which transforms the counter-tetrad of β into α, and so carries the system b, β into the system α, a.

PART II. MULTIPLE PRODUCTS.

I. *Multiple Products are Complete Invariants.*

6. With two dyadics AA', BB' is connected a form $\overline{AB}\,\overline{A'B'}$ which Gibbs called the double product of the two dyadics.* It is formed by multiplying the dyadics distributively, each pair of terms combining to form a product in which the prefactor is product of prefactors and postfactor product of postfactors. Gibbs showed that this double multiplication is distributive with respect to a resolution of either dyadic or is invariantive, as is readily seen upon expansion.

So with a system of dyadics are a series of multiple products given by the various ways in which prefactors and postfactors can be independently combined. From their construction it is evident that such forms retain their significance when the prefactors and postfactors are transformed separately and therefore belong to the class of functions that Pasch called *complete invariants.*† If the dyadics appear as transformations operating on the elements of a certain field, since a transformation of postfactors amounts to a transformation of that field, it follows that the geometric interpretation of a multiple product must involve an arbitrary initial field. If, for example, a system of collineations and correlations in the plane operate upon four points, the mul-

* *Loc. cit.*, p. 306. The function here considered is a double product only in the sense that it is formed by a certain double process. It is neither the scalar nor the vector, but the combinatorial double product. All of these have certain properties in common which characterize double multiplication.

† "Vollkommene Invariante," *Math. Annalen*, Bd. 52, p. 128.

tiple products will give results independent of the initial 4-point, *i. e.*, invariants and covariants of the resulting tetrads. Illustrations of this property will appear in the discussions that follow.

II. *Apolarity of Collineations.*

7. Consider two contragredient collineations $a\alpha$ and βb; the first an operator on points, the second an operator on lines. They have a double product invariant $(a\beta)(\alpha b)$. When this vanishes, the collineations will be called *apolar*.

To see the meaning of this, write

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3,$$
$$\beta b = \mu_1 \beta_1 b_1 + \mu_2 \beta_2 b_2 + \mu_3 \beta_3 b_3,$$

and take $\Delta b = \Delta\alpha$ as reference triangle. That is, place

$$(b_i \alpha_i) = 1, \quad (b_i \alpha_k) = 0, \qquad (i \neq k).$$

We then have

$$(a\beta)(\alpha b) = \lambda_1 \mu_1 (a_1 \beta_1) + \lambda_2 \mu_2 (a_2 \beta_2) + \lambda_3 \mu_3 (a_3 \beta_3).$$

This obviously vanishes when, for each value of the subscript, a_i is on β_i. Two triangles so related that each point of the one lies on the corresponding line of the other will be called *incident*. Now Δa and $\Delta\beta$ are the correspondents of the reference triangle with respect to $a\alpha$ and βb. Hence, apolarity is the condition under which two collineations can send a triangle into a pair of incident triangles.*

A collineation apolar to the identical collineation sends certain triangles into incident or inscribed triangles. Such a collineation has been called normal.

Write the given collineation

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3,$$

and the identical collineation

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,$$

where $\Delta a = \Delta\alpha$ is reference triangle. The apolarity condition is then

$$\lambda_1 (a_1 \alpha_1) + \lambda_2 (a_2 \alpha_2) + \lambda_3 (a_3 \alpha_3) = (a\alpha) = 0.$$

Hence the condition for normal collineation is the vanishing of what Gibbs called the *scalar* † of the dyadic.

8. Another interpretation for apolarity is obtained by using the system of counter-tetrads explained in Art. 4. The vanishing of the invariant

* M. Pasch, *Math. Annalen*, Bd. 23, p. 431. † *Loc. cit.*, p. 275.

$(a\beta)(\alpha b)$ gives the condition that the collineation

$$a(\alpha b)\beta$$

be normal. Hence, if $a\alpha$ and βb operating on a certain tetrad and counter-tetrad give a 4-point a_i and a 4-line β_i, the collineation which transforms b_i (the counter-tetrad of β_i) into a_i will be normal. According to (9) the collineation which carries b_i into a_i is

$$\Sigma(a_2 a_3 a_4)(\beta_2\beta_3\beta_4)a_1\beta_1.$$

Therefore, the condition that the collineations $a\alpha$ and βb be apolar is

$$\Sigma(a_2 a_3 a_4)(\beta_2\beta_3\beta_4)(a_1\beta_1)=0, \tag{11}$$

where a_i and β_i correspond respectively through $a\alpha$ and βb to a tetrad of points and its counter-tetrad of lines.

It is to be observed that (11) is linear in each of the quantities a_i and β_i. Hence, if all of those quantities except a point a_k are given, it will lie on a definite line; and if all except a line β_k are given, it will pass through a definite point. Therefore, a 4-point and 4-line subject to the condition (11) determine a tetrad of lines through a_i and a tetrad of points on β_i, consisting, in fact, of the evectants of (11) with respect to a_i and β_i.

If we write a_i and β_i in such form that the equations (2) and (7) hold, the apolarity condition (11) can be written

$$(a_2a_3a_4)(a_1\beta_1)-(a_1a_3a_4)(a_2\beta_2)+(a_1a_2a_4)(a_3\beta_3)-(a_1a_2a_3)(a_4\beta_4)=0.$$

Placing $(a_2a_3a_4)\doteq 4$, we get for the evectant with respect to a_1

$$4\gamma_1=4\beta_1-(a_2\beta_2)\overline{a_3a_4}+(a_3\beta_3)\overline{a_2a_4}-(a_4\beta_4)\overline{a_2a_3}.$$

Placing α_i as counter-tetrad of a_i, this can be written in the form

$$4\gamma_1=4\{\beta_1-\tfrac{1}{4}[(a_1\beta_1)\alpha_1+(a_2\beta_2)\alpha_2+(a_3\beta_3)\alpha_3+(a_4\beta_4)\alpha_4]\},$$

as is readily seen upon multiplying the right-hand members of both equations by each of the points a_i and using the identity

$$(a_1\beta_1)+(a_2\beta_2)+(a_3\beta_3)+(a_4\beta_4)=0,$$

to which (11) reduces. Since the expression in brackets is symmetrical with respect to the numbers 1, 2, 3, 4, we can finally write

$$\gamma_i=\beta_i-\eta, \tag{12}$$

where

$$\eta=\tfrac{1}{4}[(a_1\beta_1)\alpha_1+(a_2\beta_2)\alpha_2+(a_3\beta_3)\alpha_3+(a_4\beta_4)\alpha_4].$$

From (12) and (8) we have

$$\gamma_2-\gamma_1=\beta_2-\beta_1=\overline{b_3b_4},$$

where b is the counter-tetrad of β. Hence, if we order to the lines γ_i the

points b_i, it follows that each pair of lines intersect on the join of the remaining pair of points. Such a 4-point and 4-line may be called *chiastic.**

From the symmetry of (11) in a_i and β_i we are now able to write the evectant with respect to β_i in the form

$$c_i = a_i - y, \tag{13}$$

where

$$y = \tfrac{1}{4}\,[(a_1\beta_1)b_1 + (a_2\beta_2)b_2 + (a_3\beta_3)b_3 + (a_4\beta_4)b_4].$$

Therefore, the 4-point c_i and the 4-line α_i are chiastic.

The geometric interpretation of apolarity then leads to the following statement of a theorem of Pasch: †

Two apolar collineations transform any 4-point and associated 4-line into a 4-point a_i and a 4-line β_i such that there is a 4-line through a_i chiastic to the counter-tetrad of β_i, and a 4-point on β_i chiastic to the counter-tetrad of a_i.

From (13) it is observed that the tetrads c_i and a_i are perspective, y being the center of perspective. Since counter-tetrads are chiastic, this is a special case of a theorem of Pasch which states that any pair of 4-points chiastic to the same 4-line are perspective.

9. It has already been observed that the apolarity of the collineations $a\alpha$ and βb amounts to the vanishing of the linear invariant (scalar) of

$$a(\alpha b)\beta.$$

For brevity we shall write

$$s_1 = a\alpha, \qquad s_2 = b\beta,$$
$$\sigma_1 = \alpha a, \qquad \sigma_2 = \beta b;$$

and generally we shall designate a collineation by s, and its inverse in contragredient form by σ. The collineation written above is then $s_1 s_2$. As a convenient abbreviation we shall denote the linear invariant of any collineation by the symbol of that collineation placed in parentheses. Thus the apolarity of $a\alpha$ and βb is given symbolically by

$$(s_1 s_2) = 0. \ddagger$$

From the definition it follows immediately that

$$(s_1 s_2) = (s_2 s_1) = (\sigma_1 \sigma_2) = (\sigma_2 \sigma_1).$$

Similarly, we shall sometimes find it convenient to denote the linear invariant of the product of any number of collineations $s_1, s_2, \ldots, s_r$ by

* Cf. Sir Robert Ball, "Theory of Screws," p. 306.

† *Math. Annalen*, Bd. 26, p. 211.

‡ s_1 and s_2 in this case will be called *harmonic*, retaining the word *apolar* to express the relation of s_1 and σ_2, or s_2 and σ_1.

$(s_1 s_2 \dots . s_r)$. Writing the collineations in the form $a\alpha$, $b\beta$, $c\gamma$, etc., it is immediately seen that

$$(s_1 \dots . s_{r-1} s_r) = (s_r s_1 \dots . s_{r-1}). \tag{14}$$

That is, the linear invariant of the product of any number of collineations is not affected by a cyclic permutation of those collineations.*

10. Suppose we have three collineations each of which transforms a 3-line α_i into a 3-line β_i. If a_i and b_i are the points of the triangles α and β, the collineations can be written

$$\sigma_1 = \lambda_1 \beta_1 a_1 + \lambda_2 \beta_2 a_2 + \lambda_3 \beta_3 a_3,$$
$$\sigma_2 = \mu_1 \beta_1 a_1 + \mu_2 \beta_2 a_2 + \mu_3 \beta_3 a_3,$$
$$\sigma_3 = \nu_1 \beta_1 a_1 + \nu_2 \beta_2 a_2 + \nu_3 \beta_3 a_3,$$

where λ_i, μ_i and ν_i are all different from zero. Furthermore, let

$$s = \rho_1 c_1 \alpha_1 + \rho_2 c_2 \alpha_2 + \rho_3 c_3 \alpha_3$$

be a non-singular collineation apolar to σ_1, σ_2 and σ_3. Taking α as reference triangle, the conditions required are

$$\rho_1 \lambda_1 (\beta_1 c_1) + \rho_2 \lambda_2 (\beta_2 c_2) + \rho_3 \lambda_3 (\beta_3 c_3) = 0,$$
$$\rho_1 \mu_1 (\beta_1 c_1) + \rho_2 \mu_2 (\beta_2 c_2) + \rho_3 \mu_3 (\beta_3 c_3) = 0,$$
$$\rho_1 \nu_1 (\beta_1 c_1) + \rho_2 \nu_2 (\beta_2 c_2) + \rho_3 \nu_3 (\beta_3 c_3) = 0.$$

These equations can only coexist when either

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = 0$$

or

$$\rho_1 (\beta_1 c_1) = \rho_2 (\beta_2 c_2) = \rho_3 (\beta_3 c_3) = 0.$$

The first of these expresses that the collineations are linearly related; the second that the 3-line β_i and the 3-point c_i are incident. Hence, if a non-singular collineation is apolar to three linearly independent collineations having a common triangle pair, it transforms the first of those triangles into one incident to the second. In particular, if α_i and β_i are identical, their triangle is transformed by s into an inscribed triangle and is consequently a Pasch triangle† of s.

Suppose a second set of collineations t_1, t_2 and t_3 which transform a 3-point d_i into a 3-point c_i. The same is then true of any collineation of the net

$$\rho_1 t_1 + \rho_2 t_2 + \rho_3 t_3.$$

* Since the invariants of a collineation are expressible in terms of the linear invariants of its powers, it follows that all the invariants of a product of collineations depend only on the cyclical order.

† F. Morley, *loc. cit.*, p. 295.

Suppose one of these collineations should transform a_i into a 3-point incident to β_i. By the last paragraph the conditions required are

$$\begin{aligned}\rho_1(s_1t_1)+\rho_2(s_1t_2)+\rho_3(s_1t_3)&=0,\\ \rho_1(s_2t_1)+\rho_2(s_2t_2)+\rho_3(s_2t_3)&=0,\\ \rho_1(s_3t_1)+\rho_2(s_3t_2)+\rho_3(s_3t_3)&=0.\end{aligned}$$

These equations can be satisfied if

$$\begin{vmatrix}(s_1t_1) & (s_1t_2) & (s_1t_3)\\ (s_2t_1) & (s_2t_2) & (s_2t_3)\\ (s_3t_1) & (s_3t_2) & (s_3t_3)\end{vmatrix}=0. \tag{15}$$

From the symmetry of this condition in s and t we conclude that if there is a collineation which transforms d_i into c_i and a_i into a triad incident to β_i, then there is a collineation that transforms b_i into a_i and c_i into a triad incident to δ_i.

If in the above theorem we take a_i equal to b_i and c_i equal to d_i, we have the theorem of Hun that the relation of Pasch triangle and fixed triangle in a normal collineation is mutual.

III. *The Intermediate.**

11. Take in the next place two cogredient point collineations $a\alpha$ and $b\beta$. They have a double product $\overline{ab}\,\overline{\alpha\beta}$.† This is a covariant line collineation which has been called the *intermediate* of $a\alpha$ and $b\beta$.

The geometric interpretation is easily seen. Let η (Fig. 1) be the correspondent of any line ξ two of whose points are x and y. Then, by a well-known identity,

$$\eta=\overline{ab}\,(\alpha\beta\xi)=\overline{ab}\,\{(\alpha x)(\beta y)-(\alpha y)(\beta x)\}=\overline{x'y''}-\overline{x''y'}, \tag{16}$$

in which x' and x'', y' and y'' are the correspondents of x and y with respect to $a\alpha$ and $b\beta$. Therefore, η is a line through the join of $\overline{x'y''}$ and $\overline{x''y'}$, or, as we may say, through the cross-join of the correspondents of x and y. Since x and y are any points on ξ, *η is the locus of cross-joins of pairs of points on ξ.* This, from the known construction of a polarity, amounts to saying that $\overline{x'x''}$ envelopes a conic tangent to $\overline{x'y'}$ and $\overline{x''y''}$ at their junction with η.

In case of three points x, y, z of ξ, the preceding construction involves Pascal's theorem for the hexagon inscribed in a 2-line.

* A. B. Coble, *Trans. Amer. Math. Society*, Vol. IV, p. 70. Professor Morley has used the word *Clebschian* to represent a form of this kind (*Trans.*, Vol. IV, p. 471).

† In the symbolic notation of Clebsch this would of course be written

$$(a\,b\,x)\;(\alpha\,\beta\,\xi)=0,$$

where ξ is given and x variable.

In case the two collineations are the same, η is the join of x' and y', and the intermediate reduces to the reciprocal or line form. Hence, the line equation of a given collineation is gotten by taking the double product of the dyadic with itself.

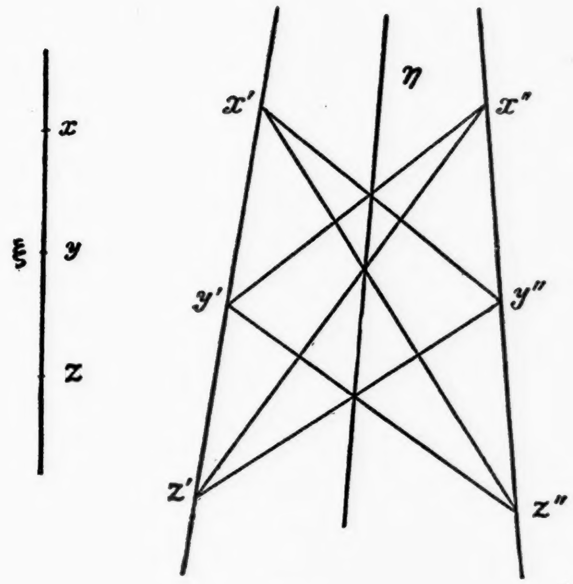

FIG. 1.

12. An identity for which we shall find frequent use is obtained by developing $(abx)\,(\alpha\beta\xi)$ according to the ordinary rule for multiplication of determinants. We thus obtain

$$(abx)\,(\alpha\beta\xi)=\begin{vmatrix}(a\alpha) & (a\beta) & (a\xi)\\ (b\alpha) & (b\beta) & (b\xi)\\ (x\alpha) & (x\beta) & (x\xi)\end{vmatrix}=(\overline{ab}\cdot\overline{\alpha\beta})\,(x\xi)+(a\beta)\,(b\xi)\,(x\alpha)$$
$$+(a\xi)\,(b\alpha)\,(x\beta)-(a\alpha)\,(b\xi)\,(x\beta)$$
$$-(b\beta)\,(a\xi)\,(\alpha x).$$

Writing

$$s_1=a\alpha,\quad s_2=b\beta,$$
$$\sigma_1=\alpha a,\quad \sigma_2=\beta b,$$

and using the notation of Art. 9 for the linear invariants, we have

$$\overline{s_1 s_2}=(\overline{s_1 s_2})\,\sigma_0+\sigma_1\sigma_2+\sigma_2\sigma_1-(s_1)\,\sigma_2-(s_2)\,\sigma_1, \tag{17}$$

where the bar over $s_1 s_2$ is used to signify the operation of forming the intermediate, and σ_0 represents the identical line collineation.

13. The intermediate belongs to a type of correspondence that occurs in any number of dimensions. And though we are at present concerned with the collineation in the plane, it may be worth the trouble to indicate the general

theory. It is very easy to extend the construction already given for the plane intermediate to the case of higher dimensions. Using the equations (16) we have for the intermediate $\overline{ab}\,\overline{\alpha\beta}$, where $a\alpha$ and $b\beta$ are point collineations in space, that to any line ξ corresponds a linear complex with reference to which $\overline{x'y''}$ and $\overline{x''y'}$ are polar lines, x' and x'', y' and y'', being correspondents with respect to $a\alpha$ and $b\beta$ of two points x and y on ξ.

In the same way for the intermediate $\overline{abc}\,\overline{\alpha\beta\gamma}$ we have

$$\overline{abc}(\alpha\beta\gamma\pi) = \overline{abc}\,\{(\alpha x)(\beta\gamma\cdot yz) + (\alpha y)(\beta\gamma\cdot zx) + (\alpha z)(\beta\gamma\cdot xy)\}, \quad (18)$$

where x, y, z are three points of the plane π. If then we take any triangle on the plane π, transform its points by the collineation $a\alpha$, transform the opposite lines by the collineation $\overline{bc}\,\overline{\beta\gamma}$, and join the corresponding elements, we get a set of three planes intersecting on the correspondent of π with respect to the collineation $\overline{abc}\,\overline{\alpha\beta\gamma}$. If $\overline{bc}\,\overline{\beta\gamma}$ is the identical line collineation, or quadratic complex to which every line belongs, the preceding construction is readily translated into a descriptive property of two complete 4-points in planes in space.

Proceeding in this way, we can construct intermediates in any number of dimensions. Another method was, however, presented by Kraus* and Muth.† Consider in the first place the plane intermediate written in the form

$$(abx)(\alpha\beta\xi) = 0.$$

This expresses the apolarity of the collineations $\overline{ax}\,\overline{\alpha\xi}$ and $\overline{bx}\,\overline{\beta\xi}$. For, on forming the double product, we get

$$x(abx)(\alpha\beta\xi)\xi = 0.$$

Now, $\overline{ax}\,\overline{\alpha\xi}$ and $\overline{bx}\,\overline{\beta\xi}$ may be considered as binary projectivities which give for points on ξ lines joining x to the correspondents through $a\alpha$ and $b\beta$. Then, since two binary apolar projectivities give rise to an involution, it follows that if we transform the points of a line ξ by the collineations $a\alpha$ and $b\beta$, the correspondent of ξ with respect to $\overline{ab}\,\overline{\alpha\beta}$ is the locus of points from which the transforms appear in involution.

Considering x and ξ as lines in space, it follows from the argument of the last paragraph that the collineation $\overline{ab}\,\overline{\alpha\beta}$ in three dimensions gives for a line ξ the complex consisting of lines which, joined to the correspondents of points on ξ, give pairs of planes belonging to an involution.

In order to interpret the triple intermediate $\overline{abc}\,\overline{\alpha\beta\gamma}$, we need a new invariant. Three plane collineations $a\alpha$, $b\beta$ and $c\gamma$ have a triple product

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* "Dissertation" (Giessen), 1886.

† *Math. Annalen*, Bd. 33.

invariant $(abc)(\alpha\beta\gamma)$. When this vanishes, the collineations have been called *harmonic.** Its vanishing simply expresses that the intermediate of two of the collineations is apolar to the third. Since we are able to construct the intermediate and to interpret the condition of apolarity, this harmonic relation may be supposed known.

The intermediate of three collineations $a\alpha$, $b\beta$ and $c\gamma$ of space can be written in the form

$$(abcx)(\alpha\beta\gamma\pi) = 0,$$

where x represents a point and π a plane. This expresses that the three projectivities $\overline{ax}\,\overline{\alpha\pi}$, $\overline{bx}\,\overline{\beta\pi}$ and $\overline{cx}\,\overline{\gamma\pi}$ are harmonic; for, on putting the triple product equal to zero, we get that equation. But $\overline{ax}\,\overline{\alpha\pi}$, $\overline{bx}\,\overline{\beta\pi}$ and $\overline{cx}\,\overline{\gamma\pi}$ are ternary correspondences which give for points on π lines joining x to their correspondents with respect to $a\alpha$, $b\beta$ and $c\gamma$. Therefore, if we construct with respect to $a\alpha$, $b\beta$ and $c\gamma$ the correspondents of points belonging to a plane π, the correspondent of π with respect to $\overline{abc}\,\overline{\alpha\beta\gamma}$ is the locus of points from which those line systems appear harmonic.

So by a process of continuous induction we can build up intermediates of any degree of complexity.

14. A collineation is singular when there is an element whose correspondent is indeterminate. Thus the intermediate $\overline{ab}\,\overline{\alpha\beta}$ in the plane is singular when a line ξ can be found such that

$$\overline{ab}\,(\alpha\beta\xi) = 0. \tag{19}$$

From the construction of the intermediate it is evident that ξ must in this case pass by $a\alpha$ and $b\beta$ into the same line η. Now, (19) is the condition for the correlation

$$a\,(\alpha\beta\xi)\,b$$

to be symmetrical, *i. e.*, to be a polarity. But $a(\alpha\beta\xi)b$ sets up a binary correlation on η consisting of pairs of points given by $a\alpha$ and $b\beta$ for points of ξ. Therefore, since every line of the plane cuts ξ, it follows that in case of a singular intermediate every line of the plane passes by $a\alpha$ and $b\beta$ into a pair of lines apolar to a definite pair of points, *i. e.*, the double points of the binary polarity on the correspondent of ξ.

Another property of a normal collineation connects with the intermediate of it and the identical collineation. In fact, from (17), the intermediate of s and s_0 is

$$\overline{ss_0} = (s)\,s_0 - \sigma.$$

* J. Kraus, *Math. Annalen*, Bd. 29, p. 234

If s is normal, $(s)=0$ and the intermediate of s and s_0 is the line form of the inverse of s. If, then, x and y are two points on a line ξ, and x' and y' are their correspondents through a normal collineation, the lines $\overline{xy'}$ and $\overline{x'y}$ intersect on the correspondent of ξ through the inverse collineation.

IV. *Apolarity of Collineation and Correlation.**

15. A collineation $a\alpha$ and a contragredient correlation bc may be apolar, *i. e.*, may satisfy the conditions $ab(c\alpha)=0$. The meaning of this is easily seen. Let

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3,$$
$$bc = \mu_1 b_1 c_1 + \mu_2 b_2 c_2 + \mu_3 b_3 c_3,$$

and take $\Delta c = \Delta\alpha$ as reference triangle. The condition of apolarity is then

$$\overline{ab}(c\alpha) = \lambda_1\mu_1\overline{a_1 b_1} + \lambda_2\mu_2\overline{a_2 b_2} + \lambda_3\mu_3\overline{a_3 b_3} = 0. \tag{20}$$

This equation expresses that the triangles a_i and b_i are perspective. Therefore, since they are correspondents through $a\alpha$ and bc of the reference triangle, it follows that *a collineation and an apolar correlation transform any triangle into a pair of perspective triangles.*

Conversely, if a_i and b_i are perspective and $a\alpha$ is given, values μ_i can be found such that (20) holds. Taking those values as the coefficients in bc, we have a correlation apolar to $a\alpha$. Therefore, if a collineation $a\alpha$ transforms the points of a triangle α into a triangle perspective to b_i, there is a correlation apolar to $a\alpha$ which transforms the points of the triangle α into the points b_i. In particular, a collineation and correlation are apolar if they transform respectively the points and lines of a triangle into the same triad of points.

The condition that the intermediate $\overline{ab}\,\overline{\alpha\beta}$ of two collineations be apolar to a correlation cd is

$$(abc)\alpha\beta\cdot d = (abc)\{\alpha(\beta d) - \beta(\alpha d)\} = \overline{bc}(\beta d).a\alpha + \overline{ac}(\alpha d)\cdot b\beta = 0.$$

Now, $\overline{bc}(\beta d)=0$ and $\overline{ac}(\alpha d)=0$ are the conditions of apolarity of $b\beta$ and $a\alpha$ with cd. Hence, if two collineations are apolar to a correlation, so is their intermediate. By an entirely analogous process it follows that if two correlations are apolar to the same collineation, their intermediate is also.

The intermediate of a collineation or correlation with itself is the reciprocal or adjoined form. Hence, if a collineation and correlation are apolar, the same relation subsists when either or both are replaced by their adjoined forms.

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* F. Aschieri called such correspondences *harmonic*. Compare his article, "Sulle omografie binarie e ternarie," *Rend. del R. Istituto Lombardo*, (2), Vol. XXIV, p. 289.

If, for example, two collineations transform the points of a triangle α into perspective triads, we have seen that there is a correlation apolar to both collineations which transforms the lines α_i into either of those triads. The preceding paragraph then expresses that, if two collineations transform α into a pair of perspective triangles, the intermediate gives a triangle perspective to both.*

16. A correlation apolar to the identical collineation transforms any triangle into a perspective one and is therefore a polarity. The condition that bc be a polarity is then

$$\overline{bc} = 0. \tag{21}$$

Suppose a collineation $a\alpha$ is apolar to a polarity bc. We then have

$$\overline{ab}\,(\alpha c) = 0, \qquad \overline{bc} = 0. \tag{22}$$

Let ξ be a fixed line of $a\alpha$ given by the root λ of the characteristic equation, *i. e.*, such that

$$(\xi a)\,\alpha = \lambda \xi.$$

Multiplying by ξ, we get from (22)

$$(\alpha c)\,\{(\xi b)a - (\xi a)b\} = (\alpha c)\,(\xi b)a - \lambda(\xi c)b = 0, \qquad (\xi c)b - (\xi b)c = 0.$$

Combining these equations, we obtain

$$(\xi c)\,(b\,\alpha)a = \lambda(\xi c)b.$$

For a fixed line of a collineation, an apolar polarity gives a fixed point corresponding to the same root of the characteristic equation. If the characteristic equation has three distinct roots, the fixed triangle is then self-conjugate or polar with respect to any apolar polarity.

From (17) we have for the adjoined form of a collineation s

$$\overline{ss} = (\overline{ss}) + 2\sigma^2 - 2\,(s)\,\sigma, \tag{23}$$

where σ is the inverse of s. Since a polarity is symmetrical, if it is apolar to s, it is apolar to σ. In that case we have also seen that it is apolar to $\overline{ss}$ and identity. Therefore, from (23) it follows that if a polarity is apolar to s, it is apolar to σ^2. All collineations that are covariants of σ are, however, expressible linearly in terms of σ^0, σ and σ^2.† Consequently, *a polarity apolar to a collineation is apolar to all of its covariants.*

If a collineation s transforms the points of a triangle α into a perspective triad, there is a polarity which transforms the lines of α into the same triad. That polarity is obviously apolar to s and to all of its covariants. Further,

* Cf. Muth, *Math. Annalen*, Bd. 40, p. 98.

† Clebsch and Gordan, *Math. Annalen*, Bd. 1, p. 373.

there is a polarity which leaves α fixed and is apolar to s. Therefore we have Muth's theorem that if a collineation s transforms a triangle into a perspective one, then all of the covariants of s transform it into triangles perspective to each other and to the original triangle.*

17. For a correlation to be apolar to a collineation requires the identical vanishing of a line and therefore subjects the coefficients of either to three linear conditions. There is not then, in general, a correlation apolar to each of three given collineations. The condition for such is the vanishing of the determinant of the nine equations expressing the three apolarities. This invariant, the explicit form of which does not concern us, has been called Δ.† It is a combinant symmetrical in the coefficients of the three collineations, and of the third degree in each.

We will now consider the peculiarities of a system of three collineations for which Δ vanishes. In the net

$$s = \rho_1 s_1 + \rho_2 s_2 + \rho_3 s_3 \tag{24}$$

there are a single infinity of singular collineations. The singular points lie on a cubic that we may call C; the singular lines envelop a cubic that we may call Γ. It is well known that the adjoined form of a singular collineation consists of the product of singular line and singular point. And we have seen that a correlation apolar to a collineation is apolar to its adjunct. Therefore, *if a correlation is apolar to the collineations* s_1, s_2 *and* s_3, *the singular lines and points of* (24) *are correspondents in that correlation, and consequently the cubics* Γ *and* C *are reciprocal through it.*

With a point of C is associated in two ways a line of Γ. In the first place, the point a appears as singular point in a definite collineation of (24) which has a singular line β. Secondly, it is transformed by those collineations into the points of a definite line γ.

To say that a correlation is apolar to each of three collineations, amounts to saying that those collineations operating on the inverse of that correlation give polarities. Such a transformation of the collineations does not affect the cubic Γ, which is therefore the Cayleyan of the three polarities. We saw above, however, that the line β passes by the inverse of the apolar correlation into the point a. Therefore, β passes by the three polarities into points of γ, and consequently β and γ are corresponding lines of Γ.

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* Muth, *loc. cit.*, p. 97. † Rosanes, *Crelle*, Bd. 95, p. 254.

Conversely, suppose with each point of C are associated a pair of corresponding lines of Γ. Two collineations of (24) whose common polar triangles are not singular may be written

$$s_1 = \lambda_1 b_1 \alpha_1 + \lambda_2 b_2 \alpha_2 + \lambda_3 b_3 \alpha_3,$$
$$s_2 = \mu_1 b_1 \alpha_1 + \mu_2 b_2 \alpha_2 + \mu_3 b_3 \alpha_3.$$

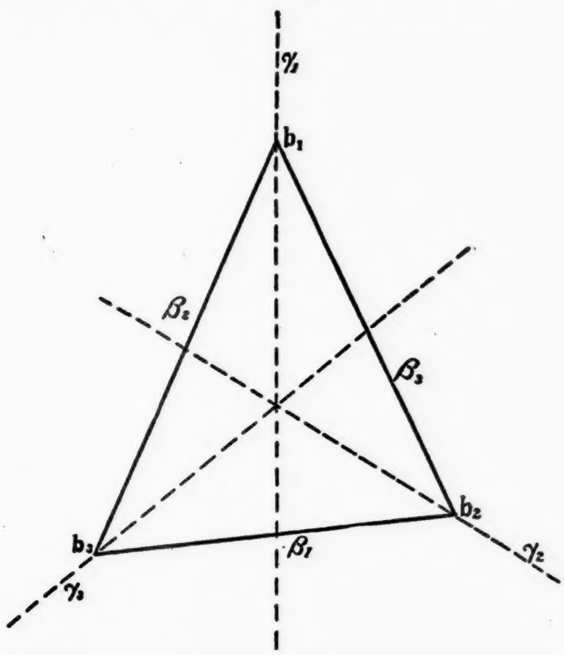

FIG. 2.

The lines β_i are singular in the collineations whose singular points are a_i. The points a_1, a_2, a_3 pass by the net of collineations respectively into points of $\gamma_1, \gamma_2, \gamma_3$, where γ_i is incident to b_i. By supposition $\beta_1\gamma_1$, $\beta_2\gamma_2$ and $\beta_3\gamma_3$ are three pairs of corresponding lines with respect to the curve. If we should start with b_1, we could construct a complete 4-point, all of whose lines touch the curve. It would contain those three pairs of lines as diagonal pairs, and therefore $\gamma_1, \gamma_2, \gamma_3$ pass through a point (Fig. 2). The triad a_i, then, passes by two of the collineations into the triad b_i, and by the third into one perspective to b_i. According to Art. 15, the three collineations, therefore, have a common apolor correlation.

The vanishing of the invariant Δ is the necessary and sufficient condition that with any point of C should be associated (in the above way) a pair of corresponding lines of Γ, and, dually, with any line of Γ should be associated a pair of corresponding points of C.

The similarity of a set of collineations for which Δ vanishes to a net of polarities is noticeable. It is due to the fact that a set of polarities are apolar to the identical collineation, or, what amounts to the same thing, that a net of collineations with vanishing Δ may be transformed into a net of polarities in such a way as to have either the initial or the resultant field invariant.

PART III. THE ALTERNANT.

I. *Introduction.*

18. In contrast with the symmetrical forms just considered are a series of combinants that we will call *alternants*. The alternant of n collineations s_i is defined by the equation

$$[s_1 \ldots . s_n] = \begin{vmatrix} s_1 & s_1 & \ldots . & s_1 \\ s_2 & s_2 & \ldots . & s_2 \\ \cdots & \cdots & \cdots & \cdots \\ s_n & s_n & \ldots . & s_n \end{vmatrix}, \tag{25}$$

where the determinant is supposed to be developed in the order of its rows; *i. e.*, in each term of the development the first letter is taken from the first column, the second from the second column, etc. This determinant is readily seen to follow the ordinary rules so far as its rows are concerned. If, for example, a linear relation exists between the collineations $s_1, \ldots ., s_n$, the alternant is zero. Using the ordinary rule of signs, the determinant may be developed as the sum of products by their minors of determinants of r-th order in the first r columns. The alternant can not, however, in general be developed in terms of minors taken from its first r rows.

Obviously, there will be a marked difference according as the order of the alternant is odd or even. If the order n is even, for every term of the form $s_i \ldots . s_j s_k$ there will be a term $-s_k s_i \ldots . s_j$, where the intervening letters in both are the same. The linear invariant is therefore

$$\Sigma \,\}\, (s_i \ldots . s_j s_k) - (s_k s_i \ldots . s_j) \,\{.$$

The alternant of an even number of collineations is therefore normal.

Again, if n is even, the alternant can be written in the form

$$\Sigma\, (P s_i Q - Q s_i P), \tag{26}$$

where P and Q are products not containing s_i. This is harmonic* with s_i, since

$$\Sigma \,\}\, (s_i P s_i Q) - (s_i Q s_i P) \,\{ = 0.$$

The alternant of an even number of collineations is consequently harmonic with each of them.

* See Art. 9, foot-note.

From (26), if s_i is the identical collineation, we see that *the alternant of an even number of collineations vanishes when one of those is the identical collineation.*

In case of an odd alternant, since the members of a cyclic group are all of the same sign, we have

$$([s_1 \ldots . s_n]) = n(s_1[s_2 \ldots . s_n]). \tag{27}$$

If, then, the alternant of an odd number of collineations is normal, each collineation is harmonic with the alternant of the remaining $n-1$.

Write the alternant in the form

$$[s_1 \ldots . s_n] = \Sigma s_i S_i,$$

where S_i is the minor of s_i in the first column of the alternant. If n is odd and one of the collineations, s_1 for example, is the identical collineation, since the first minors are even, all those containing s_1 vanish and the alternant takes the form

$$[s_1 \ldots . s_n] = s_0[s_2 \ldots . s_n] = [s_2 \ldots . s_n].$$

The alternant of an odd number of collineations containing the identical collineation is then equal (except for algebraic sign) to the alternant of the remaining $n-1$.

II. *The Alternant of Two Ternary Collineations.*

19. We shall usually write the collineations in the form

$$s_1 = a\alpha, \qquad s_2 = b\beta.$$

The alternant is then

$$[s_1 s_2] = s_1 s_2 - s_2 s_1 = (\alpha b)\, a\beta - (\beta a)\, b\alpha, \tag{28}$$

The covariant collineations of a collineation s are linear functions of s_0, s and s^2, where s_0 is the identical collineation. Since each of these is commutative with s, it follows that *the alternant of a collineation and any of its covariants vanishes identically.*

The invariant relations of the alternant and covariants of s_1 and s_2 may be summed up in two general theorems.

(i) *The alternant is apolar to all the covariants of* s_1 *or* s_2.

For, let $c\gamma$ be any covariant of s_1. The condition of apolarity with the alternant is

$$0 = (\alpha b)(a\gamma)(\beta c) - (\beta a)(b\gamma)(\alpha c) = b\beta \cdot [(a\gamma)\alpha c - (\alpha c)\gamma a],$$

where the dot is used to represent the process of forming the double product, or bilinear invariant. Since the expression in brackets is the alternant of $a\alpha$ and a covariant $c\gamma$, the function vanishes as was required.

(ii) *The alternant is apolar to the intermediate of one of the collineations and any covariant of the other.*

For, let $c\gamma$ again be a covariant of $a\alpha$. The intermediate of this with $b\beta$ is

$$\overline{bc}\,\overline{\beta\gamma}.$$

The condition of apolarity with the alternant is

$$0=(\alpha b')(abc)(\beta'\beta\gamma)-(\beta' a)(b'bc)(\alpha\beta\gamma),$$

where $b'\beta'$ is a new symbol for $b\beta$. Interchanging $b\beta$ and $b'\beta'$ and adding, the last expression becomes

$$\tfrac{1}{2}\{(\beta'\beta\gamma)(\overline{b'b}\,\overline{\alpha\cdot c\alpha})-(b'bc)(\overline{\beta'\beta}\,\overline{a\cdot\gamma\alpha})\}=\tfrac{1}{2}\overline{\beta'\beta}\,\overline{b'b}\cdot[\gamma\overline{\alpha\cdot ca}-\overline{a\cdot\gamma}\alpha c].$$

The expression in brackets expands into

$$(\alpha a)\gamma c-(\alpha c)\gamma a-(a\alpha)\gamma c+(a\gamma)\alpha c=(a\gamma)\alpha c-(\alpha c)\gamma a,$$

which is zero, since it is the alternant of $a\alpha$ and a covariant.

20. Since all covariants of s are expressible linearly in terms of s_0, s, s^2, the alternant is found to be apolar to the eight collineations

$$\sigma_0,\ \sigma_1,\ \sigma_1^2,\ \sigma_2,\ \sigma_2^2,\ \overline{s_1 s_2},\ \overline{s_1^2 s_2},\ \overline{s_1 s_2^2}, \tag{29}$$

where, as in Art. 9, $\overline{s_1 s_2}$ is the intermediate of s_1 and s_2 and σ is the same connex as s but considered reciprocally. If the eight collineations are linearly independent, the eight apolarity conditions are sufficient uniquely to determine the alternant. It is our purpose, in the next place, to see whether or when such is the case.

By the formula (17) we have

$$\overline{s_1 s_2^2}=(\overline{s_1^2 s_2^2})+\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_1^2-(s_1^2)\sigma_2^2-(s_2^2)\sigma_1^2.$$

Since the alternant $[s_1 s_2]$ is apolar to σ_0, σ_1^2 and σ_2^2, we see that the condition of apolarity of $[s_1 s_2]$ and $\overline{s_1^2 s_2^2}$ is the vanishing of the linear invariant of the collineation

$$(s_1 s_2-s_2 s_1)(s_1^2 s_2^2+s_2^2 s_1^2).$$

Hence, by direct expansion of this last-named invariant, we obtain

$$\begin{aligned}[s_1 s_2]\cdot\overline{s_1^2 s_2^2}&=(s_1 s_2 s_1^2 s_2^2)+(s_1 s_2^3 s_1^2)-(s_2 s_1^3 s_2^2)-(s_2 s_1 s_2^2 s_1^2)\\&=(s_1 s_2 s_1^2 s_2^2)-(s_2 s_1 s_2^2 s_1^2),\end{aligned} \tag{30}$$

since by Art. 9 $(s_1 s_2^3 s_1^2)$ and $(s_2 s_1^3 s_2^2)$ are equal, both passing by a cyclic permutation into $(s_1^3 s_2^3)$.

Again, we have

$$\overline{s_1^2 s_2}=(\overline{s_1^2 s_2})+\sigma_1^2\sigma_2+\sigma_2\sigma_1^2-(s_1^2)\sigma_2-(s_2)\sigma_1^2.$$

Since $[s_2^2 s_1]$ is apolar to σ_0, σ_2 and σ_1^2, by the same argument as before, the

apolarity condition of $[s_2^2 s_1]$ and $s_1^2 s_2$ is found to be the vanishing of the linear invariant of the collineation

$$(s_2^2 s_1 - s_1 s_2^2)\,(s_1^2 s_2 + s_2 s_1^2).$$

Expanding and making use of the cyclic permutation, we then obtain

$$\begin{aligned}[s_2^2 s_1]\cdot\overline{s_1 s_2^2} &= (s_2^2 s_1^3 s_2) + (s_2^2 s_1 s_2 s_1^2) - (s_1 s_2^2 s_1^2 s_2) - (s_1 s_2^3 s_1^2)\\ &= (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2).\end{aligned} \tag{31}$$

In like manner, making use of the identity

$$\overline{s_1 s_2} = (\overline{s_1 s_2}) + \sigma_1\sigma_2 + \sigma_2\sigma_1 - (s_1)\,\sigma_2 - (s_2)\,\sigma_1,$$

we obtain the apolarity condition of $[s_1^2 s_2^2]$ and $\overline{s_1 s_2}$ as the linear invariant of

$$(s_1^2 s_2^2 - s_2^2 s_1^2)\,(s_1 s_2 + s_2 s_1)$$

under the form

$$[s_1^2 s_2^2]\cdot\overline{s_1 s_2} = (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2). \tag{32}$$

Since $[s_1 s_2]$ is a normal collineation, the adjoined form is given by (17) as

$$\overline{[s_1 s_2]\,[s_1 s_2]} = (\overline{[s_1 s_2]\,[s_1 s_2]}) + 2[\sigma_1\sigma_2]^2.$$

The discriminant of $[s_1 s_2]$ is then

$$\begin{aligned}\Delta_{11} &= \tfrac{1}{6}\,[s_1 s_2]\cdot\overline{[s_1 s_2]\,[s_1 s_2]} = \tfrac{1}{3}\,[s_1 s_2]\cdot[\sigma_1\sigma_2]^2 = \tfrac{1}{3}\,([s_1 s_2]^3)\\ &= \tfrac{1}{3}\,\{(s_1 s_2 s_1 s_2 s_1 s_2) - (s_1 s_2 s_1 s_2^2 s_1) - (s_1 s_2^2 s_1^2 s_2) - (s_2 s_1^2 s_2 s_1 s_2)\\ &\qquad - (s_2 s_1 s_2 s_1 s_2 s_1) + (s_2 s_1 s_2 s_1^2 s_2) + (s_2 s_1^2 s_2^2 s_1) + (s_1 s_2^2 s_1 s_2 s_1)\}\\ &= (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2),\end{aligned} \tag{33}$$

in which the notation Δ_{11} is used to show that it is the discriminant of an alternant involving s_1 and s_2 each to the first degree.

A comparison of (30), (31) and (32) with (33) shows that the various invariants there considered are equal to each other and to the discriminant of $[s_1 s_2]$.

Now suppose that Δ_{11} does not vanish and that there exists a linear relation of the form

$$\rho_0\sigma_0 + \rho_1\sigma_1 + \rho_2\sigma_1^2 + \rho_3\sigma_2 + \rho_4\sigma_2^2 + \rho_5\overline{s_1 s_2} + \rho_6\overline{s_1^2 s_2} + \rho_7\overline{s_1 s_2^2} + \rho_8\overline{s_1^2 s_2^2} = 0. \tag{34}$$

Since, according to (29), $[s_1 s_2]$ is apolar to the first eight collineations in that sequence, but by (30) and (33) is not apolar to the last, it follows that ρ_8 is zero. Likewise, operating in turn with $[s_1^2 s_2]$, $[s_2^2 s_1]$ and $[s_1^2 s_2^2]$, we find that $\rho_7 = \rho_6 = \rho_5 = 0$. Hence the relation (34) must be of the form

$$\rho_0\sigma_0 + \rho_1\sigma_1 + \rho_2\sigma_1^2 + \rho_3\sigma_2 + \rho_4\sigma_2^2 = 0. \tag{35}$$

Replacing σ_i by s_i and forming the alternant with s, we obtain

$$\rho_3\,[s_1 s_2] + \rho_4\,[s_1 s_2^2] = 0. \tag{36}$$

Forming the bilinear invariant with $\overline{s_1^2 s_2^2}$, this gives

$$\rho_3 \Delta_{11} = 0.$$

Hence, $\rho_3 = 0$. Likewise on operating with $\overline{s_1^2 s_2}$, it is seen that $\rho_4 = 0$. Similarly, $\rho_2 = \rho_1 = \rho_0 = 0$.

Hence, if the discriminant Δ_{11} is different from zero, no linear relation of the type (34) can exist. If, however, the discriminant is zero, since the nine collineations are apolar to $[s_1 s_2]$, they must satisfy a linear relation. Therefore, *the vanishing of the discriminant of the alternant is the necessary and sufficient condition for the existence of a linear relation of the type* (34).

Since, in the usual case, the discriminant of the alternant is not zero, it follows that in general the apolarity conditions of (29) are independent and give an invariant determination of the alternant. When the discriminant is zero, all the collineations of the net

$$\lambda_1 [s_1 s_2] + \lambda_2 [s_1^2 s_2] + \lambda_3 [s_1 s_2^2] + \lambda_4 [s_1^2 s_2^2] \tag{37}$$

satisfy those conditions and the determination is not unique.

21. When the discriminant vanishes, there is always a collineation of (37) that vanishes identically; *i. e.*, the four alternants satisfy a linear relation. For, since all the collineations of (37) are apolar to all those in (34), it follows that the two sets must contain four linear relations. If one of these belongs to (37), the point at issue is settled. If not, there must be four equations of the type (34). Either one of those is of the form (35), and the collineation in question is (36); or it is possible to solve for one of the intermediates, and so obtain an equation

$$\overline{s_1 s_2} = \lambda_0 \sigma_0 + \lambda_1 \sigma_1 + \lambda_2 \sigma_1^2 + \lambda_3 \sigma_2 + \lambda_4 \sigma_2^2.$$

Developing $\overline{s_1 s_2}$ by (17), inverting, and forming the alternant with s_1, gives

$$[s_1^2 s_2] - (s_1)[s_1 s_2] = \lambda_3 [s_1 s_2] + \lambda_4 [s_1 s_2^2],$$

which is the relation desired.

Suppose, conversely, that the four alternants satisfy a linear relation

$$\lambda_1 [s_1 s_2] + \lambda_2 [s_1^2 s_2] + \lambda_3 [s_1 s_2^2] + \lambda_4 [s_1^2 s_2^2] = 0.$$

In this equation there must be at least one coefficient, for instance λ_1, that is different from zero. Operating on the equation with $\overline{s_1^2 s_2^2}$, we see that Δ_{11} must then vanish. Therefore, the vanishing of the discriminant is the necessary and sufficient condition for a linear relation between the four alternants.

22. The symmetry resulting when Δ_{11} vanishes suggests that it is an invariant common to the four alternants. In order to prove that such is the

case, take in the first instance the alternant $[s_1^2 s_2]$. According to (33), the discriminant has the form

$$\Delta_{21} = (s_1^2 s_2 s_1^4 s_2^2) - (s_2 s_1^2 s_2^2 s_1^4).$$

From the characteristic equation for s_1 we have

$$s_1^4 = \lambda_0 + \lambda_1 s_1 + \lambda_2 s_1^2.$$

Substituting this value for s_1^4, we have

$$\Delta_{21} = \lambda_1 \{ (s_1^2 s_2 s_1 s_2^2) - (s_2 s_1^2 s_2^2 s_1) \} = -\lambda_1 \Delta_{11}. \tag{38}$$

Making use of

$$s_2^4 = \mu_0 + \mu_1 s_2 + \mu_2 s_2^2$$

and following out the same argument, we obtain the discriminants of $[s_1 s_2^2]$ and $[s_1^2 s_2^2]$ in the form

$$\left.\begin{aligned} \Delta_{12} &= -\mu_1 \Delta_{11}, \\ \Delta_{22} &= \lambda_1 \mu_1 \Delta_{11}. \end{aligned}\right\} \tag{39}$$

If λ_1 is zero, the characteristic equation for s_1^2 is

$$\{s_1^2\}^2 = \lambda_0 + \lambda_2 s_1^2.$$

The equation is quadratic, and hence the collineation is a perspectivity. Therefore, λ_1 and μ_1 are respectively the invariants whose vanishing expresses that s_1^2 and s_2^2 are perspectivities. From (38) and (39) we see, then, that *the alternant of a perspectivity and any collineation is singular.*

If λ_1 and μ_1 are not zero, Δ_{11} is a combinant of the two systems

$$\lambda_0 s_0 + \lambda_1 s_1 + \lambda_2 s_1^2 \quad \text{and} \quad \mu_0 s_0 + \mu_1 s_2 + \mu_2 s_2^2.$$

It must, then, express a property of the fixed triangles of those systems. What that property is, we shall see later.

23. The alternant $[s_1 s_2]$ is a combinant of the net

$$\lambda s_0 + \mu s_1 + \nu s_2. \tag{40}$$

In forming it any two independent collineations may then be chosen. Let s be a singular collineation, x its singular point, and ξ its singular line. The alternant can be written

$$s s_1 - s_1 s,$$

where s_1 is some other collineation of the net. Since sx is zero, the correspondent of x with respect to the alternant is

$$x' = s s_1 x.$$

Now, s transforms every point (and in particular $s_1 x$) into a point on ξ. Therefore, the alternant transforms the singular point of any collineation of (40) into a point of the associated singular line. For varying λ, the

singular points and lines obtained are the fixed points and associated fixed lines in the collineation $\mu s_1 + \nu s_2$. Therefore, *the fixed triangles of all the collineations of the pencil* $\mu s_1 + \nu s_2$ *are Pasch triangles of the alternant.*

The significance of the apolarity relations satisfied by the alternant is here suggested. In fact, two collineations (one in points, the other in lines) that send a triangle into a pair of incident triangles are apolar. Now, if s is any collineation in (40), the triangle that s leaves fixed is sent by the alternant into an inscribed triangle. Therefore, the alternant is apolar to s and to all of its covariants.

Two collineations have in general a common pair of polar triangles. In terms of these they may be written

$$s_1 = \lambda_1 b_1 \alpha_1 + \lambda_2 b_2 \alpha_2 + \lambda_3 b_3 \alpha_3,$$
$$s_2 = \mu_1 b_1 \alpha_1 + \mu_2 b_2 \alpha_2 + \mu_3 b_3 \alpha_3.$$

Their adjoined forms and intermediate are respectively

$$\overline{s_1 s_1} = 2\{\lambda_1 \lambda_2 \beta_3 a_3 + \lambda_2 \lambda_3 \beta_1 a_1 + \lambda_3 \lambda_1 \beta_2 a_2\},$$
$$\overline{s_2 s_2} = 2\{\mu_1 \mu_2 \beta_3 a_3 + \mu_2 \mu_3 \beta_1 a_1 + \mu_3 \mu_1 \beta_2 a_2\},$$
$$\overline{s_1 s_2} = (\lambda_1 \mu_2 + \lambda_2 \mu_1)\beta_3 a_3 + (\lambda_2 \mu_3 + \lambda_3 \mu_2)\beta_1 a_1 + (\lambda_3 \mu_1 + \lambda_1 \mu_3)\beta_2 a_2.$$

Hence, $\overline{s_1 s_1}$, $\overline{s_2 s_2}$ and $\overline{s_1 s_2}$ have a common pair of polar triangles, *i. e.*, the common pair of s_1 and s_2 considered contragrediently. Since $[s_1 s_2]$ is apolar to each of those collineations, according to Art. 10, it transforms the triad a_i into one incident to β_i. Taking any two collineations and the associated intermediate belonging to (29), we obtain a pair of polar triangles such that the points of the first pass by the alternant into a triad incident to the second. These relations are therefore the geometric equivalent of the eight apolarity conditions.

24. In Arts. 20–22 the entire theory seemed to hinge on the vanishing or non-vanishing of the discriminant of the alternant. It is our purpose, in the next place, to consider the geometrical interpretation of that invariant.

For that purpose suppose a polarity c^2 to be apolar to s_1 and s_2. Designating the collineations respectively by $a\alpha$ and $b\beta$, the conditions required are

$$\overline{ac}\,(\alpha c) = \overline{bc}\,(\beta c) = 0. \tag{41}$$

There are six equations in all, three of them linear in the coefficients of $a\alpha$, three linear in the coefficients of $b\beta$, and all linear in the coefficients of c^2. Therefore, if we eliminate the coefficients of c^2 from these equations, we get an invariant of the third degree in the coefficients of $a\alpha$ and $b\beta$ whose vanishing is the necessary and sufficient condition for the equations (41).

Now, we have seen that a polarity apolar to a collineation is apolar to all of its covariants, and that a correlation apolar to two collineations is apolar to their intermediate. Therefore, c^2 is apolar to the following nine collineations:

$$\sigma_0, \ \sigma_1, \ \sigma_1^2, \ \sigma_2, \ \sigma_2^2, \ \overline{s_1 s_2}, \ \overline{s_1^2 s_2}, \ \overline{s_1 s_2^2}, \ \overline{s_1^2 s_2^2}. \tag{42}$$

Since the nine collineations are apolar to the same polarity, they must satisfy a linear relation; in fact, must satisfy three linear relations. Therefore, according to Art. 20, the discriminant of the alternant is zero. But the discriminant is of the third degree in the coefficients of $a\alpha$ and $b\beta$. Therefore, the vanishing of the discriminant of the alternant is the necessary and sufficient condition for two collineations to have a common apolar polarity.

If a polarity is apolar to a collineation, there are two cases to be considered, according as the polarity is singular or is not.

If the polarity is singular, either it consists of the square of a point or its adjoined form consists of the square of a line. In both cases, if the polarity is apolar to a collineation, its fixed triangle contains the double element. Hence, if a singular polarity is apolar to each of two collineations, their fixed triangles have an element in common. In that case all the collineations of (42) have a fixed point or line in common.

In general, however, if a polarity is apolar to a collineation, the fixed triangle (or, a fixed triangle) of the collineation is self-conjugate with respect to the polarity. If two triangles are self-conjugate with respect to the same polarity, they lie on a conic. And, conversely, if two fixed triangles lie on a conic, they are self-conjugate with respect to a polarity, which is consequently apolar to their associated collineations. Therefore, we see that *the vanishing of the discriminant of the alternant is the necessary and sufficient condition for any two collineations formed linearly from those of* (42) *to have fixed triangles lying on a conic.*

25. As we have seen, the conditions for a polarity c^2 apolar to $a\alpha$ and $b\beta$ are

$$(\alpha c)\,\overline{ac} = (\beta c)\,\overline{bc} = 0.$$

Multiplying these equations by $b\beta$ and $a\alpha$ and writing in the variables to avoid confusion, we have

$$(\alpha c)\,\{(\beta c)(a\xi) - (\beta a)(c\xi)\}\,(b\eta) \equiv 0,$$
$$(\beta c)\,\{(\alpha c)(b\xi) - (\alpha b)(c\xi)\}\,(a\eta) \equiv 0.$$

If ξ and η are equal, by subtraction we get

$$\{(\alpha b)(\beta c)(a\xi) - (\beta a)(\alpha c)(b\xi)\}\,(c\xi) \equiv 0. \tag{43}$$

Comparing this with the form

$$\{(\alpha b)\, a\beta - (\beta a)\, b\,\alpha\}\, c^2,$$

we see that in the present case the product of the alternant with c^2 is a correlation having no particular coincidence conic, *i. e.*, a null-system or line.

Suppose for the moment we designate the alternant by $d\delta$. The equation (43) is then

$$(d\xi)(\delta c)(c\xi) = 0, \tag{44}$$

where ξ is any line whatever. Let η be a fixed line of the alternant. We then have

$$(d\eta)\,\delta = \lambda\eta,$$

which, substituted in the preceding equation, gives

$$(d\eta)(\delta c)(c\eta) = \lambda(\eta c)^2 = 0.$$

Hence, the fixed lines of the alternant touch the conic of c^2.

If again we follow out the argument of this article with γ^2, the double product of c^2 with itself, we obtain, as correlative to (44),

$$(\gamma d)(x\delta)(x\gamma) = 0, \tag{45}$$

where x is any point whatever. Taking x as a fixed point of the alternant, it follows as before that the fixed lines of the alternant touch γ^2.

Now, we have found that when c^2 is apolar to s_1 and s_2, it is apolar to a set of nine covariants. Therefore, *all the alternants that can be formed of those covariants are singular, all their fixed lines are tangent to, and all their fixed points lie on,* c^2.

It was shown by Study that the fixed points of the alternant of two binary collineations consist in the common harmonic pair of the fixed points of those collineations. The fixed triangles of two ternary collineations are not in general polar with respect to a conic. If such is, however, the case, we have just seen that the fixed lines of the alternant are tangent to, and the fixed points lie on, that conic.

A Geometrical Application of the Theory of the Binary Quintic.

By Florence P. Lewis.

Introduction.

If five points on a non-degenerate conic are given, certain curves arise from the polarization of the binary quintic determined by them. These are covariant curves of the five points. It is the object of this paper to discuss certain of these curves, and sets of points and lines associated with them. Similar processes may be applied to the quintic of five points on the cubic norm curve in space. A brief treatment of the covariant forms of the binary quintic from this standpoint is given in § 7, and the results are correlated as far as possible with those already obtained in the plane.

For the algebraic theory of the quintic, reference is made to Salmon's "Higher Algebra" and to "Algebra of Invariants" by Grace and Young. The notations of both works are used.

§ 1. *A Pencil of Conics.*

Let the quintic be represented on the base conic N, and let the reference triangle be formed by the tangents to N at the points given by the canonizant $C_{3,3}=0$. The quintic may then be written in Salmon's canonical form,

$$f\equiv a_0t^5+a_1(1-t)^5-a_2, \tag{1}$$

and the conic in the form

$$x_0=t^2, \qquad x_1=(1-t)^2, \qquad x_2=1. \tag{2}$$

Any point of the plane, in terms of the parameters of points of contact of tangents from it to N, is

$$x_0=t_1t_2, \qquad x_1=(1-t_1)(1-t_2), \qquad x_2=1; \tag{3}$$

and any line of the plane, in terms of its intersections with N, is

$$\xi_0=t_1+t_2-2, \qquad \xi_1=-(t_1+t_2), \qquad \xi_2=t_1+t_2-2t_1t_2. \tag{4}$$

The point $(1, 1, 1)$ is given by the Hessian of the canonizant, the $C_{6,2}\equiv a_0^2a_1^2a_2^2\{t^2+(1-t)^2+1\}$ of f. Every quadratic determines at once a point and a line of the plane, these being pole and polar with respect to N.

If the quintic be polarized thus:

$$a_0 t_1 t_2^2 t_3^2 + a_1(1-t_1)(1-t_2)^2(1-t_3)^2 - a_2 = 0, \tag{5}$$

the locus of the intersection of tangents at t_2 and t_3 for a fixed t_1 is a conic whose equation is

$$a_0 t_1 x_0^2 + a_1(1-t_1)x_1^2 - a_2 x_2^2 = 0. \tag{6}$$

By varying t_1 we obtain a pencil of conics apolar to N (in lines) and on the four points $\pm\sqrt{a_0^{-1}}$, $\pm\sqrt{a_1^{-1}}$, $\pm\sqrt{a_2^{-1}}$. These four points are a set orthic to N. When $t_2 = t_3$, the conic t_1 meets N, and the points of intersection are given by the quartic polar of t_1 as to f. t_1 is a root of $f=0$ when, and only when, point t_1 of N is on conic t_1 of the pencil. If t_1 is a root of the canonizant (*i. e.*, $t_1 = 0, 1, \infty$), the corresponding conic degenerates into lines meeting at a reference point; hence, the reference triangle ABC (Fig. 1) is the diagonal triangle

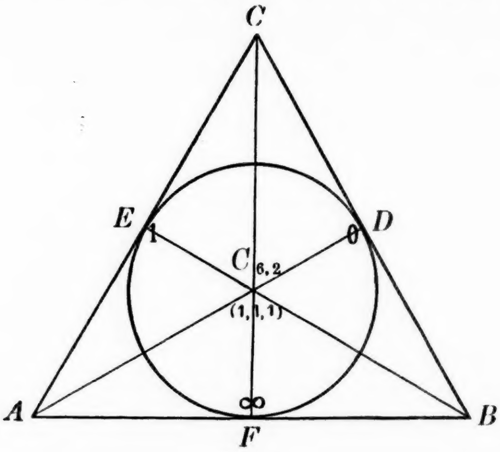

FIG. 1.

of the base-points of the pencil. Since the roots of the Hessian of f are double points of quartic polars, there are six members of the pencil tangent to N, and their points of contact are the points $H=0$. No member has double contact unless an invariant of f vanishes. Two values of t_1 (given by $C_{2,2}=0$) have quartic polars which are self-apolar; the corresponding conics are apolar to N both in points and in lines.*

Conics of the pencil and points of N are in one-to-one correspondence. We can construct geometrically the point corresponding to a given conic, and *vice versa*. The point $(1, 1, 1)$ is given by the quadratic $C_{6,2}=0$; its polar lines as to all conics of the pencil pass through the point $Q(a_0^{-1}, a_1^{-1}, a_2^{-1})$ given by the quadratic $C_{2,2} \equiv a_1 a_2(1-t) + a_0 a_2 t - a_0 a_1 t(1-t) = 0$. The polar of Q as to any member of the pencil therefore passes through $(1, 1, 1)$, and its coordinates are

* W. F. Meyer, "Apolarität und rationale Curven," p. 150.

$$\eta_0=t_1, \qquad \eta_1=1-t_1, \qquad \eta_2=1. \qquad (7)$$

If the conic t_1 is given, we construct this line η and take its polar point with respect to the triangle DEF whose vertices are the canonizant points on N. This point is t_1. The proof follows at once from consideration of the degenerate members. For, it has been seen that the t_1's of the degenerate conics are 0, 1, ∞, which give the points D, E, F on N; and the polar of Q as to the degenerate conic on A is the line $A-(1, 1, 1)-D$, whose polar point as to the triangle DEF is D. By the fundamental theorem on projective correspondences, since the construction holds for three members of the pencil it holds for all. It is obviously reversible. The points $C_{2,2}=0$ and $C_{6,2}=0$ are mutually related to the pencil.

The above correspondence of point and conic is easily verified analytically.

§ 2. *Rational Quartic K of Class 3.*

Let f be polarized thus:

$$a_0t_1^3t^2+a_1(1-t_1)^3(1-t)^2-a_2=0, \qquad (8)$$

and consider the locus of the line joining the roots of the quadratic in t. We obtain a rational line cubic K whose equations are

$$\xi_0=a_0t_1^3, \qquad \xi_1=a_1(1-t_1)^3, \qquad \xi_2=-a_2. \qquad (9)$$

ξ passes through a given point $x_0=t_2t_3$, $x_1=(1-t_2)(1-t_3)$, $x_2=1$, provided t_1 satisfies

$$a_0t^3t_2t_3+a_1(1-t)^3(1-t_2)(1-t_3)-a_2=0. \qquad (10)$$

This is the cubic polar of t_2t_3 as to f, and cubic polars are all cubics apolar to the canonizant. Hence, the necessary and sufficient condition that three t's be parameters of lines on a point is that they be apolar to $C_{3,3}$. The curve has three cusps; for, if t_2, t_3 be taken as two roots of $C_{3,3}=0$, the cubic polar becomes a perfect cube, the cube of the third root. Hence the points A, B, C are the cusps. Since $C_{3,3}$ is the cubic polar of $C_{2,2}$, the cusp tangents meet at $C_{2,2}=0$ or Q of § 1. For a given t_1 the line ξ is identical with the polar of point t_1 on N as to the corresponding member of the pencil (6).

By differentiating the functions ξ of (9) we obtain the equations of K as a point quartic:

$$x_0=a_1a_2(1-t)^2, \qquad x_1=a_0a_2t^2, \qquad x_2=a_0a_1t^2(1-t)^2. \qquad (11)$$

The line equations of N are

$$\xi_0=(1-t), \qquad \xi_1=t, \qquad \xi_2=-t(1-t). \qquad (12)$$

Hence, the quadratic of points of contact of tangents from a given point t_1 of K to N is:

$$a_1a_2(1-t_1)^2(1-t)+a_0a_2t_1^2t-a_0a_1t_1^2(1-t_1)^2t(1-t)=0. \qquad (13)$$

This is the Hessian of the cubic polar of t_1 as to f; namely,

$$a_0 t_1^2 t^3 + a_1(1-t_1)^2(1-t)^3 - a_2.$$

The curve K may therefore be regarded as produced from N by a transformation T which sends the point t of N into the intersection of tangants at the Hessian of the cubic polar of t as to f. Hence, to find the parameters of the intersections of K and N, we ask for a point t_1 such that the Hessian of its cubic polar is a perfect square; that is, we look for $t_1 t_2$ such that $t_1^2 t_2^2$ is a quartic apolar to f. If f be written symbolically, this covariant is found by elimination of t_1 between $(\alpha t_1)^3(\alpha t)^2=0$ and $(\alpha t_1)^2(\alpha t)^3=0$. f factors out of the eliminant, and the remaining factor is a $C_{4,8}$. Now the Hessian of the cubic polar $(\alpha t_1)^2(\alpha t)^3$ is $|\alpha\beta|^2(\alpha t_1)^2(\beta t_1)^2(\alpha t)(\beta t)$. Equated to zero, this function gives a convenient representation of the curve, t_1 being the parameter and a point of the curve being given by the two values of t. K cuts N when the two t's are equal. This gives

$$C_{4,8} \equiv |\alpha\beta|^2|\gamma\delta|^2|\alpha\gamma|\,|\beta\delta|\,(\alpha t)^2(\beta t)^2(\gamma t)^2(\delta t)^2=0, \tag{14}$$

where all of the symbols refer to the quintic. The roots of $C_{4,8}$ occur in four pairs, and the square of each pair is a quartic apolar to f. Taking t_2 and t_3 of (5) as a pair, it is evident that the points of the plane determined by these four quadratics are the base-points of the pencil (6).

The common lines of K and N have as parameters the six Hessian points of f, and their points of contact with N are the roots of the Steinerian. This follows at once from the fact that the lines of K are cut out by quadratic polars of points t as to f. To find the double line of K, we recall that parameters of lines on a point form cubics apolar to $C_{3,3}$; hence, the parameters of the double line are given by the Hessian of $C_{3,3}$; *i. e.*, $C_{6,2} \equiv a_0^2 a_1^2 a_2^2(t^2-t+1)=0$. A root of this is $-\omega$, and the corresponding line of K is the line (a_0, a_1, a_2). Hence, *the double line of K is the polar line of $C_{2,2}$ as to the triangle ABC.*

K in x coordinates is

$$\sum^3 a_1^2 a_2^2 x_1^2 x_2^2 - 2\sum^3 a_0^2 a_1 a_2 x_0^2 x_1 x_2 = 0. \tag{15}$$

§ 3. *The Involutory Quadratic Transformation T.*

Mention was made in the preceding section of a point transformation T which transforms N into the quartic K. This transformation may be arrived at in three ways. It is:

(a) The transformation which sends any point x of the plane into the point y given by the Hessian of the cubic polar (as to f) of the quadratic of x.

(b) The transformation which sends x into a point y such that the quadratics of x and y from a quartic apolar to f.

(c) The transformation which sends x into the intersection of its polar lines as to the pencil (6) of § 1.

That (a) and (b) are equivalent follows from the fact that a quadratic and the Hessian of its cubic polar as to f obviously form a quartic apolar to f, and that when two roots of an apolar quartic (which are not roots of $C_{3,3}=0$) are given, the remaining two are uniquely determined. The matter may be stated thus: If, of two quadratics Q_1 and Q_2, Q_2 is the Hessian of the cubic polar of Q_1 as to a quintic f, then Q_1 is a factor of the Hessian of the cubic polar of Q_2. As an instance, we note that the cubic polar of $C_{2,2}$ is $C_{3,3}$, whose Hessian is $C_{6,2}$. The cubic polar of $C_{6,2}$ is $C_{7,3}$, whose Hessian must therefore contain $C_{2,2}$ as a factor.

To show that (a) and (c) are equivalent, we derive a formula for (c). The conic (6), when polarized, gives

$$a_0t_1x_0y_0+a_1(1-t_1)x_1y_1-a_2x_2y_2=0. \tag{16}$$

If y be given, this line for all t_1's passes through $(a_iy_i)^{-1}$, $i=0, 1, 2$. Hence, (c) is the transformation

$$x_i=(a_iy_i)^{-1}, \qquad i=0, 1, 2. \tag{17}$$

Now let t_1t_2 be roots of the quadratic of x, and form the cubic polar $a_0t_1t_2t^3+a_1(1-t_1)(1-t_2)(1-t)^3-a_2$ and its Hessian $a_1a_2\,(1-t_1)(1-t_2)(1-t)+a_0a_2t_1t_2t-a_0a_1t_1t_2t(1-t_1)(1-t_2)(1-t)$. If the roots of this quadratic in t are t_3 and t_4, we find as the corresponding point of the plane

$$y_0=t_3t_4=a_1a_2(1-t_1)(1-t_2),$$
$$y_1=(1-t_3)(1-t_4)=a_0a_2t_1t_2,$$
$$y_2=a_0a_1t_1t_2(1-t_1)(1-t_2).$$

Using (3), this reduces to $y_0=a_1a_2x_1x_2$, $y_1=a_0a_2x_0x_2$, $y_2=a_0a_1x_0x_1$, which is identical with the form (17).

From (c), the fixed points of T are the base-points of pencil (6). From (b), a fixed point determines a quadratic t_1t_2 such that $t_1^2t_2^2$ is apolar to f. It was shown in § 2 that t_1, t_2 are then intersections of K with N; that is, K cuts N at the points of contact of tangents to N from the fixed points of T. That A, B, C are the singular points of T is obvious from the fact that $C_{3,3}$ (whose roots taken in pairs give A, B, C) is apolar to f. For, if t_1, t_2, t_3 are roots of $C_{3,3}=0$, $t_1t_2t_3t$ is apolar to f for an arbitrary t; that is, the point A transforms into any point of the line BC. From (c), it may be seen that corresponding points x, y as to T are base-points of the involution cut on the line $\overline{xy}$ by the conics (6).

§ 4. *Rational Quintic O and a Theorem Concerning its Double Points.*

If we polarize f as follows:

$$a_0t_1^4t+a_1(1-t_1)^4(1-t)-a_2=0, \tag{18}$$

and consider the locus of the intersection of tangents to N at t_1 and t, we obtain a rational curve with parameter t_1 of order 5 and class 8. This quintic O is the locus of the intersection of the tangent to N at t_1 with the corresponding line of K; for, the t_1 and t which satisfy $(\alpha t_1)^4(\alpha t)=0$, (18), are apolar to the quadratic $(\alpha t_1)^3(\alpha t)^2$ (8) which determines the line t_1 of K. When t_1 is a root of $f=0$, the equations $(\alpha t_1)^4(\alpha t)=0$ and $(\alpha t_1)(\alpha t)^4=0$ are both satisfied by t_1; hence, two points of O on the tangent to N at t_1 have come into coincidence at t_1, and *O is therefore five times tangent to N at the points $f=0$.* This accounts for ten intersections and ten common lines. To find the remaining six common lines, let h and s be a pair of corresponding roots of Hessian and Steinerian of f, and let the tangents at these points on N meet at R. Since the quartic polar of s as to f has a double root at h, Rs is tangent to O at R, and the six lines are accounted for. Moreover, since h^3s is a quartic apolar to f, the point h is transformed into R by T (using § 3,(b)), and R is a point of K. It has been seen (§ 2) that Rs is the tangent to K at R. *O and K are therefore six times tangent at the points R, and their common tangents at these points are tangent to N.*

O has six double points with parameters which are pairs of solutions of $(\alpha t_1)^4(\alpha t)=0$ and $(\alpha t_1)(\alpha t)^4=0$. Elimination of t_1 gives a $C_{5,17}$ which obviously contains f as a factor. The remaining factor $C_{4,12}$ is the product of the six quadratics which determine the points. Since the number of constants used to determine them is not twelve but ten, this set of six points are subject to two invariant conditions. Concerning them we will prove the following theorem:

If P is a double point of O, there exists a point γ on N and on line PQ, such that the conic C_γ of (6) *is tangent to PQ at P, and the line η_γ* (7) *passes through P. γ is the polar of the two parameters of P as to $C_{3,3}$ and satisfies a certain $C_{6,6}=0$.* (Q is point $C_{2,2}=0$, as in § 1.) (19)

To prove this, let t_1, t_2 be the parameters of P, and let γ be determined as the polar of t_1t_2 with respect to $C_{3,3}$. Since t_1 and t_2 are a pair of solutions of $(\alpha t_1)^4(\alpha t_2)=0$ and $(\alpha t_2)^4(\alpha t_1)=0$, the cubic polar of t_1t_2 as to f is $t_1t_2\gamma$; that is, the lines of K having parameters t_1, t_2, γ pass through P. Since the cubic $t_1t_2\gamma$ is apolar to itself, we have $t_1t_2t_1t_2\gamma$ apolar to f, or $(\alpha t_1)^2(\alpha t_2)^2(\alpha t)=0$ when $t=\gamma$. But this is the condition that the conic C_γ should pass through P. From the fact that the cubic polar $t_1t_2\gamma$ is apolar to $C_{3,3}$, it follows that η_γ (which is the polar line of γ as to triangle DEF) passes through P. Now η_γ is the polar of Q as to conic C_γ; and line K_γ (which has been shown to pass

through P) is the polar of γ as to C_γ (§ 2). Hence, the points Q and γ are on the tangent to C_γ at P, which establishes the theorem.*

The six parameters γ are roots of a covariant $C_{6,6} \equiv C_{5,1}C_{1,5} - C_{3,3}^2$. This follows immediately from the theorem, as will be shown in the following section.

§ 5. *Curves on the Double Points of O: Cubic Ω and Rational Cubic Ψ.*

The theorem (19) of the preceding section gives certain curves on the double points P of O and having interesting relations to the system of covariants of f.

Since the points P are points of contact of tangents from $C_{2,2}$ to members of pencil (6), consider the locus of such points. This is a non-rational cubic curve Ω on the four base-points of the pencil and tangent at these points to the lines joining them to $C_{2,2}$, which latter point is on the curve.† Ω is apolar to N and cuts it in $C_{4,6} \equiv [f, C_{3,3}]^1$. Its equation is

$$\sum^{3} a_0 x_0^2 (x_1 - x_2) = 0. \tag{20}$$

The curve passes through A, B, C; the tangents at these points and the tangent at $C_{2,2}$ meet at $C_{6,2}$ on the curve. On any line through $C_{2,2}$ there are two points of the curve which by the transformation T are interchanged (§ 3,(c)); hence, Ω is by T transformed into itself. $C_{2,2}$ and $C_{6,2}$ are symmetrically related to the pencil. Interchanging their rôles, we have a second cubic on $A, B, C, C_{6,2}, C_{2,2}$ and the four base-points; these nine points are therefore the base-points of a pencil of cubics each of which transforms into itself by T.

For convenience, the coordinates of certain lines which are readily expressed in terms of the parameter t of a point S on N are given here:

$$\left.\begin{array}{llll}
(1)\ \text{Tangent to } N: & 1-t, & t, & -t(1-t). \\
(2)\ \text{Line } SC_{2,2}: & \dfrac{(1-t)^2}{a_2} - \dfrac{1}{a_1}, & \dfrac{1}{a_0} - \dfrac{t^2}{a_2}, & \dfrac{t^2}{a_1} - \dfrac{(1-t)^2}{a_0}. \\
(3)\ \text{Line } \eta: & t, & 1-t, & -1. \\
(4)\ \text{Line } t \text{ of } K: & a_0 t^3, & a_1(1-t)^3, & -a_2. \\
(5)\ \text{Line } \zeta: & a_0 t, & a_1(1-t), & -a_2.
\end{array}\right\} \tag{21}$$

η, it will be recalled, is the polar of S as to triangle DEF, and the polar of $C_{2,2}$ as to conic t of pencil (6); ζ is the polar of $C_{6,2}$ as to the same conic. The

* In the above discussion we considered a pencil of conics correlated to points of a conic N. Joining points of N to $Q(C_{2,2}=0)$, we have a (1, 2) correspondence of conics of the pencil and lines on $C_{2,2}$. When a conic touches its line, the point of contact is one of the six points P. When $C_{2,2}$ is replaced by an arbitrary fixed point A of the plane, the six points of contact are (presumably) a general set of six, since the number of their coordinates is now 12. Placing A at $C_{2,2}$ produces the specialization peculiar to the double points of O.

† Salmon-Fiedler, "Geometrie der höheren ebenen Kurven," p. 269.

equation of Ω is obtained by eliminating t between the equation of η and that of conic t of (6).

Any two of the above lines intersect in a point whose locus is a rational curve. The following combinations are noteworthy:

(a) (1)(2): Conic N.
(b) (1)(4): Quintic O.
(c) (2)(3): Cubic Ψ with double point at $C_{6,2}$.
(d) (2)(4): A quintic with triple point at $C_{2,2}$.
(e) (3)(4): Quartic Θ with triple point at $C_{6,2}$.
(f) (3)(5): Conic F on $A, B, C, C_{2,2}$ and $C_{6,2}$.

That (c), (d) and (e) pass through the double points P of O follows at once from the theorem (19). When t is γ of the theorem, the lines (2), (3) and (4) meet at P; hence, the determinant of the coordinates (2), (3) and (4) is the sextic whose roots are the six γ's. This sextic is reducible to $C_{6,6} \equiv C_{5,1}C_{1,5} - C^2_{3,3}$.*

The cubic Ψ is defined by lines (2) and (3), from which its parametric equations are at once obtained. Elimination of t gives

$$\sum^{3} a_0^{-1}\{x_1(x_0-x_1)^2 - x_2(x_0-x_2)^2\} = 0. \qquad (22)$$

Since η always passes through $C_{6,2}$, and line (2) passes through this point when t satisfies $[C_{6,2}, C_{2,2}]^1 = C_{8,2} = 0$, Ψ has a double point at $C_{6,2}$ with the parameters $C_{8,2}=0$. It cut N in $C_{3,3}=0$ (the points D, E, F) and in $C_{5,3} = [C_{2,2}, C_{3,3}]^1 = 0$. η passes through $C_{2,2}$ when t satisfies $[C_{2,2}, C_{3,3}]^2 = C_{5,1} = 0$. Hence, Ψ passes through $C_{2,2}$ with $C_{5,1}$ as parameter. Ω and Ψ have three intersections at $C_{6,2}$ and $C_{2,2}$, and their remaining intersections are the six points P. Ω and a given point $C_{2,2}$ on it determine these points, since $C_{6,2}$ and A, B, C are thereby given; varying the position of $C_{2,2}$ on the curve gives ∞^1 such sets.

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* Let f be represented by $C_{1,5} = (at)^5 = (\epsilon t)^5 = (\delta t)^5 = (\zeta t)^5$. Then $C_{2,2} = |a\delta|^4 (at)(\delta t)$. Let the canonizant be represented by $(\beta t)^3$. Then $(\beta t)^3 = C_{3,3} = [C_{2,2}, C_{1,5}]^2 = |a\delta|^4 |a\zeta||\delta\zeta|(\zeta t)^3$. $C_{5,1} = [C_{3,3}, C_{2,2}]^2 = [(\beta t)^3, |a\delta|^4 (at)(\delta t)]^2 = |a\delta|^4 |a\beta||\delta\beta|(\beta t)$. Also let $(\gamma t_1) = \gamma_0 + \gamma_1 t_1 = 0$, γ being real. The three lines η, K and $SC_{2,2}$ cut from N the three quadratics $(\beta t_1)(\beta t)^2 = 0$, $(at_1)^3 (at)^2 = 0$ and $|a\delta|^4 (at_1)(\delta t)(\gamma t) = 0$ respectively. The three lines meet in a point when the three quadratics are in involution; *i.e.*, when the determinant of the coefficients of t vanishes. The condition to be satisfied by t_1 is

$$\begin{array}{r} (\beta t_1) \\ (at_1)^3 \\ |\zeta\delta|^4(\delta t_1) \end{array} \begin{vmatrix} \beta_0^2 & 2\beta_0\beta_1 & \beta_1^2 \\ a_0^2 & 2a_0a_1 & a_1^2 \\ \zeta_0\gamma_0 & \zeta_0\gamma_1 + \zeta_1\gamma_0 & \zeta_1\gamma_1 \end{vmatrix} = 0.$$

Since $(\gamma t_1) = 0$, $|\beta\gamma| = (\beta t_1)$ and $|a\gamma| = (at_1)$. Expanding the determinant and dropping the subscript from t_1,

$$|\zeta\delta|^4 (at)^3 (\beta t)(\delta t)|a\beta|\{|\beta\gamma||\zeta a| + |a\gamma||\zeta\beta|\} = 0.$$

Using the identity $|a\beta|(\delta t) \equiv |a\delta|(\beta t) - |\beta\delta|(at)$,

$$|\zeta\delta|^4 (at)^3 |a\delta||\zeta a|(\beta t)^3 + |\zeta\delta|^4 |a\delta||\zeta\beta|(at)^4(\beta t)^2 - |\zeta\delta|^4|\beta\delta||\zeta a|(at)^4(\beta t)^2 - |\zeta\delta|^4|\beta\delta||\zeta\beta|(\beta t)(at)^5 = 0.$$

Since $\zeta \equiv \delta$, the second and third terms cancel. The first and last terms give $C^2_{3,3} - C_{5,1}C_{1,5} = 0$.

The locus of points of contact of tangents from S to the corresponding conic of (6) is a septimic on the points P. Its equation is

$$\sum^{3} a_0 x_0 (a_2 x_2^2 - a_1 x_1^2)^3 = 0,$$

gotten by eliminating t from (6) of § 1 and line (4) of the present section.

§ 6. *Quartic Θ with Triple Point.*

The locus of the intersection of lines (3) and (4) of § 5 (21) is a rational quartic Θ. It has certain projective peculiarities which will be discussed.

The parametric equations of Θ are:

$$\left.\begin{aligned} x_0 &= (1-t)\,[a_1(1-t)^2 - a_2], \\ x_1 &= t\,[a_2 - a_0 t^2], \\ x_2 &= t(1-t)\,[a_1(1-t)^2 - a_0 t^2]. \end{aligned}\right\} \tag{23}$$

Elimination of t gives the form

$$\sum^{3} a_0 x_0 (x_2 - x_1)^3 = 0. \tag{24}$$

To obtain a symbolic expression, we recall that line η cuts N in the quadratic polar of t as to $C_{3,3}$; line K cuts from N the quadratic polar of t as to f (§ 2). If we write $C_{3,3} \equiv (\zeta t)^3$ and $f \equiv (\alpha t)^5 \equiv (\beta t)^5 \equiv (\gamma t)^5 \equiv (\delta t)^5$, the point x of the curve is given by the Jacobian of the quadratics in t', $(\zeta t)(\zeta t')^2$ and $(\delta t)^3 (\delta t')^2$; that is, $|\zeta\delta|\,(\zeta t)(\delta t)^3 (\zeta t')(\delta t') = 0$. Since $C_{3,3} \equiv |\alpha\beta|^4 |\alpha\gamma|\,|\beta\gamma|\,(\gamma t)^3$, this becomes $|\alpha\beta|^4 |\alpha\gamma|\,|\beta\gamma|\,|\delta\gamma|\,(\delta t)^3 (\gamma t)(\delta t')(\gamma t') = 0$, where t is the parameter and the two t''s are points on N whose tangents meet at point t of the quartic. η always passes through $C_{6,2}$; line K passes through this point when $[C_{6,2}, f]^2 \equiv C_{7,3} = 0$. Hence, the curve has a triple point at $C_{6,2}$ with parameters $C_{7,3} = 0$. The tangents at the triple point are the lines η having these parameters.

The vertices of the reference triangle ABC are on the curve, and its sides make sections which break into harmonic pairs.* This is seen at once from the equations (23). It is natural to ask for the number of triangles thus related to the curve. The contravariant of the general ternary quartic $(\alpha x)^4$ which, when equated to zero, gives the locus of lines making sections such that $g_3 = 0$, is $|\alpha\beta\xi|^2 |\beta\gamma\xi|^2 |\gamma\alpha\xi|^2 = 0$.† When the quartic has a triple point, this breaks into a cubic Φ and the triple point counted three times. In the present case, the question asked above is most conveniently treated by the use of the polar conic of a point on the curve, as will now be shown.

The necessary and sufficient condition that a binary quartic with distinct roots have $g_3 = 0$, is that it contain as a factor the quadratic polar of each of

* The four points on a line section of a rational quartic are not in general projective with the binary quartic of their parameters; but when the curve has a triple point, the parameters are projective with lines on the triple point, which are again projective with the points of a line section.

† Clebsch, "Vorlesungen über Geometrie," p. 280. g_3 in Salmon's notation is the $C_{3,0}$ of the quartic.

its roots. To prove this, let the quartic be written $(\alpha t)(\beta t)(\gamma t)^2$, where α and β are real and γ symbolic. For the quadratic polar of (αt) we have

$$\begin{aligned}[(\alpha t)(\beta t)(\gamma t)^2, (\alpha t)^2] &= 2|\beta\alpha|\,|\gamma\alpha|(\alpha t)(\gamma t)+|\gamma\alpha|^2(\alpha t)(\beta t)\\ &=(\alpha t)\{2|\beta\alpha|\,|\gamma\alpha|(\gamma t)+|\gamma\alpha|^2(\beta t)\}.\end{aligned}$$

Since $|\beta\alpha|\neq 0$, (βt) becomes a factor when, and only when, $|\gamma\alpha|\,|\gamma\beta|=0$; *i. e.*, when $(\alpha t)(\beta t)$ is harmonic to $(\gamma t)^2$. Hence, the polar conic of a point t on the curve cuts it in points whose joins with t make harmonic line sections. This holds for the general rational quartic. If the curve has a triple point, three intersections of conic and curve are at the triple point (which proves that the contravariant of the preceding paragraph contains the triple point three times); two are at t and are to be disregarded, since the line section made by the tangent has a double root; the remaining three, when joined to t, give the lines of the cubic Φ on that point. We ask that a point t_1 with two of these points, say t_2 and t_3, form a mutually related set; *i. e.*, if t_1 gives t_2 and t_3, t_2 shall in the same way give t_1 and t_3, etc. Now the polar conic of a point y as to Θ is

$$\sum^3 a_0 y_0 (y_2-y_1)(x_2-x_1)^2+\sum^3 a_0 x_0 (x_2-x_1)(y_2-y_1)^2=0. \tag{26}$$

Let the functions (23) of t_1 and t be substituted for y and x respectively. After cancellation of the triple-point parameters and a factor $(t-t_1)^2$ corresponding to the point of contact of conic and curve, the result factors thus:

$$\begin{aligned}&[a_0 t_1 t(t+t_1-tt_1)+a_1(1-t_1)(1-t)(t_1 t-1)+a_2(1-t_1-t)]\\ &\quad\cdot[a_1 a_2(t_1+t_2-2)-a_0 a_2(t+t_1)+a_0 a_1(t+t_1-2t_1 t)]=0.\end{aligned} \tag{27}$$

Let these factors in order be denoted by ϕ_1 and ϕ_2. ϕ_1 is recognized as the quadratic polar of t_1 as to

$$C_{10,4}=[C_{7,3}, C_{3,3}]^1=a_0^3 a_1^3 a_2^3\{a_0 t^3(t-2)+a_1(1-t)^3(1+t)+a_2(2t-1)\}, \tag{28}$$

which is a self-apolar quartic. For the binary quartic $(\alpha t)^4\equiv(\beta t)^4\equiv(\gamma t)^4$, the condition that $(\alpha t_1)^2(\alpha t_2)^2=0$, $(\alpha t_1)^2(\alpha t_3)^2=0$ and $(\alpha t_2)^2(\alpha t_3)^2=0$ be simultaneously true is $|\alpha\beta|^4(\gamma t_1)^4=0$;* hence, *the necessary and sufficient condition that t_1 and the roots of its quadratic polar be mutually related for every t_1 is that $(\alpha t)^4$ be self-apolar.* (29)

Since $C_{10,4}$ is self-apolar, we have proved a characteristic property of Θ: *Through any point of the curve are two lines making harmonic sections with the curve and such that the line joining two of their further intersections also makes a harmonic section.*

There are thus ∞^1 triangles of the kind sought. These will be called Φ triangles. On each side of such a triangle are two further intersections whose

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* Since t_2 and t_3 are roots of $(\alpha t_1)^2(\alpha t)^2=0$, we have $t_2 t_3=\dfrac{a_1^2(\alpha t_1)^2}{a_0^2(\alpha t_1)^2}$, $t_2+t_3=\dfrac{-2a_0 a_1(\alpha t_1)^2}{a_0^2(\alpha t_1)^2}$. Eliminate t_2 and t_3 from $(\alpha t_2)^2(\alpha t_3)^2\equiv[\beta_0^2 t_2 t_3+\beta_0\beta_1(t_2+t_3)+\beta_1^2]^2=0$. The result is $|\beta\alpha|^2|\beta\gamma|^2(\alpha t_1)^2(\gamma t_1)^2=0$, which must be $|\alpha\beta|^4(\gamma t_1)^2$ since $(\alpha t)^4$ has no other $C_{3,4}$.

parameters must satisfy $\phi_2=0$. ϕ_2 is the polarized form of $C_{2,2}\equiv a_1a_2(1-t)+a_0a_2t-a_0a_1t(1-t)$. For each t_1 this factor gives a unique line of Φ and could be used to obtain a parametric representation of Φ.

We consider next the involution on Θ cut out by conics on its triple point.* The base-points of this involution are given by a quintic covariant of f which we call $\bar{f}$ and whose covariants are indicated in like manner. The points $\bar{f}=0$ are cut out by a covariant conic on the triple point. Since a conic consisting of two lines on the triple point cuts out two arbitrary points, $C_{7,3}$ is the canonizant of $\bar{f}$. (Invariant factors are neglected when the function is to be equated to zero.) Line sections of Θ are quartics apolar to $\bar{f}$. Since a root of the Hessian of a quintic is a triple root of an apolar quartic, $\bar{H}=0$ gives the parameters of the flex-points, and $\bar{S}=0$ (the Steinerian of $\bar{f}$) the further intersections of the inflexional tangents with the curve. When a line section for which $g_3=0$ has a double root, it has a triple root, and the line is a flex-line. We therefore obtain the flex-parameters by putting $t=t_1$ in ϕ_1 and ϕ_2. It follows that $\bar{H}$ is $\rho C_{2,2}C_{10,4}$, where ρ is an invariant function or numerical. We now have these relations between the covariants of f and of $\bar{f}$:

$$C_{7,3}=\rho\bar{C}_{3,3}, \qquad C_{2,2}=\rho\bar{C}_{6,2}, \qquad C_{2,2}C_{10,4}=\rho\bar{H}.$$

Since $C_{2,2}$ is the Hessian of $C_{7,3}$ (§3), $\bar{f}$ is such that its Hessian contains its $\bar{C}_{6,2}$ as a factor. Referring to the syzygy

$$6HC'-9i\tau^2-j(4jB+6\tau\alpha)=0$$

derived by Dr. A. E. Landry,† it is evident that when j (the canonizant) does not contain τ ($C_{6,2}$) as a factor, *i. e.*, when the discriminant of the canonizant does not vanish, $B=0$ is the necessary and sufficient condition that H contain τ. Hence, $\bar{B}=0$. (This B is the invariant of degree 8.)

It may be verified at once that the section made by the line $(1,1,1)$, which is the polar of the triple point as to the triangle ABC, is $C_{10,4}=0$. We have thus four points of inflexion on a line. The remaining flexes are given by $C_{2,2}=0$. We wish now to show that the flex-tangents at these points meet on the curve. Recurring to the expression ϕ_1, (27), it can be shown that t_1 with the roots t_2 and t_3 forms a cubic apolar to both $C_{3,3}$ and $C_{7,3}$. It is necessary first to prove a theorem on apolar cubics.

When two binary cubics are apolar, their Jacobian is a self-apolar quartic, and triads apolar to both cubics are made up of a point and its quadratic polar as to the Jacobian.

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* Clebsch, "Vorlesungen," p. 461.

† "A Geometrical Application of Binary Syzygies," *Transactions of the American Mathematical Society*, Vol. I (1909), p. 106.

Let the cubics be $(\alpha t)^3$ and $(\beta t)^3$, where $|\alpha\beta|^3=0$. Their Jacobian is $|\alpha\beta|(\alpha t)^2(\beta t)^2$. If t_1 is a root of a cubic apolar to $(\alpha t)^3$ and $(\beta t)^3$, the other two roots are given by the Jacobian of the quadratics $(\alpha t_1)(\alpha t)^2$ and $(\beta t_1)(\beta t)^2$; namely, $|\alpha\beta|(\alpha t_1)(\beta t_1)(\alpha t)(\beta t)$. We wish to prove that this quadratic is the same as the quadratic polar of t_1 as to $|\alpha\beta|(\alpha t)^2(\beta t)^2$. Polarizing $|\alpha\beta|(\alpha t)^2(\beta t)^2$ twice, we have

$$(1) \qquad |\alpha\beta|\{4(\alpha t)(\alpha t_1)(\beta t)(\beta t_1)+(\alpha t)^2(\beta t_1)^2+(\alpha t_1)^2(\beta t)^2\}=0.$$

We have identically:

$$(2) \qquad (\alpha t)(\beta t_1)\equiv|\alpha\beta|\,|t_1 t|+(\beta t)(\alpha t_1),$$

$$(3) \qquad (\alpha t_1)(\beta t)\equiv|\alpha\beta|\,|t t_1|+(\beta t_1)(\alpha t).$$

Substitute in the second and third terms of (1) after multiplying (2) and (3) by $(\alpha t)(\beta t_1)$ and $(\alpha t_1)(\beta t)$ respectively. The result, after simple reductions, is

$$(4) \qquad 6|\alpha\beta|(\alpha t)(\alpha t_1)(\beta t)(\beta t_1)+|\alpha\beta|^3|t_1 t|^2=0.$$

Since $|\alpha\beta|^3=0$, this reduces to its first term, and the second part of the theorem is proved. To see that the Jacobian is a self-apolar quartic, it is sufficient to note that the roots of a cubic apolar to two cubics are in an involution and to apply (29).

Now $C_{3,3}$ and $C_{7,3}$ are apolar, and $C_{10,4}$ is their Jacobian. We have thus proved that *the* ∞^1 *cubics which give vertices of* Φ *triangles are the self-apolar pencil* $C_{3,3}+\lambda C_{7,3}$. (30)

The product $C_{2,2}C_{5,1}$ is a member of this pencil; for, $[C_{3,3}, C_{2,2}]^2\equiv C_{5,1}$ and $[C_{7,3}, C_{2,2}]^2\equiv 0$. Hence, the triangle $C_{2,2}C_{5,1}$ is a Φ triangle. Since the polar conic of a point of inflexion is the flex-tangent and a line on the triple point, two Φ lines through a flex-point coincide with the flex-tangent, which is therefore a side of the Φ triangle. Hence, when two flex-points are vertices, the flex-lines must meet at the third vertex, which is a point of the curve. We have now proved a second important property of Θ:

The six inflexions break up into sets of four and two; the four are on a line, and the tangents at the remaining two meet on the curve. The parameters of the two sets are given by $C_{10,4}=0$ *and* $C_{2,2}=0$ *respectively.*

The intersection of the two flex-lines, the point $C_{5,1}$ on Θ, is on the line joining $C_{2,2}$ and $C_{6,2}$ as points determined by quadratics on N; for, $[C_{2,2}C_{3,3}]^2\equiv C_{5,1}$, which says that the line η of $C_{5,1}$ passes through point $C_{2,2}$. Lines through $C_{5,1}$ on Θ cut the curve in three further points which form a pencil $C+\lambda C'$, where C is a cubic and C' its cubic covariant.

We consider now tangents to Θ at vertices of a Φ triangle. Tangents at the reference points are easily seen to meet at point $C_{2,2}$; tangents at the vertices $C_{2,2}C_{5,1}$ meet at $C_{5,1}$; and tangents at the triple point $C_{7,3}$ (a Φ triangle with

coincident vertices) meet there. It will be shown that the tangents at vertices of each Φ triangle meet at a point, and that the locus of such points is the straight line joining $C_{2,2}$ and $C_{6,2}$. Beginning from the converse standpoint, we find the sextic of points of contact of tangents from a point x of the plane. The tangent at y on the curve is

$$3\overset{3}{\Sigma} a_0 y_0 (y_2-y_1)^2 (x_2-x_1) + \overset{3}{\Sigma} a_0 x_0 (y_2-y_1)^3 = 0.$$

Substituting for y the functions (23), the result reduces to

$$3C_{3,3}C_{2,2}a_0a_1a_2[x_0t+x_1(1-t)-x_2]+C_{7,3}[a_0t^3x_0+a_1(1-t)^3x_1-a_2x_2]=0, \quad (31)$$

which gives the points of contact of tangents from x. When x is on the join of $C_{6,2}$ (or $1, 1, 1$) and $C_{2,2}$ $(a_0^{-1}, a_1^{-1}, a_2^{-1})$, we have $x_i = a_i^{-1}+\mu$, $i=0, 1, 2$. Substituting these values of x in (31), the sextic becomes

$$3a_0a_1a_2C_{3,3}C_{2,2}\left[\frac{t}{a_0}+\frac{1-t}{a_1}-\frac{1}{a_2}\right]+C_{7,3}[t^3+(1-t)^3-1]$$
$$+\mu C_{7,3}[a_0t^3+a_1(1-t)^3-a_2]=0.$$

In terms of covariants this is

$$3C_{3,3}[C_{2,2}C_{5,1}-C_{7,3}]+\mu(a_0a_1a_2)^{-1}C_{7,3}^2=0.$$

Since $C_{4,0}C_{3,3} \equiv C_{2,2}C_{5,1}-C_{7,3}$, we obtain

$$3[C_{2,2}C_{5,1}+C_{7,3}][C_{2,2}C_{5,1}-C_{7,3}]+\mu(a_0a_1a_2)^{-1}C_{4,0}C_{7,3}^2=0,$$

or

$$3C_{2,2}^2C_{5,1}^2-[3-(a_0a_1a_2)^{-1}\mu C_{4,0}]C_{7,3}^2=0.$$

The factors of this are of the form $C_{2,2}C_{5,1}+\lambda C_{7,3}$, which is identical with the pencil (30) giving the vertices of Φ triangles. Hence:

The six points of contact of tangents drawn from a point of the line $C_{6,2}C_{2,2}$ *break into two sets of three, and each set forms a* Φ *triangle.*

It is interesting to note that Φ triangles may be obtained by drawing the lines of K (§ 2) from a point on the line $C_{6,2}C_{2,2}$ and taking their intersections with the corresponding lines η (§ 5, (21)). For, if the line K is made to pass through $x_i = a_i^{-1}+\mu$, the resulting cubic is $3a_0a_1a_2C_{3,3}-\mu C_{7,3}$, which is again the pencil (30). Triads of this pencil on N are points of contact of triangles circumscribed about N and inscribed in the conic F on the five points $A, B, C, C_{2,2}$ and $C_{6,2}$. For, the conic on the vertices of triangles whose sides touch N at $C_{3,3}=0$ and $C_{7,3}=0$ is on the Hessian point of each cubic. The conic F cuts N in $C_{10,4}=[C_{3,3}, C_{7,3}]^1$.

The triangle $C_{2,2}C_{5,1}$ on Θ (formed by the two flex-lines which meet on the curve and the line joining their flex-points) is a natural triangle of reference for the curve. Let the triple point be the unity point as before, and take the line joining the flexes as $x_1=0$. The curve may be generated by the inter-

section of a line on $(1,1,1)$ with a correlated line on $(0,1,0)$, the intersection of the flex-lines. The coordinates of the variable lines may be taken as

$$\left.\begin{array}{lccc} \eta: & t, & 1-t, & -1, \\ \xi: & at^3, & 0 & -b. \end{array}\right\} \tag{32}$$

The curve is then written

$$\left.\begin{array}{c} x_0=b(1-t), \qquad x_1=at^3-bt, \qquad x_2=at^3(1-t), \\ ax_0(x_2-x_1)^3+bx_2(x_1-x_0)^3=0. \end{array}\right\} \tag{33}$$

The reference scheme uses eight coordinates, and the ratio $a:b$ gives the nine necessary to determine the curve. Since three line sections are known, the fundamental quintic to which all line sections are apolar is readily found:

$$\bar{f}=a(t^5-5t^4+10t^3)-b(10t^2-5t+1),$$

from which $\bar{B}=I_8=0$ is verified. The conic on the points $\bar{f}=0$ is

$$x_0^2+6x_1^2+x_2^2+3x_1(x_0+x_2)-14x_0x_2=0.$$

The parameters of the four flexes on a line were given by the Jacobian of cubics of any two Φ triangles. We now have $\bar{C}_{3,3}=C_{7,3}=at^3-b$ and $C_{2,2}C_{5,1}=t(1-t)$. The Jacobian is $b-2t+2at^3-at^4$, which is again cut out by the line $(1,1,1)$. This verifies (what could be seen otherwise):

The line on four flexes is the polar of the triple point as to all Φ triangles. Moreover, *the conic on $\bar{f}=0$ is tangent to this line.*

Φ is now $a_0\xi_0(\xi_1+\xi_2)^2-b\xi_2(\xi_0+\xi_1)^2=0$, and the locus of self-apolar line sections is $4\xi_0\xi_1-(\xi_1+\xi_2)(\xi_0+\xi_1)=0$. Parametric equations of Φ are obtained by making use of the fact that when t_1 is a vertex of a Φ triangle the quadratic of intersections other than vertices on the opposite side is the quadratic polar of t_1 as to $C_{7,3}$. This is proved by consideration of the values $t_1=0, 1, \infty$. The quadratic is $at_1t^2-b=0$. Using (33), the line joining the two points on Θ given by this quadratic is found to have coordinates

$$\xi_0=-b(1-t), \qquad \xi_1=t(at-b), \qquad \xi_2=-at^2(1-t),$$

the subscript having been dropped from t_1. These equations express a side of a Φ triangle in terms of the parameter t of the opposite vertex.

We can now prove a property of the six points of the curve other than vertices on the sides of a Φ triangle.* It has been seen that the two points on one side of the triangle are given by the quadratic $at_1t^2-b=0$. Calling the roots t and $-t$, we write in line coordinates the points t and $-t$ of Θ given by (33). The product of these points is rational in t^2. Eliminating t^2 by means of $at^2-b=0$, we have, for the points on the side opposite t_1,

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*This property is made the basis of the discussion of the curve by R. A. Roberts (*Proceedings of the London Mathematical Society*, Vol. XVI).

$$a\xi_0^2t_1^3(at_1-b)-ab\xi_1^2t_1(1-t_1)^2-b\xi_2^2(at_1-b)$$
$$+2\xi_0\xi_1abt_1^2(1-t)-2\xi_1\xi_2abt_1(1-t_1)=0. \quad (34)$$

We wish to show that three such degenerate line conics are members of a pencil when t_1, t_2 and t_3 are vertices of a Φ triangle. t_1, t_2 and t_3 must satisfy an equation of the form $C_{2,2}C_{5,1}+\lambda C_{7,3}=0$. In the present notation this is $at^3-\lambda t^2+\lambda t-b=0$. The conditions are therefore (the s's being the symmetric functions of the three t's) $s_1=s_2$ and $s_3=b/a$. The second condition is used to eliminate b from (34). After division by a^2, the result is

$$\xi_0^2t_1^2(t_1-s_3)-\xi_1^2t_1(1-t_1)^2-\xi_2^2(t_1-s_3)+2\xi_0\xi_1t_1^2(1-t_1)-2\xi_1\xi_2t_1(1-t_1)=0.$$

The matrix of three such conics, when $s_1=s_2$, vanishes identically, and the conics therefore belong to a pencil. Hence:

The six further points of the curve on the sides of a Φ triangle are six points on four lines.

§ 7. *The Binary Quintic Represented on a Rational Cubic in Space.*

Let the ρ^3 in space be written in points:

$$x_0=t^3, \qquad x_1=3t^2, \qquad x_2=3t, \qquad x_2=1, \quad (35)$$

and in planes:

$$\xi_0=1, \qquad \xi_1=-t, \qquad \xi_2=t^2, \qquad \xi_3=-t^3. \quad (36)$$

Any point of space in terms of parameters of planes from it to the curve is

$$x_0=s_3, \qquad x_1=s_2, \qquad x_2=s_1, \qquad x_3=1; \quad (37)$$

and any plane in terms of its intersections with the curve is

$$\xi_0=1, \qquad \xi_1=-\tfrac{1}{3}s_1, \qquad \xi_2=\tfrac{1}{3}s_2, \qquad \xi_3=-s_3, \quad (38)$$

where s_1, s_2 and s_3 are the elementary symmetric functions of three t's.

Let the quintic be written in canonical form as before (§ 1, (1)), and let the canonizant points A, B, C ($t=0, 1, \infty$) determine a plane π and a point p. From (37) and (38) it is seen that the condition that two cubics be apolar is the same as the condition that the point of each lie on the plane of the other. Since a cubic is self-apolar, p is on π. π is the plane $x_1=x_2$, and p has coordinates (0, 1, 1, 0).

From any point of π are three planes of ρ^3 whose intersections with π are lines of a rational plane cubic. At A, B and C the three planes coincide; hence, the curve has three cusps and the intersection of the cusp-tangents is p. It is a rational quartic in points. Since the intersection of two consecutive planes of ρ^3 is the tangent line, the same curve is the locus of the intersection of tangent lines of ρ^3 with π. It is obviously the section of the developable quartic surface which ρ^3 determines, and is independent of the ratios $a_0:a_1:a_2$. To find its double line, we look for a line of π which contains two planes of ρ^3.

The bisecant line from p meets ρ^3 in the Hessian of the canonizant cubic. Since these two t's and any t are apolar to $C_{3,3}$, the planes of these points and any plane of ρ^3 meet on π; that is, the two planes meet on π and give the double line. The points of contact are cut out by the two tangent lines to ρ^3. The parameters are given by $C_{6,2}=0$.

A transformation of the plane is determined by quartics on ρ^3 apolar to f, as follows: let $t_1t_2t_3t_4$ be apolar to f, and let lines t_1t_2 and t_3t_4 meet π in P and Q, respectively; then P and Q are corresponding points of the transformation. That this transformation is one-to-one and involutory is clear from the fact that from a point of the plane but one bisecant line can be drawn, and that the two values of t thus found uniquely determine the second pair in the apolar quartic. It is the same transformation as is gotten by taking the cubic polar of t_1t_2 as to f and finding the intersection of planes of ρ^3 at these points. Since cubic polars are apolar to $C_{3,3}$, the point so found will lie in π. To express analytically this transformation, we take as reference triangle the triangle ABC, and as unity point the transform of p. If the bisecant line from P of π meets ρ^3 at t_1 and t_2, the transformed point Q is given by the cubic

$$a_0t^3t_1t_2+a_1(1-t)^3(1-t_1)(1-t_2)-a_2=0.$$

The coordinates of Q are therefore

$$\left.\begin{aligned} x_0&=s_3=-a_1(1-t_1)(1-t_2)+a_2,\\ x_1&=x_2=s_2=-3a_1(1-t_1)(1-t_2),\\ x_3&=a_0t_1t_2-a_1(1-t_1)(1-t_2). \end{aligned}\right\} \tag{39}$$

Now the bisecant from p gives points $C_{6,2}=0$ on ρ^3; *i. e.*, t_1 and t_2 satisfy $t^2-t+1=0$. Using these values in (39), we find $x_0=a_1-a_2$, $x_1=x_2=3a_1$, $x_3=a_1-a_2$. This is the transform of p, and is to be taken as the unity point $y_0=y_1=y_2$ in the plane. Since the space coordinates of A, B, and C are known from (35), the transformation from space to plane is readily found:

$$\left.\begin{aligned} x_0&=a_1y_1-a_2y_2, & a_0y_0&=(x_1-3x_3),\\ x_1&=x_2=3a_1y_1, & a_1y_1&=x_1,\quad x_1=x_2,\\ x_3&=a_1y_1-a_0y_0, & a_2y_2&=(x_1-3x_0). \end{aligned}\right\} \tag{40}$$

Applying this substitution to the equations (39), we find for Q:

$$y_0=t_1t_2,\qquad y_1=(1-t_1)(1-t_2),\qquad y_2=1. \tag{41}$$

It is thus proved that *if the bisecant line from P meets ρ^3 in t_1t_2, the transformed point Q is at the intersection of tangents t_1t_2 to the conic $y_0=t^2$, $y_1=(1-t)^2$, $y_2=1$.* The position of the conic is determined by the reference scheme as described above. The coordinates of P in terms of t_1t_2 are found from the equations of two planes on the chord t_1t_2 of ρ^3. Changing to y coordinates by (40), we find for the intersection of the chord with π:

$$y'_0=[a_0t_1t_2]^{-1}, \qquad y'_1=[a_1(1-t_1)(1-t_2)]^{-1}, \qquad y'_2=[a_2]^{-1}. \qquad (42)$$

Comparison with (41) shows that P and Q satisfy (17). Hence, *the transformation of the present paragraph is identical with the quadratic transformation T of* § 3.

When $t_1=t_2$, P traces the quartic K and Q the conic N (see § 2, following (13)). Point p named by its tangent to N is $C_{2,2}=0$, and its transform $(1,1,1)$ is $C_{6,2}=0$ as before. If from the fixed points of T in π the bisecants are drawn to ρ^3, the four quadratics $(at_1)^3(at)^2=0$, $(at)^3(at_1)^2=0$, or $C_{4,8}=0$ (14), are obtained. The quadric cone formed by projecting ρ^3 from a point t on it cuts π in a conic on ABC which is the transform of line t on N.

The rational plane quintic O (§ 4) was defined as the locus of the intersection of a tangent to N with the corresponding line of K (§ 2). For this we may now substitute the following: Given ρ^3 and a plane cutting it, inscribe any

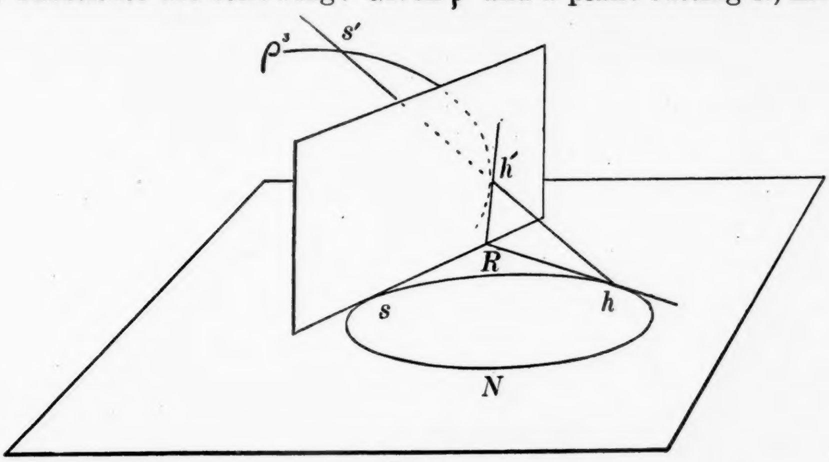

FIG. 2.

conic in the triangle ABC of the three points in the plane; correlate points of conic and cubic in such a way that the point of contact of the conic with the side BC corresponds to A on ρ^3, etc.; the plane t of ρ^3 and the tangent to the conic at the corresponding point t meet at a point whose locus is O. The number of coordinates used in the plane is 8 for the triangle and p, and 2 for the conic. The binary quintic is not mentioned, but it is at once recovered through the fact that p gives $C_{2,2}$ on N, and $(1,1,1)$ gives (by its bisecant) $C_{2,2}$ on ρ^3.

Certain relations between the points given by the Hessian and Steinerian covariants on ρ^3 and N are of interest. (a) Let h and s be a pair of corresponding roots of the Hessian and Steinerian of f on N, and h' and s' the points having the same parameters on ρ^3. (See Fig. 2.) By the definition of K (§ 2) the line of K having h as parameter is tangent to N at S; and it has been shown in the present section that a line t of K is the intersection of π

with the plane t of ρ^3. Hence, *the six planes of* ρ^3 *at the points* $H=0$ *cut the plane of the canonizant in six lines of a conic, and the points of contact are the points* $S=0$ *of a correlated quintic on the conic.* (β) The tangent to ρ^3 at h' meets π at R, the point with parameter h on K. This point has been shown to be the intersection of tangents to N at s and h (§ 4, first paragraph). If we join s' to a variable point t' on ρ^3, the line meets π in a conic whose transform by T is the line Rs, as was shown above; this is in accord with the fact that the cubic polar of $s't'$ always has a root at h'. When $t'=h'$, the cubic polar has a second root at h', and the point determined on π by this cubic is on the intersection of two consecutive planes at h'; that is, the point is R. Now the chord $s't'$ meets π in the transform by T of the point given by the cubic polar of $s't'$. Since h^3s is a quartic apolar to f, the transform of R is h on N. Hence, chord $s'h'$ contains h. *If the points* h *on* N *were given, we should obtain the pairs* $s'h'$ *by drawing bisecants to* ρ^3 *from them; the planes of* ρ^3 *at points* h' *will then determine points* s *on* N. (γ) Since Rs is tangent to N and K, the conic into which Rs transforms by T is tangent to K and N: to N at h, and to K at the transform of s. Therefore, *six conics on* ABC *are tangent to* K *and* N; *the points of contact on* N *are at* $H=0$, *and the points of contact on* K *are cut out by tangents to* ρ^3 *at* $S=0$.

Two parameters t_1, t_2 on ρ^3 give a cubic polar whose plane (named by t_1, t_2) passes through p. The locus of the intersection of chord t_1t_2 with plane t_1t_2 is a surface Ω whose points are in one-to-one correspondence with the points of π, the point t_1t_2 of the surface being made to correspond to the point y of (41). (It will be recalled that this y is the transform by T of the intersection of chord t_1t_2 with π.) Let

$$t_1+t_2=\sigma_1, \qquad t_1t_2=\sigma_2. \tag{43}$$

Equations (41) are equivalent to

$$y_0=\sigma_2, \qquad y_1=1-\sigma_1+\sigma_2, \qquad y_2=1. \tag{44}$$

Two planes on the chord are

$$\left.\begin{aligned} 3x_0-\sigma_1x_1+\sigma_2x_2 \qquad &=0,\\ -x_1+\sigma_1x_2-\sigma_2x_3&=0.\end{aligned}\right\} \tag{45}$$

Using (10) with change of subscripts and (38), the plane t_1t_2 is found to be

$$a_0\sigma_2x_0+a_1(1-\sigma_1+\sigma_2)(-x_0+x_1-x_2+x_3)-a_2x_3=0. \tag{46}$$

Elimination of σ_1 and σ_2 from (45) and (46) gives the equation of the surface. To express analytically the above-mentioned correspondence between points of Ω and π, we eliminate σ_1 and σ_2 between (44), (45) and (46). Or, eliminating x from the equations (45) and (46) and $(x\xi)=0$, then using (44), we find as the equation of the point of Ω corresponding to y:

$$\begin{vmatrix} 3y_2 & -(y_0-y_1+y_2) & y_0 & 0 \\ 0 & -y_2 & y_0-y_1+y_2 & -3y_0 \\ a_0y_0-a_1y_1 & a_1y_1 & -a_1y_1 & a_1y_1-y_2y_2 \\ \xi_0 & \xi_1 & \xi_2 & \xi_3 \end{vmatrix} = 0. \tag{47}$$

The coefficients of the ξ's in this equation are the coordinates of x; that is, we have $x_i = f_i(y_0, y_1, y_2)$, where f_i are homogeneous cubic functions of y. *The transformation from plane to surface is then such that a plane section* $(x\xi)=0$ *is the map of the plane cubic* $(f\xi)=0$.

The four cubics $f_i(y_0, y_1, y_2)=0$ have six points in common; they are in fact cubic adjoints of O (§ 4). To see this, we recall that the parameters of a double point of O are subject to the conditions $(\alpha t_1)^4(\alpha t_2)=0$ and $(\alpha t_2)^4(\alpha t_1)=0$. If t_1 and t_2 are roots of $(\beta t)^2=0$, this is equivalent to the requirement that $(\beta t)^2$ be contained as a factor in its cubic polar $|\alpha\beta|^2(\alpha t)^3$. In Salmon's notation this becomes

$$a_0t_1t_2t^3 + a_1(1-t_1)(1-t_2)(1-t)^3 - a_2 \equiv \rho(b_0+2b_1t+b_2t^2)(c_0+c_1t).$$

Eliminating c_0, c_1 and ρ from the four equations to which this identity gives rise, and substituting for the b's their equivalents in terms of the point y determined by the quadratic, we obtain four equations in y which a double point must satisfy. These are precisely the four cubics found above.

The surface Ω contains ρ^3, and this curve is the map of O; for, if t_1 and t satisfy $(\alpha t_1)^4(\alpha t)=0$, the cubic polar of the quadratic tt_1 contains t_1 as a factor, and the intersection of chord tt_1 with plane tt_1 is the point t_1 on ρ^3. The corresponding point tt_1 of π is on O by definition of the curve.

Surface Ω meets π when chord t_1t_2 meets plane t_1t_2 on π. Let chord t_1t_2 meet π at P as before. The plane t_1t_2 (*i.e.*, the plane cutting ρ^3 in the cubic polar of t_1t_2) contains Q, the transform of P by T, and p. Therefore, the surface meets π when P and Q are in line with p. The locus of such points is the plane cubic Ω. (See § 5. Point $C_{2,2}$ of § 5 is p.) Making $x_1=x_2$ in the equation of the surface and using the transformation (40) gives equation (20) of § 5. That A, B and C, the fixed points of T, and p are on this curve is seen at once from the present standpoint. The double points of O were given by quadratics which are factors in their cubic polars. If t_1t_2 is such a quadratic on ρ^3, the plane of the cubic polar contains the line t_1t_2; P and Q are then on the section of the plane with π, which says that the double points Q are on curve Ω. Since the line t_1t_2 just referred to lies in its plane, it is a line of the surface Ω. Six lines of the surface are thus accounted for. The six conics and fifteen lines determined by the double points of O map into the remaining twenty-one lines.

A theorem proved by Stahl* states that adjoint cubics of a rational quintic which are on two fixed points t_1 and t_2 of the curve meet further at the intersection of tangents to the perspective conic with the same parameters. In the above mapping process, the point of π whose parameters with respect to the conic N are t_1 and t_2, was made to correspond to the third intersection with Ω of the chord t_1t_2 on ρ^3. *Stahl's theorem follows at once from the fact that cubics on two fixed points of O map into plane sections on a fixed chord of ρ^3 which have necessarily the same third intersection with Ω.*

§ 8. *Further Consideration of Rational Quintic O of* § 4.

The quintic O was defined as the locus of the intersection of tangents to N at points t and t' when $(\alpha t)^4(\alpha t')=0$. It was seen that the parameters of its double points satisfy $(\alpha t)^4(\alpha t')=0$ and $(\alpha t')^4(\alpha t)=0$, from which, by elimination of t', we obtain $C_{1,5}\cdot C_{4,12}=0$. The first factor is accounted for by the fact that O is five times tangent to N at the points $f=0$ (§ 4). The parameters of the double points then satisfy a $C_{4,12}=0$.

To express this $C_{4,12}$ in terms of the irreducible system, we let 0 and ∞ be two of its roots. Then f takes the form

$$f\equiv\alpha_0t^5+10\alpha_2t^3+10\alpha_3t^2+\alpha_5. \tag{48}$$

$C_{4,12}$ must be of the form $C^2_{2,6}+\lambda C_{2,2}C^2_{1,5}$. We determine λ so that this shall have roots $0, \infty$ when $\alpha_1=\alpha_4=0$. In symbols,

$$C_{2,6}=|\alpha\beta|^2(\alpha t)^2(\beta t)^2, \qquad C_{2,2}=|\alpha\beta|^4(\alpha t)(\beta t), \qquad C_{1,5}=(\alpha t)^5.$$

Putting $t=0$ gives for the determination of λ:

$$[|\alpha\beta|^2\alpha_0^3\beta_0^3]^2+\lambda|\alpha\beta|^4\alpha_0\beta_0(\alpha_0^5)^2=0,$$

whence, when $\alpha_0^4\alpha_1=\alpha_1^4\alpha_0=0$, $\lambda=-\frac{2}{3}$. Hence,

$$C_{4,12}=C^2_{2,6}-\tfrac{2}{3}C_{2,2}C^2_{1,5}. \tag{49}$$

The above form (48) of f isolates one of the six double points by giving it the parameters $(0, \infty)$. Certain properties of the curve are easily discussed from this standpoint. We take N in the form

$$x_0=t^2, \qquad x_1=2t, \qquad x_2=1. \tag{50}$$

The triangle of reference is now formed by two tangents from a double point to N and the line joining their points of contact. The equation of the conic on the remaining five points is found as follows: The adjoint cubics, found by the method of § 7 (paragraph following (47)), now take the form

$$\begin{vmatrix} \alpha_0x_0+\alpha_2x_2 & 3(\alpha_2x_1+\alpha_3x_2) & 3(\alpha_2x_0+\alpha_3x_1) & \alpha_3x_0+\alpha_5x_2 \\ x_2 & -x_1 & x_0 & 0 \\ 0 & x_2 & -x_1 & x_0 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \end{vmatrix}=0. \tag{51}$$

*"Zur Erzeugung der ebenen rationalen Curven," *Mathematische Annalen*, Vol. XXXVIII (1891).

By proper choice of the η's, this will break into a conic and a line on the double point $(0,1,0)$; for convenience, we make it contain x_0 as a factor. The resulting equations for the η's are

$$4\alpha_2\eta_3-\alpha_5\eta_0=0, \qquad 6\alpha_3\eta_3-\alpha_5\eta_1=0, \qquad \eta_2=0,$$

from which is obtained the conic

$$6\alpha_0\alpha_3x_0^2+(\alpha_0\alpha_5-16\alpha_2\alpha_3)x_1^2+6\alpha_2\alpha_5x_2^2-24\alpha_3^2x_1x_2 \\ -24\alpha_2^2x_0x_1-(20\alpha_2\alpha_3+\alpha_0\alpha_5)x_0x_2=0. \tag{52}$$

The product of the tangents from $(0,1,0)$ to conic (52) is found to be

$$x_0^2[6\alpha_0\alpha_3(\alpha_0\alpha_5-16\alpha_2\alpha_3)-144\alpha_2^4]+x_2^2[6\alpha_2\alpha_5(\alpha_0\alpha_5-16\alpha_2\alpha_3)-144\alpha_3^4] \\ +x_0x_2[32\alpha_2^2\alpha_3^2-\alpha_0^2\alpha_5^2-4\alpha_0\alpha_2\alpha_3\alpha_5]=0. \tag{53}$$

To find the cubic adjoints with double point at $(0,1,0)$, we make the polar conic of this point, as to cubic (51), degenerate into two lines of the form $ax_0^2+bx_0x_2+cx_2^2=0$. The resulting two conditions on η_i give $\eta_0=\eta_3=0$. The pencil of rational cubics is

$$\eta_1[x_0^2(\alpha_0x_0-2\alpha_2x_2)-x_1x_2(4\alpha_3x_0+\alpha_5x_2)] \\ +\eta_2[x_2^2(\alpha_5x_2-2\alpha_3x_0(-x_0x_1(\alpha_0x_0+4\alpha_2x_2)]=0. \tag{54}$$

Two members of the pencil have a cusp at $(0,1,0)$. To find the cusp tangents, we require that the polar conic

$$\alpha_0\eta_2x_0^2-\alpha_5\eta_1x_2^2-(4\alpha_3\eta_1-4\alpha_2\eta_2)x_0x_2=0$$

be a perfect square. Elimination of η_1 and η_2 between this quadratic and its discriminant gives for the cusp-lines:

$$2\alpha_0\alpha_3x_0^2+\alpha_0\alpha_5x_0x_2+2\alpha_2\alpha_5x_2^2=0. \tag{55}$$

There is a simple relation between these lines, the tangents to O at the same point, and the tangents to N. The equations for O are now:

$$\left.\begin{aligned} x_0&=4\alpha_2t^4+6\alpha_3t^3+\alpha_5t, \\ x_1&=-\alpha_0t^5-2\alpha_2t^3+2\alpha_3t^2+\alpha_5, \\ x_2&=-\alpha_0t^4-6\alpha_2t^2-4\alpha_3t. \end{aligned}\right\} \tag{56*}$$

The line equations are found from these by differentiation, and the values $t=0$, ∞ give for the tangents to O at its double point:

$$(4\alpha_3x_0+\alpha_5x_2)(\alpha_0x_0+4\alpha_2x_2)=0. \tag{57}$$

The same pair of lines are obtained by differentiating (55) partially with respect to x_0 and x_2. Hence, *the tangents to O are the harmonic conjugates of the tangents to N with respect to the cusp-lines of cubic adjoints, all six lines being drawn from the same double point of O.*

From the standpoint of the space cubic ρ^3 (§ 7), the above result is interpreted as follows: The six double points $(1, 2, \ldots, 6)$ of O map into six lines

*This is found as the intersection of line t of N with the corresponding line t of K. See § 4. The coordinates of the lines are respectively $(1, -t, t^2)$ and $(a_0^2(at)^3, a_0a_1(at)^3, a_1^2(at)^3)$.

of the surface Ω; and the conics $C_1, \ldots, C_6$ on five of the six points map into six lines, the two sets forming a double six. Let point 1 give line a_1, meeting ρ^3 at $t_1 t_2$, and let the corresponding conic C_1 give line b_1 (which does not meet ρ^3). The cusp-directions (55), being the double lines of the involution of tangents to cubic adjoints, transform into the points of a_1 on the parabolic curve of Ω. Any line on point 1, in particular the tangent t_1 to N, transforms into a plane section on b_1. Since the transform of N is the section of Ω with the developable surface determined by ρ^3, the point of contact t_1 on N becomes the third intersection of tangent t_1 to ρ^3 with Ω. Hence, interpreting f on ρ^3 without regard to N:

If t_1, t_2 are a pair of associated roots of $C_{4,12}$, (49), and give chord a_1 of ρ^3, and if b_1 is the corresponding line of the double six on Ω, the line t_1 of ρ^3 meets Ω in a further point P such that the plane b_1P cuts a_1 in the harmonic conjugate of t_1 as to the parabolic points of Ω on a_1.

The five lines from (0, 1, 0) to the remaining double points are found by elimination of x_1 from the cubics in the brackets of (54). The result is

$$\alpha_0^2 x_0^5 + 2\alpha_0\alpha_2 x_0^4 x_2 - 8\alpha_2^2 x_0^3 x_2^2 + 8\alpha_3^2 x_0^2 x_3^3 - 2\alpha_3\alpha_5 x_0 x_2^4 - \alpha_5^2 x_2^5 = 0. \tag{58}$$

The lines to the five points $f=0$ on N are found by eliminating t between $f=0$ and $x_0=t^2$, $x_2=1$:

$$\alpha_0 x_0^5 + 20\alpha_0\alpha_2 x_0^4 \alpha_2 + 100\alpha_2^2 x_0^3 x_2^2 - 100\alpha_3^2 x_0^2 x_2^3 - 20\alpha_3\alpha_5 x_0 x_2^4 - \alpha_5^2 x_2^5 = 0. \tag{59}$$

Quintics (58) and (59) are apolar. That is, *the six points have the property that lines joining one to the other five are apolar to lines joining the same point to a set of five fixed points in the plane.*

We now have $C_{2,2} = 3\alpha_2^2 t^2 + (\alpha_0\alpha_5 + 2\alpha_2\alpha_3 t) + 3\alpha_3^2$. The point $C_{2,2}$ therefore has coordinates

$$x_0 = 3\alpha_3^2, \qquad x_1 = -(\alpha_0\alpha_5 + 2\alpha_2\alpha_3), \qquad x_2 = 3\alpha_2^2.$$

The cubic polar of the quadratic $(0, \infty)$ as to f is $3\alpha_2 t^2 + 3\alpha_3 t$. Two roots of this are $(0, \infty)$, and the third is the γ of § 4. The point γ on N is therefore

$$x_0 = \alpha_3^2, \qquad x_1 = -2\alpha_2\alpha_3, \qquad x_2 = \alpha_2^2.$$

Comparison with the coordinates of $C_{2,2}$ shows that these points are in line with the double point (0, 1, 0), which agrees with the theorem of § 4. We have also the coordinates of the following lines:

Polar of $C_{2,2}$ as to N:	$6\alpha_2^2$,	$(\alpha_0\alpha_5 + 2\alpha_2\alpha_3)$,	$6\alpha_3^2$.
Polar of (0, 1, 0) as to conic (52):	$12\alpha_2^2$,	$(-\alpha_0\alpha_5 + 16\alpha_2\alpha_3)$,	$12\alpha_3^2$.

These lines meet on $x_1=0$, the polar of (0, 1, 0) as to N. Now the line $C_{2,2}$ is determined by a covariant of f and is therefore independent of the choice of the double point to be used as (0, 1, 0). Hence:

The intersections of the polars of each double point as to N, and as to the conics on the other five, are six points on a line.

The line on which the above six points lie (*i.e.*, the polar of $C_{2,2}$ as to N) is a covariant line of O which appears also in the case of the general rational plane quintic. It makes the unique line section apolar to all line sections.* Since O is defined $(\alpha t)^4(\alpha t')=0$, a point of the curve falls on the line $C_{2,2}=0$ when quadratic tt' is apolar to $C_{2,2}$. Hence, the parameters of points on this line satisfy $[f, C_{2,2}]=C_{3,5}=0$. In the present notation we have

$$\begin{aligned}C_{3,5}\equiv\alpha_0(\alpha_0\alpha_5+2\alpha_2\alpha_3)t^5+6(\alpha_0\alpha_3^2-4\alpha_2^3)t^4+2\alpha_2(\alpha_0\alpha_5-16\alpha_2\alpha_3)t^3\\ -2\alpha_3(\alpha_0\alpha_5-16\alpha_2\alpha_3)t^2-6(\alpha_2^2\alpha_5-4\alpha_3^3)t-\alpha_5(\alpha_0\alpha_5+2\alpha_2\alpha_3).\end{aligned}$$

The apolarity of this with the line sections $x_i=0$, (56), is easily verified.

Parametric equations for the conic C_1 on double points 2, 3, 4, 5, 6 may be found by use of a corollary to Stahl's theorem on the rational quintic (see end of § 7). The corollary states that C_1 and the perspective conic N are in triangular relation. Consider triangles with vertices on C_1 and sides touching N. We name a point of C_1 by the t of the point of contact on the opposite side of the triangle. C_1 is given by equation (52). Line $x_0=0$ cuts C_1 in points whose equation is

$$(\alpha_0\alpha_5-16\alpha_2\alpha_3)\xi_2^2+6\alpha_2\alpha_5\xi_1^2+24\xi_1\xi_2=0.$$

To find points of contact of tangents from these points to N, write $\xi_0=1$, $\xi_1=-t$, $\xi_2=t^2$. The resulting quadratic gives the parameters of the points of C_1 on $x_0=0$. The quadratic $x_2=0$ is found similarly. The tangent to N at $t=1$ is $x_0-x_1+x_2=0$. The tangent at a variable point t, $x_0-x_1t+x_2t^2=0$, meets this at $x_0=t$, $x_1=1+t$, $x_2=1$. Substitution of these values in (52) gives the quadratic of the section $x_0-x_1+x_2=0$ with C_1. From this, using quadratics $x_0=0$ and $x_2=0$ already found, the section $x_1=0$ is determined. The equations for C_1 thus found are

$$\left.\begin{aligned}x_0&=(\alpha_0\alpha_5-16\alpha_2\alpha_3)t^2-24\alpha_3^2t+6\alpha_2\alpha_5,\\ x_1&=24\alpha_2^2t^2+(52\alpha_2\alpha_3-\alpha_0\alpha_5)t+24\alpha_3^2,\\ x_2&=6\alpha_0\alpha_3t^2-24\alpha_2^2t+(\alpha_0\alpha_5-16\alpha_2\alpha_3).\end{aligned}\right\}\qquad(60)$$

That these quadratics are not to be multiplied by factors independent of t is verified by substitution in (52).

The polar of the sixth double point (0, 1, 0) as to C_1 is $12(\alpha_2^2x_0+\alpha_3^2x_2)-(\alpha_0\alpha_5-16\alpha_2\alpha_3)x_1=0$, whence the quadratic of points of contact of tangents from this point is

*If the general rational quintic is represented by $x_i=f_i(t)$, a line section ξ apolar to all line sections must satisfy $[f_i, \xi_0f_0+\xi_1f_1+\xi_2f_2]^5=0$. The conditions on ξ, namely $\xi_1(f_0, f_1)^5+\xi_2(f_0, f_2)^5=0$, $\xi_0(f_1, f_0)^5+\xi_2(f_1, f_2)^5=0$ and $\xi_0(f_2, f_0)^5+\xi_1(f_2, f_1)^5=0$ give $\xi_0:\xi_1:\xi_2=(f_1, f_2)^5:(f_2, f_0)^5:(f_0, f_1)^5$, which is unique when the f's are general.

$$t^2[12a_2^2(a_0a_5-16a_2a_3)-6\cdot 12a_0a_3^3]$$
$$+t[(a_0a_5-15a_2a_3)(52a_2a_3-a_0a_5)+(24)^2a_2^2a_3^2]$$
$$+[12a_3^2(a_0a_5-16a_2a_3)-6\cdot 12a_2^3a_5]=0.$$

The discriminant of this quadratic is the function whose vanishing is the condition that the six points lie on a conic.

To find the quintic of the five double points on C_1, we find the intersections of C_1 with any two adjoint cubics. The two sextics in t will have a quintic factor in common, which is the required quintic. This is found to be

$$F(t)=a_0(a_0a_5+8a_2a_3)t^5+12(a_0a_3^2-8a_3^2)t^4+16a_2(a_0a_5-19a_2a_3)t^3$$
$$+16a_3(a_0a_5-19a_2a_3)t^2+12(a_5a_2^2-8a_3^3)t+a_5(a_0a_5+8a_2a_3).$$

$F(t)$ is apolar to $f=a_0t^5+10a_1t^3+10a_2t^2+a_5$.

The six double points of O are completely determined as the fixed points of a certain $(3,1)$ correspondence. Let f be written $a_0t^5+a_1(1-t)^5-a_2$, and take N in the form $x_0=t^2$, $x_1=(1-t)^2$, $x_2=1$, as in § 1. Let t_1t_2 on N give point y, and let t_3, t_4, t_5 be roots of the cubic polar of t_1t_2 as to f. Lines of N at points t_3, t_4, t_5 form a triangle whose vertices are three points x corresponding to y. If t_3 and t_4 were given, we should find y as the intersection of the quadratic polars of these points as to f; hence, one x gives one y, and the correspondence is $(3, 1)$. From the fact that a double point of O belongs to a quadratic which is a factor in its cubic polar, x and y coincide at the six double points and only there. The coordinates y must be rational functions of the coordinates x. Let x belong to the quadratic t_3t_4. The lines cutting out the quadratic polars (*i. e.*, the lines of K (§ 2) having these parameters) are

$$\begin{array}{lll} a_0t_3^3, & a_1(1-t_3)^3, & -a_2, \\ a_0t_4^3, & a_1(1-t_4)^3, & -a_2, \end{array}$$

whence

$$a_0y_0=(t_3+t_4)^2-t_3t_4+3[1-(t_3+t_4)],$$
$$a_1y_1=(t_3+t_4)^2-t_3t_4,$$
$$a_2y_2=(t_3+t_4)^2-t_3t_4+3t_3^2t_4^2-3t_3t_4(t_3+t_4).$$

Since $x_0=t_3t_4$, $x_1=(1-t_3)(1-t_4)$, $x_2=1$, and $x_0-x_1+x_2=t_3+t_4$, the transformation takes the form

$$a_0y_0=(-x_0+x_1+x_2)^2-x_1x_2,$$
$$a_1y_1=(x_0-x_1+x_2)^2-x_0x_2,$$
$$a_2y_2=(x_0+x_1-x_2)^2-x_0x_1.$$

The conics $y_i=0$ are on the point $(1,1,1)$. Sets of three points x corresponding to one y are three variable intersections of this net.

The Quartic Curve and its Inscribed Configurations.

By H. Bateman.

§ 1. *Introduction.*

Whereas the geometry of a planar cubic curve can be regarded as fairly complete, that of the quartic is far from being so. It is true that the present knowledge of the properties of the curve is very extensive, as may be seen from the admirable article by G. Kohn and G. Loria in the "Encyklopädie der Mathematischen Wissenschaften";* but there are several important questions which have still to be answered. Some of these are mentioned in Ciani's recent report.† At present the most useful form for the equation of a general quartic is that obtained by regarding the curve as the envelope of a family of conics ‡

$$\lambda^2 S_0 + 2\lambda S_1 + S_2 = 0,$$

where λ is a variable parameter. This form, however, is unsuitable for a discussion of the invariants and covariants of the curve. A canonical form consisting of the sum of the fourth powers of five linear functions can not be used for this purpose, for Clebsch § has shown that the equation of a quartic curve can be thrown into this form only when the invariant B (the catalecticant) vanishes. The sum of six fourth powers is a possible form, ‖ but the equation of the general quartic can be reduced to this form in ∞^3 ways. It is easy to deduce from this equation that three corners of the hexagon given by the six linear forms form an apolar triad with regard to the quartic, and that the sides of two such hexagons touch a curve of the third class; but the equation is not adapted to a simple discussion of other geometrical properties of the curve. It has been found by experience that it is convenient to have a number of typical forms for the equation of the curve, each form being appropriate for the study of the properties of the curve relative to some inscribed configuration.

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* Bd. III 2, Heft 4 (1909), pp. 517–570.

† "Le curve piane di quart' ordine," *Giornale di Matematiche*, t. 48 (1910), pp. 259–304.

‡ G. Salmon, "Higher Plane Curves," Dublin (1852), p. 196.

§ *Crelle's Journal*, Bd. 59 (1861), p. 125.

‖ Rosanes, *Crelle's Journal*, Bd. 76 (1873), p. 329. Scherrer, "Progr. Frauenfeld," p. 17. Scòrza, *Annali di Matematica* (3), t. 2 (1899), p. 329.

In some cases the existence of the configuration implies that the quartic curve is not general. Such cases are studied here in detail so as to prepare the way for the determination of the relations between the invariants which correspond to each particular case. A complete system of rational invariants of the general quartic curve has not yet been obtained in a form which is easy to use; but, thanks to the labors of Salmon,* Clebsch,† Maisano,‡ Gordan,§ Caporali,‖ Pascal¶ and Emmy Noether,** considerable progress has been made. In particular, Pascal has obtained criteria for certain types of degeneration, and Noether has constructed a relatively complete set of forms. Drs. Morley and Conner†† have found the relation which connects the invariants A and B when the curve contains ∞^1 configurations of fifteen points lying by threes on twenty lines, while Caporali has obtained some of the relations which are characteristic of certain other special types of curves.

A result of considerable importance for the invariant theory has been obtained by writers on Abelian functions of genus 3.‡‡ It follows from the work of Riemann and Schottky that the invariants of a quartic curve can be expressed rationally in terms of six fundamental quantities which can be regarded as irrational invariants of the curve. In particular, the twenty-eight bitangents can be represented by equations in which the coefficients are rational functions of the six fundamental quantities.§§ Cayley has shown that the quartic curve possesses a double point when the irrational invariants are connected by a certain relation. The Abelian theory has been further developed by Klein,‖‖ Wirtinger,¶¶ H. F. Baker*** and J. E. Wright.††† Frobenius‡‡‡ has obtained similar results by algebraic methods. In his second memoir he starts with

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* "Higher Plane Curves."

† *Loc. cit.*

‡ *Giornale di Matematiche*, t. 19 (1881), p. 198.

§ *Math. Ann.*, Bd. 20 (1882), p. 487.

‖ "Memorie di Geometria," Naples (1888).

¶ *Napoli Atti* (2), t. 12 (1905), p. 1.

** *Crelle's Journal*, Bd. 134 (1908).

†† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 263.

‡‡ B. Riemann, "Werke," 2d edition, p. 487. H. Weber, "Theorie der Abel'schen Functionen vom Geschlecht drei," Berlin (1876). F. Schottky, "Abriss einer Theorie der Abel'schen Functionen von drei Variabeln," Leipzig (1880); also *Acta Mathematica*, t. 27 (1903). Berlin *Berichte* (1910).

§§ See also A. Cayley, "Collected Papers," Vol. XI, p. 221; Vol. XII, p. 74. R. de Paolis, *Mem. Lincei* (3), 2 (1878).

‖‖ *Math. Ann.*, Bd. 10 (1876).

¶¶ "Untersuchungen über Thetafunktionen," Leipzig (1895). *Math. Ann.*, Bd. 40.

*** "Multiply Periodic Functions," Cambridge (1907).

††† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 271.

‡‡‡ *Crelle's Journal*, Bd. 99 (1886); Bd. 103 (1888).

Hesse's method * of deriving a general quartic curve from a net of quadric surfaces touching eight associated planes. He uses the symbol $f_{\alpha\beta\gamma\delta}$ to denote the determinant formed from the homogeneous coordinates of four of these planes, and shows that the thirty-five ratios $f_{\kappa\lambda\mu\nu} : f_{0123}$ can be expressed rationally in terms of the sixteen ratios $f_{\lambda\beta\gamma\delta} : f_{0123}$, $(\lambda = 4, 5, 6, 7)$, in which the suffixes β, γ, δ can denote any three of the numbers 0, 1, 2, 3, and the numbers $0, 1, 2, \ldots, 7$ are used to denote the eight associated planes. Now, these sixteen ratios form an orthogonal matrix, on account of the existence of relations of the type

$$\Sigma f_{\lambda\alpha\beta\gamma} f_{\lambda\alpha'\beta'\gamma'} = 0, \qquad (\lambda = 4, 5, 6, 7; \quad \alpha \neq \alpha', \ \beta \neq \beta', \ \gamma \neq \gamma'),$$

and can consequently be expressed rationally in terms of six fundamental quantities with the aid of Cayley's formulæ for the coefficients of an orthogonal linear transformation. † Frobenius remarks, however, that it is more convenient to retain the quantities $\dfrac{f_{\lambda\beta\gamma\delta}}{f_{0123}}$, in spite of the relations between them, and to regard these as the fundamental irrational invariants of the quartic. He shows in particular that, if $x_{\alpha\beta}$, $(\alpha, \beta = 0, 1, \ldots, 7)$, are twenty-eight linear forms which, when equated to zero, give equations of the bitangents, we have relations of the type $\sum\limits_{\lambda} f_{\beta\gamma\delta\lambda}\, x_{\alpha\lambda} \equiv 0$, connecting the forms belonging to four bitangents of an Aronhold system. The summation extends over the values of λ which differ from $\alpha, \beta, \gamma, \delta$.

An important consequence of this result is that, if we know the equations of seven bitangents of an Aronhold system, we can determine a set of irrational invariants. Now, the properties of a quartic curve in relation to an Aronhold system of bitangents may be discussed very conveniently with the aid of a $(1, 2)$ transformation ‡ in which the lines of a plane X' correspond to a net of cubics through seven fixed points in a plane X. A point P' in X' consequently corresponds to a pair of points P_1, P_2 in X, and when these come together, P' lies on a quartic curve § which is called the limiting curve (Grenzkurve, Uebergangskurve) ‖ of the transformation. The seven base points in the plane X correspond to seven bitangents of the limiting curve L, and these bitangents form an Aronhold system. It is important to notice that in many cases these

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* *Crelle's Journal*, Bd. 49 (1855).

† *Crelle's Journal*, Bd. 32 (1846), p. 119. "Collected Papers," Vol. II, p. 133.

‡ A. Clebsch, *Math. Ann.*, Bd. 3 (1871). M. Noether, Erlangen *Berichte*, 10 (1878). München *Abh.* (1889). R. de Paolis, *loc. cit.* G. Frobenius, *loc. cit.* L. Cremona and G. Battaglini, *Atti R. Acc. d. Linc.* (3), II, p. 152.

§ S. Aronhold, Berlin *Berichte* (1864).

‖ Also called the synoptic curve. Miss C. A. Scott, *Quarterly Journal*, Vol. XXIX.

bitangents can be derived from the seven base points by a correlation between the two planes, and so the work of calculating Frobenius's irrational invariants is much simplified.

In this memoir we shall endeavor to prepare the way for a discussion of the conditions that a quartic curve may be of a particular type by looking for a (1, 2) transformation which has the species of quartic curve as limiting curve. The appropriate transformation has already been found in a number of cases. It is known, for instance, that the quartic curve L has a double point when three of the base points lie on a line or when six lie on a conic.* L consists of two conics when six of the base points are at the corners of a complete quadrilateral; it consists of four straight lines when six of the base points are consecutive in pairs and the lines joining the three pairs meet at the seventh base point. The conditions on the seven points usually take two or more alternative forms, for it should be noticed that the limiting curve L is unaltered when the net of cubics in the plane X is transformed into another by a quadratic Cremona transformation with base points at three of the seven points.

We shall prove in this memoir that the Lüroth quartic arises from a (1, 2) transformation in which the seven base points have the same polar lines with regard to a conic and a cubic. When the cubic breaks up into three straight lines, the desmic quartic is obtained; this gives a simple verification of Schur's theorem † that the desmic quartic is a particular case of Lüroth's quartic.

The known fact that Lüroth's quartic can be derived from eight associated points which are the poles of a plane with regard to a cubic surface, ‡ is next used to obtain a construction for seven points which have the same polar lines with regard to a conic and a cubic. The desmic quartic arises when the plane meets the cubic surface in three lines.

A (1, 2) transformation is next set up by mapping the chords of a twisted cubic in space on the points of a plane. The two points in which a chord meets a fixed quadric then correspond to the point in the plane which is associated with the chord. The (1, 2) transformation between two planes is finally obtained by mapping the quadric surface on a plane by stereographic projection, using one of the points in which the quadric is met by the twisted cubic as vertex of projection.

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* Also when two of the points come together.

† *Crelle's Journal*, Bd. 95 (1883).

‡ W. Frahm, *Math. Ann.*, Bd. 7 (1874). E. Töplitz, *Math. Ann.*, Bd. 11 (1877).

A transformation which gives rise to a desmic quartic is obtained in this way. The representation also leads to a notable property of a quadratic complex of lines which contains all the lines joining five points on a twisted cubic. It also leads to the consideration of a type of uninodal quartic containing ∞^1 configurations of ten points lying by threes on ten lines, and to a new proof of the theorem* that a plane section of Weddle's surface contains ∞^1 configurations of fifteen points lying by threes on twenty lines.

The method, due to E. Godt† and E. Timerding,‡ of deriving a quartic curve, and in fact a (1, 2) transformation, from a general Cremona quadratic transformation between the lines of a plane is next studied; and quadratic transformations are found which give rise to the Lüroth and desmic quartic respectively.

Klein's quartic is next obtained as the limiting curve of a (1, 2) transformation; and a set of eight associated points in space, from which the curve can be derived, is deduced from this result. The known equations of the twenty-eight bitangents of Klein's quartic are obtained by a simple method.

Dr. Coble§ has extended the known reduction of the equations of a point conic and line conic to the sums of three squares. His extension relates to a point quartic and a line quartic. Two sets of six conics take the places of the sides and vertices of the common self-polar triangle, and each quartic is consequently represented as the sum of the squares of six quadratic forms. Now, it seems natural to try to extend other simplified forms of the equation of a conic to a quartic curve. An extension of the equation of a conic referred to a circumscribed triangle leads to the known form

$$\sqrt{x_1 x_2} + \sqrt{y_1 y_2} + \sqrt{z_1 z_2} = 0$$

of the equation of the general quartic curve. The corresponding generalization of the equation of a conic referred to an inscribed triangle leads to the problem of reducing the equation of the general quartic to the form

$$y_1 y_2 z_1 z_2 + z_1 z_2 x_1 x_2 + x_1 x_2 y_1 y_2 = 0.$$

This problem is shown to be equivalent to that of finding a triangle which is inscribed in a given cubic and circumscribed to a given conic; and, in general, this problem possesses a limited number of solutions. In a special case, however, ∞^1 quadrilaterals can be circumscribed to the conic and completely in-

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* H. Bateman, *Proc. London Math. Soc.*, Ser. 2, Vol. III (1905). Morley and Conner, *loc. cit.*

† "Dissertation," Göttingen (1873). Clebsch-Lindemann, "Vorlesungen," p. 1007.

‡ *Math. Ann.*, Bd. 53 (1900), p. 193.

§ *Trans. Am. Math. Soc.*, Vol. IV (1903).

scribed in the cubic. I have shown that in this case the corresponding quartic curve is a desmic quartic.

It is known that Caporali's quartic possesses ∞^1 sets of twenty-four points with the property that each set can be divided into a group of three quartets such that any two quartets belonging to different groups lie on a pair of lines. Now, I have discovered a second type of quartic curve containing ∞^1 sets of twenty-four points possessing the same property, but these configurations of twenty-four points have other properties which differ from those possessed by the configurations inscribed in Caporali's quartic.

The parts of the thesis which have been reserved for later publication are devoted to the following topics:

1) The analytical study of Hesse's configuration.
2) Discussion of the case "D" with the aid of elliptic functions.
3) A set of eleven points in a space of four dimensions with the property that each point is the vertex of a quadric cone passing through the other ten.
4) A new property of the four-nodal cubic surface.
5) A diagram showing the twenty-eight bitangents of a quartic curve, with their symbols in Hesse's notation.

§ 2. *Lüroth's Quartic.*

Lüroth* has shown that a quartic curve which passes through all the intersections of five lines is not general, for it is the covariant S of a quartic of Clebsch's type, *i. e.*, a quartic whose equation can be expressed as the sum of five fourth powers; and since this type of quartic depends on only thirteen constants, it follows that the equation of Lüroth's quartic involves only thirteen independent constants. Lüroth has shown, moreover, that when one pentagon† is known to be completely inscribed in a given quartic curve, there are ∞^1 pentagons with the same property, and their sides all touch a conic.‡

A particular $(1, 2)$ transformation which has the Lüroth quartic as limiting curve may be obtained as follows:

Let a point P correspond to the two points Q, Q' in which the polar of P with regard to a conic C_2 meets the polar conic of P with respect to a cubic C_3. There is evidently a $(1, 2)$ correspondence between P and Q; for when Q is given, P is the point of intersection of the polar lines of Q with regard to the

* *Math. Ann.*, Bd. 1 (1869), Bd. 13 (1878).

† I use the word "pentagon" here to mean the figure formed by five lines; it has ten vertices.

‡ See also Darboux, "Sur une classe remarquable de courbes et de surfaces algébriques," Paris (1896), p. 186. W. K. Clifford, "Math. Papers," p. 205.

conic and the cubic. To find the locus of P when Q and Q' come together, we take the equations of the cubic and the conic in the forms *

$$(a x^3) = 0, \qquad (b x^2) = 0, \qquad (x) \equiv 0,$$

where the symbol $(a x^3)$ is used to denote the sum of four terms with different suffixes. The four lines $x_1, x_2, x_3, x_4 = 0$ are the common tangents of the ∞^1 conics apolar to C_2 and C_3; they form a quadrilateral whose six corners lie on the Hessian of C_3. Let (y_1, y_2, y_3, y_4) be the coordinates of P. Its polar with regard to C_2 is $(b y x) = 0$, and its polar conic with regard to C_3 is $(a y x^2) = 0$. When Q and Q' come together, the line QQ' touches this polar conic at the point Q (z_1, z_2, z_3, z_4), and so its equation must be equivalent to $(a y z x) = 0$. Since $(x) = 0$, we must have a set of relations of the type

$$a_r y_r z_r = \lambda b_r y_r + \mu, \qquad (r = 1, 2, 3, 4).$$

Using the relations $(z) \equiv 0$, $(b y z) = 0$, we find that

$$\lambda\left(\frac{b}{a}\right) + \mu\left(\frac{1}{a y}\right) = 0, \qquad \lambda\left(\frac{b^2 y}{a}\right) + \mu\left(\frac{b}{a}\right) = 0.$$

Hence,

$$\left(\frac{b^2 y}{a}\right)\left(\frac{1}{a y}\right) = \left(\frac{b}{a}\right)^2.$$

Putting $\left(\frac{b^2 y}{a}\right) + \left(\frac{b}{a}\right)^2 a_5 y_5 \equiv 0$, and using the symbol $[c]$ to denote the sum of five terms with different suffixes, we see that the locus of P, when Q and Q' come together, is the Lüroth quartic $\left[\frac{1}{a y}\right] = 0$. Since the equation of the Hessian of C_3 is $\left(\frac{1}{a y}\right) = 0$, it appears that the Hessian H_3 touches the Lüroth quartic L at the six corners of the quadrilateral.

This indicates a method by which a suitable C_3 can be determined when L is given. Since there are ∞^1 pentagons completely inscribed in L, it appears that there are ∞^1 suitable C_3's.

Returning to the transformation, we notice that, if P describes a straight line $(\xi y) = 0$, the two corresponding points Q describe a cubic curve whose equation is obtained by eliminating y_1, y_2, y_3, y_4 from the four equations

$$(y) = 0, \quad (\xi y) = 0, \quad (b y x) = 0, \quad (a y x^2) = 0.$$

It is easy to see that the equation of the cubic can be derived from that of the line by writing

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* A reduction to these forms is given by R. A. Roberts, *Proc. London Math. Soc.*, Ser. 1, Vol. XXI (1889), p. 62.

$$\begin{aligned}
y_1 &= a_3 b_2 x_2 x_3^2 - a_2 b_3 x_2^2 x_3 + a_2 b_4 x_2^2 x_4 - a_4 b_2 x_2 x_4^2 + a_4 b_3 x_3 x_4^2 - a_3 b_4 x_3^2 x_4, \\
y_2 &= a_1 b_3 x_3 x_1^2 - a_3 b_1 x_3^2 x_1 + a_3 b_4 x_3^2 x_4 - a_4 b_3 x_3 x_4^2 + a_4 b_1 x_1 x_4^2 - a_1 b_4 x_1^2 x_4, \\
y_3 &= a_2 b_1 x_1 x_2^2 - a_1 b_2 x_1^2 x_2 + a_1 b_4 x_1^2 x_4 - a_4 b_1 x_1 x_4^2 + a_4 b_2 x_2 x_4^2 - a_2 b_4 x_2^2 x_4, \\
y_4 &= a_2 b_3 x_2^2 x_3 - a_3 b_2 x_2 x_3^2 + a_3 b_1 x_3^2 x_1 - a_1 b_3 x_3 x_1^2 + a_1 b_2 x_1^2 x_2 - a_2 b_1 x_1 x_2^2;
\end{aligned}$$

and these are the equations of the transformation.

The cubic curve corresponding to a given line t can be generated by the points of intersection of corresponding members of a pencil of conics and a pencil of lines. The conics are the polar conics of points on t with regard to C_3, and so pass through the four poles of t with regard to C_3. The lines are the polars of points on t with respect to C_2; they all pass through T, the pole of t with respect to C_2. This point T evidently lies on the cubic τ_3 corresponding to t, and the four tangents from T to τ_3 are the polars with respect to C_2 of the four points in which t meets L. *Hence, the invariant of* τ_3, *i. e., the cross ratio of the four tangents which can be drawn to it from any point of the curve, is equal to the cross ratio of the four points in which* τ *meets* L.

The four cubics $y_1 = 0$, $y_2 = 0$, $y_3 = 0$, $y_4 = 0$ all pass through the seven points which have the same polar lines with regard to C_2 and C_3. These seven points are the base points of the (1, 2) transformation; and their corresponding lines, *i. e.*, their polars with regard to C_2, are by a well-known theorem bitangents of the limiting curve L. Hence, we have the following theorem:

The seven points which have the same polar lines with regard to a conic C_2 *and a cubic* C_3 *are such that these polar lines form an Aronhold set of bitangents of a Lüroth quartic.*

It appears from this result that the seven points can not be chosen arbitrarily; they depend on thirteen independent constants instead of fourteen. This would not be expected *à priori*, because an arbitrary cubic and conic give fourteen arbitrary constants.

It is not easy to find the relation between the seven base points. At Prof. Morley's suggestion I have considered the cubic C_3 and two points as given, and found the locus of the other five.

Take the two given points and the intersection of their polar lines with regard to C_3 as corners of the triangle of reference. The equations of C_3 and C_2 then take the forms

$$\phi \equiv a x^3 + b y^3 + c z^3 + 3 f y^2 z + 3 g y z^2 + 3 k z x^2 + 3 l x^2 y + 6 n x y z = 0,$$
$$\psi \equiv \lambda x^2 + b g y^2 + c f z^2 + 2 f g y z = 0,$$

where λ is arbitrary. The possible conics C_2 all have double contact at two points on the line $x = 0$.

A point (x, y, z) has the same polar line with regard to C_3 and C_2, if the three derivatives of ϕ are proportional to the three derivatives of ψ; hence, it lies on the two cubic curves

$$f(gy + cz)(lx^2 + by^2 + gz^2 + 2fyz + 2nzx)$$
$$= g(by + fz)(kx^2 + fy^2 + cz^2 + 2gyz + 2nxy),$$
$$\lambda x(lx^2 + by^2 + gz^2 + 2fyz + 2nzx)$$
$$= g(by + fz)(ax^2 + 2nyz + 2kzx + 2lxy).$$

The first equation does not contain λ, and so represents the required locus when two points and C_3 are given. The locus is a cubic curve which circumscribes the triangle of reference and passes through the four poles of $x = 0$ with regard to C_3. It also passes through the two points in which the polar conic of one of the given points meets its polar line. Since the term in xyz is absent from the equation, it appears that the corners of the triangle of reference form a conjugate triad; *i. e.*, the mixed polar of two points passes through the third. It is clear that this cubic locus is the cubic τ_3 corresponding to the line $x = 0$ in the transformation. The associated point T is the point $y = 0$, $z = 0$, and this point is coresidual to the seven base points of the transformation.

If in the previous notation the equation of C_3 is $(ax^3) = 0$, where $(x) \equiv 0$, and (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4) are the two given points, the equation of the locus is

$$(axy^2)(ax^2z)(ayz^2) = (axz^2)(ax^2y)(ay^2z).$$

This equation was obtained by Professor Morley.

Returning to the previous work, we notice that as λ varies the second equation represents a pencil of cubics which pass through four fixed points on τ_3, viz., the two given points and the two points in which the polar conic of one of these points meets its polar line. It appears, then, that the different possible sets of five points associated with the two given ones and C_3 are cut out by a pencil of cubics through four fixed points on τ_3, and so form an involution on τ_3.

§ 3. *Derivation of the Desmic Quartic from a* (1, 2) *Transformation.*

Let us now consider the case when the cubic C_3 consists of three straight lines. Taking these lines as sides of the reference triangle and using the equation of the conic in the form

$$F \equiv (a, b, c, f, g, h \between x, y, z)^2 = 0,$$

we find that the polar conic of a point $P(X, Y, Z)$ with regard to the cubic is

$$Xyz + Yzx + Zxy = 0,$$

while the polar line of P with regard to the conic is $Ux + Vy + Wz = 0$, where $U = aX + hY + gZ$, $V = hX + bY + fZ$, $W = gX + fY + cZ$.

The line touches the conic if

$$\sqrt{XU} + \sqrt{YV} + \sqrt{ZW} = 0. \tag{1}$$

This, however, is Humbert's equation for a desmic quartic curve.* It evidently represents a quartic curve having the lines $X = 0$, $Y = 0$, $Z = 0$, $U = 0$, $V = 0$, $W = 0$ as bitangents. Now, when $X = 0$, we have $bY^2 = cZ^2$; hence, it appears that the points of contact of the three bitangents $X = 0$, $Y = 0$, $Z = 0$ are the corners of a complete quadrilateral whose sides are the lines

$$X\sqrt{a} \pm Y\sqrt{b} \pm Z\sqrt{c} = 0.$$

Humbert has shown that there are altogether six triads of bitangents of the curve which possess this property. The lines $U = 0$, $V = 0$, $W = 0$ form another of these triads.

It should be mentioned that since a, b, c, f, g, h are quite arbitrary, the equation we have obtained represents the general desmic quartic, and so we have the result that the desmic quartic may be derived from a transformation associated with a conic C_2 and a cubic C_3 which consists of three straight lines. This gives a simple verification of Schur's theorem that the desmic quartic is a particular case of Lüroth's quartic. This result may also be deduced from Humbert's remark that the sides of the quadrilateral formed by the lines (1) meet the quartic again in collinear points. This remark also enables us to prove that there are more than one system of pentagons completely inscribed in the desmic quartic; for, if we can show that the quadrilaterals belonging to two triads of bitangents do not touch a conic, we can be sure that the pentagons derived from these quadrilaterals by adding an associated line, belong to different systems. Now, in the case of the two triads of bitangents whose equations have already been obtained, the sides of the two quadrilaterals are given by

$$X\sqrt{a} \pm Y\sqrt{b} \pm Z\sqrt{c} = 0, \quad U\sqrt{bc - f^2} \pm V\sqrt{ca - g^2} \pm W\sqrt{ab - h^2} = 0.$$

Conics touching the sides of these quadrilaterals have equations of the types

$$lX^2 + mY^2 + nZ^2 = 0, \quad \lambda U^2 + \mu V^2 + \nu W^2 = 0,$$

respectively.

These equations are the same only if

$$\lambda gh + \mu bf + \nu fc = \lambda ga + \mu fh + \nu cg = \lambda ah + \mu hb + \nu gf = 0;$$

i. e., if

$$g^2h^2(f^2 - bc) + abfg(ch - fg) + cahf(bg - hf) = 0.$$

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* *Liouville's Journal*, t. 6 (1890), p. 423.

Since this equation is not generally satisfied, we must infer that there are different systems of pentagons associated with the two triads of bitangents. Now, Humbert has shown that there is an equation of the form (1) associated with any two of the six triads of bitangents; hence, we are led to the conclusion that *there are at least six different systems of pentagons completely inscribed in the desmic quartic.*

To determine the nature of the involution on the conic touching the sides of all the pentagons of one system we use Darboux's method. Let $y^2 = xz$ be the equation of the conic; then the equations of the five sides of a circumscribed pentagon may be written in the form

$$\alpha_s^2 x + 2\alpha_s y + z = 0, \qquad (s = 0, 1, \ldots, 4).$$

If now we put

$$x_s = \frac{\alpha_s^2 x + 2\alpha_s y + z}{f'(\alpha_s)}, \qquad (s = 1, \ldots, 4),$$

where $f(\alpha) = (\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha - \alpha_4)$, the condition $(x) \equiv 0$ is satisfied.

The equation of a desmic quartic derived from a cubic C_3 consisting of the three lines $(x^3) = 0$ and the conic C_2 whose equation is $(bx^2) = 0$, is easily found to be

$$(b^2 x)\left(\frac{1}{x}\right) = (b)^2.$$

Taking $(b^2 x) = 0$ to be the line α_0, and noticing that (b) is arbitrary since any multiple of (x) can be added to $(b^2 x)$ without altering the equation of the quartic, we find that this equation takes the form

$$\sum_1^4 \frac{f'(\alpha_s)}{\alpha_s^2 x + 2\alpha_s y + z} + \frac{\lambda}{\alpha_0^2 x + 2\alpha_0 y + z} = 0.$$

Introducing Darboux's coordinates $x : 2y : z = 1 : \theta + \phi : \theta\phi$, we find that the equation may be written in the form

$$\sum_1^4 \frac{f'(\alpha_s)}{\alpha_s - \theta} + \frac{\lambda}{\alpha_0 - \theta} = \sum_1^4 \frac{f'(\alpha_s)}{\alpha_s - \phi} + \frac{\lambda}{\alpha_0 - \phi}.$$

Hence, the pencil of quintics giving the parameters of sets of five tangents which intersect on a desmic quartic is of the form

$$(\alpha_0 - \theta) f(\theta) \left[\sum_1^4 \frac{f'(\alpha_s)}{\alpha_s - \theta} + \frac{\lambda}{\alpha_0 - \theta} + \mu\right] = 0,$$

where μ is a variable parameter.

The desmic quartic is also obtained when the cubic C_3 and the conic C_2 are represented by the equations

$$x^3 + y^3 + z^3 + 6mxyz = 0, \qquad ax^2 + by^2 + cz^2 = 0,$$

respectively.

The polar conic and polar line of (X, Y, Z) are now

$$X(x^2+2myz)+Y(y^2+2mzx)+Z(z^2+2mxy)=0,$$
$$aXx+bYy+cYz=0,$$

respectively. The conic touches the line if

$$\begin{aligned}a^2X^2(YZ-m^2X^2)+b^2Y^2(zX-m^2Y^2)+c^2Z^2(XY-m^2Z^2)\\+2bcYZ(m^2YZ-mX^2)+2caZX(m^2ZX-mY^2)\\+2abXY(m^2XY-mZ^2)=0,\end{aligned}$$

or

$$\begin{aligned}[(a^2-2mbc)X+(b^2-2mca)Y+(c^2-2mab)Z]XYZ\\=m^2[a^2X^4+b^2Y^4+c^2Z^4-2bcY^2Z^2-2caZ^2X^2-2abX^2Y^2].\end{aligned}$$

This, however, is one of Humbert's equations for a desmic quartic. The lines $X=0$, $Y=0$, $Z=0$ are bitangents whose points of contacts lie at the corners of the quadrilateral formed by the four lines

$$\sqrt{a}X\pm\sqrt{b}Y\pm\sqrt{c}Z=0.$$

It appears, then, that if we want the $(1, 2)$ transformation to give rise to a desmic quartic, it is not necessary for the cubic C_3 to break up into three lines.

§ 4. *Derivation of the* (1, 2) *Transformation from Eight Associated Points and vice versa.**

Let a point P whose coordinates are (X, Y, Z) relative to a fixed triangle in a plane E be made to correspond to a quadric surface whose equation is

$$XS_0+YS_1+ZS_2=0, \tag{1}$$

where S_0, S_1, S_2 are of the second degree in the homogeneous coordinates x, y, z, t. As P varies, the quadric always passes through a set of eight associated points $A_0, A_1, \ldots, A_7$.

Let the two generators through A_0 of the quadric corresponding to P meet a fixed plane M in two points Q, Q' and consider the $(1, 2)$ transformation by which P is derived from Q. As P describes a straight line, Q and Q' describe a cubic curve; viz., the projection of the biquadratic common to a pencil of quadrics through $A_0, A_1, \ldots, A_7$. If $a_1, \ldots, a_7$ are the projections of $A_1, \ldots, A_7$ on the plane M when A_0 is taken as vertex of projection, the net of cubics through $a_1, \ldots, a_7$ is the image of the net of lines in the plane E. The points Q and Q' come together when the quadric corresponding to P is a cone, and then by Hesse's theorem the locus of P is a quartic curve which is the limiting curve L of the $(1, 2)$ transformation.

* The leading ideas of this section are due to Hesse, Clebsch and Frobenius. See the memoirs cited above.

If the quadric (1) is always the polar quadric of P with regard to a cubic surface S_3, the eight associated points are the poles of the plane E with regard to S_3, and L is a plane section of the Hessian H_4 of the cubic surface. Now, when the equation of S_3 is written in Sylvester's canonical form $(ax)^3 = 0$ of the sum of five cubes, the equation of H_4 is $(1/ax) = 0$, and it appears that any plane section is a Lüroth quartic.

*The Lüroth quartic can be derived by Hesse's method from eight associated points which are the poles of a plane with regard to a cubic surface.**

The condition to be satisfied in order that three quadrics through eight associated points may be polar quadrics of three points with regard to a cubic surface is discussed by Töplitz.† The invariant Λ must vanish. ‡ It appears that neither the cubic surface nor the three poles are uniquely determined when $\Lambda = 0$. The plane of the three poles and the five planes associated with the canonical form of S_3 osculate a twisted cubic. § Dr. Coble has shown, with the aid of Töplitz's invariant, that when six of the eight points are given, the other two lie on a quartic surface.

To deduce an equation for Lüroth's quartic from Frahm's result, we take four poles of the plane E as vertices of the tetrahedron of reference. If, now, the equation of the cubic surface is

$$(a, b, c, d, f, g, h, k, l, m, u, \alpha, v, \beta, w, \gamma, p, q, r, s \between x, y, z, t)^3 = 0,$$

the polar planes of the corners of the reference tetrahedron coincide with the plane $X + Y + Z + T = 0$, if

$$a = l = k = u, \quad m = b = f = v, \quad h = g = c = w, \quad \alpha = \beta = \gamma = d.$$

The equation of the polar quadric of an arbitrary point (X, Y, Z, T) of this plane is now found to be

$$\begin{aligned} &X[(s-p)yz + (a-q)zx + (a-r)xy + (a-d)xt + (r-d)yt + (q-d)zt] \\ &+ Y[(b-p)yz + (s-q)zx + (b-r)xy + (r-d)xt + (b-d)yt + (p-d)zt] \\ &+ Z[(c-p)yz + (c-q)zx + (s-r)xy + (q-d)xt + (p-d)yt + (c-d)zt] = 0. \end{aligned}$$

This represents a cone if

$$\begin{aligned} &[X(s-p) + Y(b-p) + Z(c-p)]^{\frac{1}{2}} [X(a-d) + Y(r-d) + Z(q-d)]^{\frac{1}{2}} \\ &\pm [X(a-q) + Y(s-q) + Z(c-q)]^{\frac{1}{2}} [X(r-d) + Y(b-d) + Z(p-d)]^{\frac{1}{2}} \\ &\pm [X(a-r) + Y(b-r) + Z(s-r)]^{\frac{1}{2}} [X(q-d) + Y(p-d) + Z(c-d)]^{\frac{1}{2}} = 0. \end{aligned}$$

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* W. Frahm, *Math. Ann.*, Bd. 7 (1874).

† *Math. Ann.*, Bd. 11. See also H. S. White and Miss K. G. Miller, *Bull. Amer. Math. Soc.*, Vol. XV (1909), p. 347.

‡ Cf. Salmon's "Geometry of Three Dimensions," 4th edition, pp. 209–210.

§ A. C. Dixon, *Proc. London Math. Soc.*, Ser. 2, Vol. VII (1909), p. 150. Töplitz, *loc. cit.*

Hence, this is an equation for a Lüroth quartic. The six lines obtained by equating the square brackets to zero are bitangents of the curve.

The tangent plane to the quadric (2) at the point $X = Y = Z = 0$ or A_0 is easily found to be $X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y} + Z\frac{\partial F}{\partial z} = 0$, where $F \equiv (a-d)x^2 + (b-d)y^2 + (c-d)z^2 + 2(p-d)yz + 2(q-d)zx + 2(r-d)xy = 0$; and this is the polar plane of (X, Y, Z, T) with regard to the quadric cone $F = 0$.

Hence, if we make the plane M the same as E, it appears that the point P corresponds to the two points Q, Q' in which its polar plane with regard to the cone meets the section of its polar quadric with regard to S_3.

Let C_2 be the section of the quadric cone and C_3 that of the cubic surface S_3 by the plane E; then the tranformation is clearly derived from C_2 and C_3 by the method of § 2. The seven base points $a_1, a_2, \ldots, a_7$ consequently have the same polar lines with regard to C_2 and C_3; hence, we have the following theorem:

If we project from one of the eight poles of a plane with regard to a cubic surface, the other seven poles project into seven points which have the same polar lines with regard to a conic and a cubic.

Taking each of the eight poles in turn, we find that there are eight Aronhold systems of bitangents of a Lüroth quartic which possess the property of being the polar lines of seven points with regard to a conic and a cubic. We are not justified yet in asserting that every Aronhold system of bitangents possesses this property.

To obtain a set of eight associated points $A_0, \ldots, A_7$ when $a_1, \ldots, a_7$ are given, we proceed as follows: It is known, that eight associated points are the vertices of two tetrahedra which are self-polar with regard to a quadric.* Consequently, if one of these tetrahedra be chosen as tetrahedron of reference, the coordinates of the vertices of the other can be represented by

$$(x_1, y_1, z_1, t_1), \quad (x_2, y_2, z_2, t_2), \quad (x_3, y_3, z_3, t_3), \quad (x_4, y_4, z_4, t_4),$$

where the relations of the type

$$u\,x_r x_s + v\,y_r y_s + w\,z_r z_s + k\,t_r t_s = 0, \qquad (r \neq s),$$

are satisfied. By using the known properties of an orthogonal matrix, we may deduce from these six relations of the type

$$\sum_1^4 \alpha_r x_r t_r = \sum_1^4 \alpha_r y_r t_r = \sum_1^4 \alpha_r z_r t_r = \sum_1^4 \alpha_r y_r z_r = \sum_1^4 \alpha_r z_r x_r = \sum_1^4 \alpha_r x_r y_r = 0.$$

Now let $t = 0$ be the plane containing the seven points $a_1, \ldots, a_7$, and let three of these points be at the corners of the reference tetrahedron. The coordinates of the other four may then be represented by

* Hesse, "Analytische Geometrie des Raumes" (1861), 3d edition, Leipzig (1876), p. 214.

$$(x_1, y_1, z_1, 0), \quad (x_2, y_2, z_2, 0), \quad (x_3, y_3, z_3, 0), \quad (x_4, y_4, z_4, 0).$$

When these are given, our six equations determine the ratios of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and of (t_1, t_2, t_3, t_4) uniquely. Hence, when $a_1, \ldots, a_7$ are given, we can calculate Frobenius' irrational invariants for the associated quartic curve, and if we know the relation between $a_1, \ldots, a_7$ for a particular type of quartic, we can (theoretically) find the corresponding relation between the eight associated points.

§ 5. *Derivation of a* (1, 2) *Transformation by Mapping the Chords of a Twisted Cubic on a Plane.*

Let the polar planes of a point P with regard to a pencil of quadrics meet in a line l. This line meets a given quadric S in two points Q, Q' which may be said to correspond to P. If, moreover, P is restricted to lie in a certain plane X, there is generally only one position of P for a given position of Q, for the polar planes of Q with regard to the quadrics of the pencil intersect in a line l which meets X in P. There is consequently a (1, 2) transformation between the points of the plane X and the points of the quadric S. If P moves along a straight line, the corresponding line l generates a quadric surface which meets S in a biquadratic C_4. This curve C_4 passes through six fixed points $A_1, \ldots, A_6$ on S, viz., the points in which S is met by the twisted cubic K_3 which is the locus of the poles of X with regard to the quadrics of the pencil.

If now we project the points of S on a plane Y, taking A_1 as vertex of projection, the biquadratics C_4 project into cubics through seven fixed points, viz., the projections of $A_2, \ldots, A_6$ and the two points where the generators through A_1 meet Y.

The two points Q, Q', and consequently their projections on the plane Y, come together when the line l touches S; and then the locus of P is a quartic curve L, the limiting curve of the (1,2) transformation between the two planes X and Y, or between X and S. The twenty-eight bitangents of L are derived from the six points $A_1, \ldots, A_6$, the twelve generators through these points and the ten pairs of plane sections of S which pass through the six points. The curve on S which corresponds to L is the intersection of S with the Weddle surface which is the locus of the vertex of a quadric cone through $A_1, \ldots, A_6$; for Geiser* has shown that this surface is also the locus of a point Q such that $A_1, \ldots, A_6, Q, Q'$ form a set of eight associated points. Now, this is what happens in the present case; for when Q and Q' coincide, the ∞^1 lines through P correspond

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* *Crelle's Journal*, Bd. 67 (1867).

to ∞^1 quadrics through $A_1, \ldots, A_6, Q, Q'$, and since S also passes through these points, it follows that the eight points form an associated set.

To find the equation of L, we shall suppose that the pencil of quadrics is given by the equation

$$(a+\lambda)x^2+(b+\lambda)y^2+(c+\lambda)z^2+(d+\lambda)t^2=0,$$

where λ is a variable parameter. If (X, Y, Z, T) are the coordinates of P, the line common to the polar planes of P has coordinates proportional to

$$(b-c)YZ, \quad (c-a)ZX, \quad (a-b)XY, \quad (a-d)XT, \quad (b-d)YT, \quad (c-d)ZT$$

and generates a tetrahedral complex. The condition that this line should touch the quadric S is of the fourth degree in X, Y, Z, T. If, in particular, the equation of S is

$$lx^2+my^2+nz^2+pt^2=0,$$

the condition is

$$\begin{aligned} lp(b-c)^2Y^2Z^2+mp(c-a)^2Z^2X^2+np(a-b)^2X^2Y^2 \\ +mn(a-d)^2X^2T^2+nl(b-d)^2Y^2T^2+lm(c-d)^2Z^2T^2=0. \end{aligned}$$

If

$$\begin{aligned} l&=(a-b)(a-c)(a-d), & m&=(b-a)(b-c)(b-d), \\ n&=(c-a)(c-b)(c-d), & p&=(d-a)(d-b)(d-c), \end{aligned}$$

the equation becomes

$$\begin{aligned} (b-c)(a-d)(Y^2Z^2+X^2T^2)+(c-a)(b-d)(Z^2X^2+Y^2T^2) \\ +(a-b)(c-d)(X^2Y^2+Z^2T^2)=0, \end{aligned}$$

and represents a desmic quartic surface. The curve L, being a plane section of this surface, is a desmic quartic curve.

If in the general case the equation of the plane X is

$$\xi x+\eta y+\zeta z+\tau t=0,$$

the equations of the twisted cubic K_3 are

$$x:y:z:t=\frac{\xi}{a+\lambda}:\frac{\eta}{b+\lambda}:\frac{\zeta}{c+\lambda}:\frac{\tau}{d+\lambda},$$

where λ is a variable parameter. Taking two points on the cubic with parameters λ and μ, we see that the coordinates of the line joining them are proportional to six quantities of the type

$$p_{23}=\frac{(d-a)\xi\tau}{(a+\lambda)(a+\mu)(d+\lambda)(d+\mu)}.$$

Comparing these with the coordinates of the line l, we see that this chord of K_3 is the line l corresponding to a point P with coordinates (X, Y, Z, T) of type

$$X = \frac{(a+\lambda)(a+\mu)}{\xi(a-b)(a-c)(a-d)},$$

and this point lies in the plane X. *Hence, a point P in the plane X corresponds to a chord of the twisted cubic K_3.* A tangent of the twisted cubic corresponds to a point P_0 of a certain conic C_2 obtained by putting $\lambda = \mu$ in the above equations. If a point P moves along a tangent to C_2, the corresponding chord l always passes through a fixed point on K_3. Hence, a point on K_3 is associated with a tangent to C_2, the chord joining two points on K_3 with the point of intersection of the two associated tangents.

It is now evident that the six bitangents of L which correspond to the points $A_1, A_2, \ldots, A_6$ are tangents to the conic C_2; they therefore form a Brianchon set.

It should be noticed that the polar planes of a point on K_3 with regard to the pencil of quadrics meet in a line which lies in the plane X and touches the conic C_2, this is why A_r corresponds to a line and not to a single point. An interesting theorem may be obtained by considering the case when the chords joining five points on K_3 all belong to a quadratic complex R_2. It is clear from the expressions for the coordinates of a line l that the points in the plane X corresponding to these chords all lie on a quartic curve C_4. Now, these points are the ten points of intersection of five tangents to C_2; hence, it follows that the curve C_4 is a Lüroth quartic. Making use of the fact that there are ∞^1 pentagons circumscribed to C_2 and inscribed in C_4, we obtain the following theorem:

If a quadratic complex R_2 and a twisted cubic K_3 are such that a set of five points can be found on K_3 whose joining lines all belong to R_2, then ∞^1 such sets of five points on K_3 can be found.

There is a similar theorem for a complex of degree n and sets of $2n+1$ points on a twisted cubic. It should be remarked that the quadratic complex R_2 can not degenerate into the complex of tangents to a quadric surface; for a quadric surface can not touch the ten lines joining five distinct points. We have no reason to believe, however, that the complex is of a special type, for I have not yet succeeded in finding a relation between its invariants.* It is clear that, if one set of five distinct points can be found whose joins belong to a quadratic complex, there are at least ∞^3 others; for we may take any one of the ∞^2 twisted cubics on the five points, and there will be ∞^1 sets of five points situated thereon. We can expect that there are ∞^5 such sets of five points.

* In the dissertation I suggested that the quadratic complex may be of a special type and contain ∞^6 sets of five points whose joins belong to the complex.

If we reciprocate with regard to one of the fundamental linear complexes of R_2, we find that when sets of five points exist whose joins belong to R_2, there are also sets of five planes whose lines of intersection belong to R_2, and the sets of points and planes are equally numerous.

It is evident, from what has gone before, that if we reciprocate with regard to one of the quadrics of our pencil, the points on the Lüroth quartic C_4 correspond to the planes joining some point on K_3 to the lines joining the different pentads of points on K_3. Now, these lines generate a ruled surface R_8 having K_3 as a fourfold curve, and our theorem tells us that the tangent cone to R_8 from any point of K_3 is the reciprocal of the Lüroth quartic C_4 with regard to some quadric of the pencil. It can also be shown that any plane section of R_8 can be derived from C_4 by a suitable quadratic transformation. Let Y be the plane of the section; then the transformation under consideration is obtained by finding where the line l corresponding to a point P in X meets Y. Now, if S, S' are two quadrics of the pencil, there is a correlation between P and the line p in which its polar plane with regard to S meets Y; similarly, there is a correlation between P and the line p' in which the polar plane with regard to S' meets Y. If p and p' meet in Q, there is a quadratic transformation* which sends P into Q. The base points in the plane Y are the points where this plane meets K_3.

We shall now make further use of this quadratic tranformation. Let $A_1, A_2, \ldots, A_6$ be six arbitrary points on K_3. The lines joining them correspond in X to the points of intersection of six tangents to the conic C_2, while they meet the plane Y in fifteen points lying on the section of the Weddle surface having $A_1, \ldots, A_6$ as nodes. This section is a quartic curve which also passes through the three base points in which K_3 meets Y. The quadratic transformation consequently sends it into a quintic curve C_5 with three nodes, and this quintic passes through the fifteen intersections of the six tangents to C_2. Now, Darboux has shown that when a quintic C_5 and a conic C_2 are such that a set of six tangents to C_2 intersect in fifteen points lying on C_5, there are ∞^1 such sets. Transforming this theorem, we are led to the conclusion that there are ∞^1 sets of points $B_1, \ldots, B_2$ on K_3 whose joining lines cut out a configuration of fifteen points on a plane section of the Weddle surface having $A_1, \ldots, A_6$ as nodes. This result has been known for some time.†

It should be noticed that the correspondence between a point P of the plane X and a chord l of the twisted cubic K_3 enables us to map the Weddle

* T. Reye, *Zeitschr. für Math. u. Phys.*, Bd. 11. "Geometrie der Lage," 3d edition, Part II, § 25.

† H. Bateman, *Proc. London Math. Soc.*, Ser. 2, Vol. III (1905), p. 288. F. Morley and J. R. Conner, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 263.

surface W on a plane by making P correspond to the pair of points in which l meets W. The ∞^3 plane sections of W are then represented by trinodal quintics which pass through the fifteen intersections of six tangents to the conic C_2.

The result that there are ∞^3 trinodal quintics through the fifteen points is unexpected, as the method of counting constants suggests only ∞^2. The result may, however, be verified as follows: Let $x:y:z=1:\theta:\theta^2$ be the parametric representation of the conic C_2; then, if $\alpha_1, \alpha_2, \ldots, \alpha_6$ are the parameters of the points of contact of the six tangents, and $\beta_1, \beta_2, \beta_3$ are three arbitrary parameters, the equation

$$\frac{\prod_1^6 (\theta-\alpha_s)}{(\theta-\beta_1)^2(\theta-\beta_2)^2(\theta-\beta_3)^2}=\frac{\prod_1^6 (\phi-\alpha_s)}{(\phi-\beta_1)^2(\phi-\beta_2)^2(\phi-\beta_3)^2}$$

in Darboux coordinates (θ, ϕ) represents a quintic curve* which passes through the fifteen intersections of the tangents to C_2 at the points $\alpha_1, \ldots, \alpha_6$ and has double points at the points of intersection of the tangents at the points $\beta_1, \beta_2, \beta_3$.†

The theorem can evidently be generalized as follows:

There are ∞^n *curves of degree* $2n-1$ *which pass through the* $n(2n-1)$ *intersections of* $2n$ *tangents to a conic, and have double points at the* $\frac{1}{2}n(n-1)$ *intersections of* n *tangents to the conic.*

Let us next consider a set of five points $B_1, B_2, \ldots, B_5$ on the curve K_3. If l is any chord of K_3, the twisted cubics that pass through $B_1, B_2, \ldots, B_5$ and meet l generate a surface of the fifth degree S_5 having K_3 as double curve. This surface has been discussed by Clebsch,‡ who shows that it can be represented on a plane in such a way that the images of the plane sections are Lüroth quartics passing through a fixed point T and the ten intersections of five lines. The surface S_5 meets a plane Y in a quintic with double points at the three base points in which K_3 meets Y. This quintic Γ_5 contains a configuration of ten points which lie by threes on ten lines, viz., the points where the joins of $B_1, \ldots, B_5$ meet Y; for, since $B_1, \ldots, B_5$ are triple points of S_5, the lines joining them lie entirely on the surface.

The quintic Γ_5 is transformed by our quadratic transformation into a Lüroth quartic C_4 passing through the intersections of five tangents§ to the conic C_2.

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* Cf. Darboux, *loc. cit.*

† Cf. A. C. Dixon, *Quarterly Journal*, Vol. XXVI (1893), p. 212.

‡ *Math. Ann.*, Bd. 1 (1869), p. 253; Bd. 4, p. 249. See also Sturm, *Geometrischen Verwandtschaften*, Bd. 4, p. 315.

§ The fact that the ten points in which the joins of five points in space meet an arbitrary plane can be transformed by a quadratic transformation into the ten intersections of five lines, has been known to Dr. Morley for some time.

Now, since there are ∞^1 configurations of ten points on C_4 which lie by fours on five tangents to C_2, it follows that there are likewise ∞^1 configurations of ten points on Γ_5 lying by threes on ten lines. *Hence, any plane section of S_5 contains ∞^1 configurations of ten points lying by threes on ten lines.*

If the plane Y passes through the line l, the curve Γ_5 breaks up into the line l and a quartic with a double point. Hence, we are led to the conclusion that a quartic with a double point can sometimes contain ∞^1 configurations of ten points lying by threes on ten lines. The corresponding Lüroth quartic has a double point.

The skew cubic which generates S_5 cuts out a series of triads of points on Γ_5 which, together with the three base points cut out by K_3, are all conjugate triads* with regard to a certain conic Γ_2. Hence, any such triad of points lies on a conic through the three base points, and so transforms into a set of three collinear points on C_4. These points are, moreover, collinear with a fixed point on C_4, viz., the point corresponding to the line l. To see this, we have only to remark that any generating cubic of S_5 lies on a quadric through K_3, and this quadric contains the line l, since it is met by it in three points. The conic through the base points which contains the triad of points cut out by this generating cubic consequently passes through the point in which l intersects Y, and so transforms into a line passing through a fixed point T of the Lüroth quartic C_4.

When the generating cubic touches the plane Y, the associated line will touch the Lüroth quartic. Hence, the ten tangents from T to C_4 are derived from the ten generating cubics which touch the plane Y. Now it is clear that the ten points of contact of these cubics lie on the conic Γ_2, and are in fact the ten points in which Γ_2 meets Γ_5. Transforming this result, we find that the points of contact of the ten tangents from T to C_4 lie on a quartic curve with three biflecnodes which lie on C_4. It follows from the plane representation of the surface S_5 that the point T can be regarded as an arbitrary point on the Lüroth quartic C_4; hence, we have the following theorem:

If tangents are drawn to the curve from an arbitrary point T on a Lüroth quartic, the ten points of contact lie on a quartic curve with three biflecnodes situated on the Lüroth quartic. The lines joining these nodes touch the conic C_2, and form with the two tangents from T to C_2 a set of five lines intersecting in ten points on the Lüroth quartic.

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* T. Reye, "Geometrie der Lage," Vol. I (1886), p. 225. A. C. Dixon, *Quarterly Journal*, Vol. XXIII (1889). G. Humbert, *Journal de l'École Polytechnique*, Cah. 64 (1894).

It is clear that the three biflecnodes can be found at once when T, C_2 and C_4 are known. The second part of the theorem follows at once when we consider the set of five points on K_3 consisting of the two points in which it is met by l and the three points in which it is met by Y.

§ 6. *Derivation of a Quartic Curve from a Quadratic Transformation between the Lines of a Plane.*

E. Godt * and E. Timerding † have shown that the general quartic curve can be derived from a quadratic transformation between the lines of a plane by considering the locus of points which lie on their corresponding conics. ‡ Let A, B, C; A', B', C' be the two sets of fundamental points of the transformation, (ξ, η, ζ), (ξ', η', ζ') the coordinates of a line referred to these triangles; then a point whose equation is

$$\xi x + \eta y + \zeta z = 0$$

corresponds to a conic whose tangential equation is

$$\frac{x}{\xi'} + \frac{y}{\eta'} + \frac{z}{\zeta'} = 0.$$

The point equation of this conic is

$$\sqrt{x x'} + \sqrt{y y'} + \sqrt{z z'} = 0.$$

Let

$$x' = a_1 x + b_1 y + c_1 z, \quad y' = a_2 x + b_2 y + c_2 z, \quad z' = a_3 x + b_3 y + c_3 z$$

be the relations connecting the two systems of point coordinates; then the point (x, y, z) lies on its corresponding conic if

$$\sqrt{x\,(a_1 x + b_1 y + c_1 z)} + \sqrt{y\,(a_2 x + b_2 y + c_2 z)} + \sqrt{z\,(a_3 x + b_3 y + c_3 z)} = 0.$$

Comparing this equation with the equations obtained in §§ 3 and 4 for the quartics of Lüroth and Humbert, we can easily find the equations of quadratic transformations which lead to these quartics.

§ 7. *Klein's Quartic Considered as the Limiting Curve of a* (1, 2) *Transformation.*

The general equation of a cubic through the seven points P_ν with coordinates

$$x_\nu = \epsilon^{4\nu}, \quad y_\nu = \epsilon^{\nu}, \quad z_\nu = \epsilon^{2\nu}, \qquad (\nu = 1, 2, \ldots, 7),$$

where $\epsilon^7 = 1$, is

$$\xi\,(x^3 - y z^2) + \eta\,(y^3 - z x^2) + \zeta\,(z^3 - x y^2) = 0.$$

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* "Dissertation," Göttingen (1873). Clebsch-Lindemann, "Vorlesungen," p. 1007.

† *Math. Ann.*, Bd. 53 (1900), p. 193.

‡ This method of generation and a number of others are included in a general scheme studied by Caporali, *Memorie di Geometria* (1888), and Segre, *Annali di Mat.* (2), t. 20 (1892).

Consider the transformation

$$\frac{X}{x^3 - y z^2} = \frac{Y}{y^3 - z x^2} = \frac{Z}{z^3 - x y^2}, \tag{1}$$

which makes a line $\xi X + \eta Y + \zeta Z = 0$ correspond to a cubic curve through the seven points P_ν. It is evidently a $(1, 2)$ transformation whose Jacobian is the sextic curve $J = x^5 y + y^5 z + z^5 x - 3 x^2 y^2 z^2 = 0$.

To find the limiting curve of the transformation, we must eliminate (x, y, z) from this last equation and the equations (1). We easily find that

$$X z + Y x + Z y = 0, \tag{2}$$

and $J = 0$ is equivalent to $X x^2 y + Y y^2 z + Z z^2 x = 0$; consequently

$$\frac{X}{z(x^2 z - y^3)} = \frac{Y}{x(x y^2 - z^3)} = \frac{Z}{y(y z^2 - x^3)}, \quad \text{or} \quad \frac{X}{zY} = \frac{Y}{xZ} = \frac{Z}{yX}.$$

Hence,

$$\frac{X^2}{Y} + \frac{Y^2}{Z} + \frac{Z^2}{X} = 0, \quad \text{or} \quad X^3 Z + Y^3 X + Z^3 Y = 0,$$

the equation of Klein's quartic.* Hence, the limiting curve L is a Klein quartic.

The line joining the two points (x, y, z) corresponding to a given point (X, Y, Z) is represented by the equation

$$X z + Y x + Z y = 0.$$

There is evidently a correlation connecting this line and the point (X, Y, Z), which will be denoted hereafter by Q. If Q lies on L, the two corresponding points P are consecutive and the line joining them envelops the curve whose tangential equation is

$$\xi^3 \zeta + \zeta^3 \eta + \eta^3 \xi = 0.$$

If P, P' are the two consecutive points, P lies on the curve J, and the line PP' may be called the principal direction at P. A curve which crosses J at P in a direction different from the principal one, corresponds to a curve which touches the limiting curve L at Q.† For instance, a line which meets J in six distinct points generally corresponds to a rational cubic touching L in six distinct points.

To find the equation of this cubic, we take the equation of the line in the form

$$\xi x + \eta y + \zeta z = 0. \tag{3}$$

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* Klein-Fricke, "Vorlesungen über die Theorie der elliptischen Modulfunktionen," Leipzig (1892), pp. 675, 678, 701. *Math. Ann.*, Bd. 14 (1879), p. 428. Haskell, AMERICAN JOURNAL OF MATHEMATICS, Vol. XIII (1890). H. F. Baker, "Multiply Periodic Functions," p. 269. E. M. Radford, *Quarterly Journal*, Vol. XXX, p. 263.

† De Paolis, *loc. cit.*

This equation and (2) give

$$\frac{x}{\eta X - \zeta Z} = \frac{y}{\zeta Y - \xi X} = \frac{z}{\xi Z - \eta Y}.$$

Now, $X y^2 + Y z^2 + Z x^2 = 0$; therefore

$$X(\zeta Y - \xi X)^2 + Y(\xi Z - \eta Y)^2 + Z(\eta X - \zeta Z)^2 = 0$$

is the equation of this cubic. The coordinates of the double point are evidently $X = \zeta$, $Y = \xi$, $Z = \eta$; this point is evidently associated with the line (3) in the correlation already mentioned.

A line (3) which passes through one of the base points P_ν corresponds to a rational cubic which breaks up into factors. One of the factors is the line

$$\varepsilon^{4\nu} Y + \varepsilon^{\nu} Z + \varepsilon^{2\nu} X = 0, \qquad (4)$$

associated with the base point P_ν; this line is a bitangent of L. The other factor is a conic

$$\varepsilon^{5\nu}\xi^2 X^2 + \varepsilon^{3\nu}\eta^2 Y^2 + \varepsilon^{6\nu}\zeta^2 Z^2 - (\eta^2 + 2\xi\eta\varepsilon^{3\nu}) YZ$$
$$- (\zeta^2 + 2\eta\zeta\varepsilon^{6\nu}) ZX - (\xi^2 + 2\zeta\xi\varepsilon^{5\nu}) XY = 0,$$

which touches the quartic L in four points. If the line passes through a second base point P_μ, the conic breaks up into a bitangent

$$\varepsilon^{4\mu} Y + \varepsilon^{\mu} Z + \varepsilon^{2\mu} X = 0,$$

corresponding to P_μ and a second bitangent

$$\varepsilon^{5\nu-2\mu}\xi^2 X + \varepsilon^{3\nu-4\mu}\eta^2 Y + \varepsilon^{6\nu-\mu}\zeta^2 Z = 0.$$

Since

$$\xi\varepsilon^{4\nu} + \eta\varepsilon^{\nu} + \zeta\varepsilon^{2} = 0, \quad \xi\varepsilon^{4\mu} + \eta\varepsilon^{\mu} + \zeta\varepsilon^{2\mu} = 0,$$

the equation of this last bitangent may be written in the form

$$(\varepsilon^{\mu} - \varepsilon^{\nu})^2 X + (\varepsilon^{2\mu} - \varepsilon^{2\nu})^2 Y + \varepsilon^{\mu}\varepsilon^{\nu}(\varepsilon^{3\mu} - \varepsilon^{3\nu})^2 Z = 0. \qquad (5)$$

The quartic curve L has seven bitangents with equations of type (4), and twenty-one with equations of type (5). Other equations for the bitangents and quadritangent conics have been obtained by Klein. It should be mentioned that many writers have used a different system of coordinates in studying the properties of the curve.*

A set of eight associated points in space from which Klein's quartic may be derived by Hesse's method, is easily deduced from our set of seven points. If we take four of the points at the corners of the tetrahedron of reference, the other four points may be represented by

$$(1, -\theta, 1+\theta, 2\theta+1), \quad (1+\theta, 1, 1, -1), \quad (-\theta, -1, \theta, \theta+1), \quad (\theta, 1+\theta, 1, \theta),$$

where $\theta = \varepsilon^3 + \varepsilon^5 + \varepsilon^6$ and consequently $\theta^2 + \theta + 2 = 0$.

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* See, for instance, A. Wiman, Stockholm *Bihang till Handlingar* 21 (1895), N. 3. A. B. Coble, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII (1906), p. 333.

§ 8. *Reduction of the Equation of a General Quartic to a Special Form.*

Salmon* has remarked that the general quartic curve can be obtained as the locus of the poles of the tangents to a conic C_2 with regard to a cubic C_3, or as the envelope of the polar conics of points on C_2 with regard to C_3.

Let $x=\theta^2$, $y=2\theta$, $z=1$ be the parametric equations of C_2; then the equation of the polar conic with regard to C_2 of a point on C_2 with parameter θ is represented by

$$A \equiv f(\theta) \equiv \theta^2 S_0 + 2\theta S_1 + S_2 = 0,$$

and the envelope of the conic is the quartic curve $Q \equiv S_0 S_2 - S_1^2 = 0$.

If we use Darboux coordinates (θ, ϕ), where $x=\theta\phi$, $y=\theta+\phi$, $z=1$ are the coordinates of an arbitrary point P and (θ, ϕ) the parameters of the points of contact of the two tangents from P to C_2, it follows that the polar conic of P, viz.,

$$N \equiv f(\theta, \phi) \equiv \theta\phi S_0 + (\theta+\phi) S_1 + S_2 = 0,$$

passes through the poles of the two tangents through P to C_2 with regard to C_3, and therefore the points of contact of the two conics $f(\theta)=0$, $f(\phi)=0$ with their envelope Q.

The conic N consists of two straight lines if P lies on the Hessian H of C_3, and then the two lines intersect in the correspoding point P' on H. The condition for this is that $H(x, y, z)=0$, or $H[\theta\phi, \theta+\phi, 1]=0$, where H is homogeneous and of the third degree.

We are now led to ask the following question: *Can we find three conics*

$$A \equiv f(\theta)=0, \quad B \equiv f(\phi)=0, \quad C \equiv f(\psi)=0,$$

such that the points of contact with Q of each of the three pairs BC, CA, AB lie on two straight lines?

The question is evidently equivalent to the following: *Can we find a triangle which is inscribed in H and circumscribed to C_2?*

The answer seems to be that we can generally find two proper triangles of the required type. For, if we eliminate ψ from the equations

$$H[\theta\psi, \theta+\psi, 1]=0, \quad H[\phi\psi, \phi+\psi, 1]=0,$$

by means of a determinant, using Sylvester's method, we are led to an equation of the ninth degree in θ and of the ninth degree in ϕ. The first three rows of the determinant contain only θ; the last three rows contain only ϕ. It is clear, then, that $(\theta-\phi)^3$ is a factor of the determinant. Rejecting this factor, we are left with an equation of the sixth degree in $\theta\phi$, $\theta+\phi$, 1, which, when written in the form $F(x, y, z)=0$, represents a curve of the sixth degree.

* "Higher Plane Curves." See also Gerbaldi, *Rend. Palermo* (1893); Ciani, *Rend. Lombardo* (1895).

The curves $F=0$, $H=0$ intersect in eighteen points, but these do not all give rise to proper triangles. If RS is a common tangent of F and C_2, R being on F, and T is a point in which the other tangent from R to C_2 meets C_3, RRT must be regarded as a degenerate triangle fulfilling the conditions,* and so R is one of the points of intersection of F and H. Since C_2 and H generally have twelve common tangents, there are generally twelve points of type R.

The points of intersection of C_2 and H do not give rise to degenerate solutions in this case (cf. Clifford, *loc. cit.*). Since we have only accounted for twelve intersections by means of degenerate cases, we must conclude that there are six other intersections which give rise to *two* proper triangles fulfilling the conditions. This number agrees with the well-known result for the case when the cubic H consists of three straight lines.† Our argument is not quite conclusive; it would be more satisfactory if the algebraic work could be carried out in detail.

There is an important exceptional case in which ∞^1 complete quadrilaterals can be inscribed in H and circumscribed to C_2. Darboux ‡ and Clifford § have shown, in fact, that if one complete quadrilateral is circumscribed to C_2 and completely inscribed in H, then ∞^1 such quadrilaterals can be found. This is to be expected, because a quadrilateral inscribed in H and circumscribed to C_2 gives four triangles fulfilling the conditions, and this is more than the proper number; consequently there must be an infinity of triangles.

It will be convenient to call this exceptional case the case "D," and to refer to the special type of quartic which arises from it as a "D" quartic. It will be shown presently that this quartic is a desmic quartic.

In the general case the result we have just obtained may be used to reduce the equation of the general quartic curve to the form

$$\frac{l}{x_1 x_2}+\frac{m}{y_1 y_2}+\frac{n}{z_1 z_2}=0.$$

Let us write $x_1 x_2=f(\phi,\psi)$, $y_1 y_2=f(\psi,\theta)$, $z_1 z_2=f(\theta,\phi)$; we can then find constants α, β, γ such that

$$\alpha y_1 y_2 z_1 z_2+\beta z_1 z_2 x_1 x_2+\gamma x_1 x_2 y_1 y_2 \equiv S_0 S_2-S_1^2.$$

To see this, let us put $S_0=1$, $S_1=t$, $S_2=t^2$; then the above identity holds if

* Cf. Clifford's proof of Poncelet's theorem, "Math. Papers," p. 17. Clifford makes a few remarks on the present problem, but arrives at a slightly different result.

† Salmon's "Conic Sections," p. 273. The reciprocal theorem is given.

‡ *Loc. cit.*

§ "Math. Papers," p. 205.

$$\alpha(t+\theta)^2(t+\phi)(t+\psi)+\beta(t+\phi)^2(t+\psi)(t+\theta)+\gamma(t+\psi)^2(t+\theta)(t+\phi)\equiv 0.$$

Now, this equation can be satisfied by putting

$$\frac{\alpha}{\phi-\psi}=\frac{\beta}{\psi-\theta}=\frac{\gamma}{\theta-\phi},$$

and it is easy to verify that we have identically

$$(\phi-\psi)\,y_1y_2z_1z_2+(\psi-\theta)\,z_1z_2x_1x_2+(\theta-\phi)\,x_1x_2y_1y_2 \equiv(\phi-\psi)(\psi-\theta)(\theta-\phi)(S_0S_2-S_1^2).$$

Hence, the equation of the general quartic curve can be expressed in the form

$$(\phi-\psi)\,y_1y_2z_1z_2+(\psi-\theta)\,z_1z_2x_1x_2+(\theta-\phi)\,x_1x_2y_1y_2=0.$$

If we change the arbitrary constants in $x_2y_2z_2$, the equation may be written

$$Y_1Y_2Z_1Z_2+Z_1Z_2X_1X_2+X_1X_2Y_1Y_2=0.$$

It is clear that the quartic curve passes through the four points of intersection of two pairs of lines such as $X_1X_2=0$, $Y_1Y_2=0$, and that the three pairs of lines give rise to twelve points.

The conics A, B, C now have the simple equations

$$Y_1Y_2+Z_1Z_2=0,\quad Z_1Z_2+X_1X_2=0,\quad X_1X_2+Y_1Y_2=0,$$

and it is easy to verify that each of these conics touches Q in four points.

It should be noticed that the four points in which A cuts $X_1X_2=0$, the four points in which B cuts $Y_1Y_2=0$, and the four points in which C cuts $Z_1Z_2=0$, all lie on the conic

$$X_1X_2+Y_1Y_2+Z_1Z_2=0.$$

This may be written in the alternative forms

$$\frac{x_1x_2}{\phi-\psi}+\frac{y_1y_2}{\psi-\theta}+\frac{z_1z_2}{\theta-\phi}=0,$$

$$(\sigma_2^2-3\sigma_1\sigma_3)\,S_0-2\sigma_3S_1+(\sigma_1^2-3\sigma_2)\,S_2=0,$$

where

$$\sigma_1=\theta+\phi+\psi,\quad \sigma_2=\phi\psi+\psi\theta+\theta\phi,\quad \sigma_3=\theta\phi\psi.$$

§ 9. *Configurations of Sixteen Points Inscribed in a Quartic Curve of Type D.*

If, in the case D, θ, ϕ, ψ, χ are the parameters of the points of contact with C_2 of the sides of one of the quadrilaterals, we have six pairs of straight lines:

$$x_1x_2\equiv f(\phi,\psi)=0,\quad y_1y_2\equiv f(\psi,\theta)=0,\quad z_1z_2\equiv f(\theta,\phi)=0,$$
$$u_1u_2\equiv f(\theta,\chi)=0,\quad v_1v_2\equiv f(\phi,\chi)=0,\quad w_1w_2\equiv f(\psi,\chi)=0.$$

The equation of the quartic curve may be thrown into the forms

$$\begin{aligned}
(\phi-\psi)\, y_1 y_2 z_1 z_2 &+ (\psi-\theta)\, z_1 z_2 x_1 x_2 + (\theta-\phi)\, x_1 x_2 y_1 y_2 = 0,\\
(\phi-\psi)\, v_1 v_2 w_1 w_2 &+ (\psi-\chi)\, w_1 w_2 x_1 x_2 + (\chi-\phi)\, x_1 x_2 v_1 v_2 = 0,\\
(\psi-\theta)\, w_1 w_2 u_1 u_2 &+ (\theta-\chi)\, u_1 u_2 y_1 y_2 + (\chi-\psi)\, y_1 y_2 w_1 w_2 = 0,\\
(\theta-\psi)\, u_1 u_2 v_1 v_2 &+ (\phi-\chi)\, v_1 v_2 z_1 z_2 + (\chi-\theta)\, z_1 z_2 u_1 u_2 = 0,
\end{aligned}$$

or into the form

$$\lambda x_1 x_2 u_1 u_2 + \mu y_1 y_2 v_1 v_2 + \nu z_1 z_2 w_1 w_2 = 0,$$

where λ, μ, ν are subject to the relation $\lambda + \mu + \nu = 0$, but are otherwise arbitrary. It should be noticed that

$$(\phi-\psi)(\theta-\chi) x_1 x_2 u_1 u_2 + (\psi-\theta)(\phi-\chi) y_1 y_2 v_1 v_2 + (\theta-\phi)(\psi-\chi) z_1 z_2 w_1 w_2 \equiv 0.$$

It is clear that the quartic curve passes through the four tetrads of points in which the six pairs of lines $u_1 u_2$, $v_1 v_2$, $w_1 w_2$, $x_1 x_2$, $y_1 y_2$, $z_1 z_2$ intersect. From what has gone before it appears that the three diagonal points of each of these tetrads lie on the Hessian H of the cubic curve C_3, while the six pairs of lines touch the Cayleyan Γ_3 of this cubic. It should be noticed that we have the identities

$$\begin{aligned}
(\phi-\psi)\, u_1 u_2 &+ (\psi-\theta)\, v_1 v_2 + (\theta-\phi)\, w_1 w_2 = 0,\\
(\phi-\psi)\, u_1 u_2 &- (\phi-\chi)\, y_1 y_2 + (\psi-\chi)\, z_1 z_2 = 0,\\
(\psi-\theta)\, v_1 v_2 &- (\psi-\chi)\, z_1 z_2 + (\theta-\chi)\, x_1 x_2 = 0,\\
(\theta-\phi)\, w_1 w_2 &- (\theta-\chi)\, x_1 x_2 + (\phi-\chi)\, y_1 y_2 = 0.
\end{aligned}$$

Consequently we may write

$$\begin{aligned}
(\psi-\theta)\, v_1 v_2 + (\phi-\chi)\, y_1 y_2 &\equiv (\psi-\chi)\, z_1 z_2 - (\theta-\phi)\, w_1 w_2 = L,\\
(\phi-\psi)\, u_1 u_2 + (\theta-\chi)\, x_1 x_2 &\equiv (\phi-\chi)\, y_1 y_2 - (\psi-\theta)\, v_1 v_2 = N,\\
(\theta-\phi)\, w_1 w_2 + (\psi-\chi)\, z_1 z_2 &\equiv (\theta-\chi)\, x_1 x_2 - (\phi-\psi)\, u_1 u_2 = M,
\end{aligned}$$

and the equation of the quartic curve takes the form

$$\frac{\lambda}{(\phi-\psi)(\theta-\chi)}(M^2-N^2) + \frac{\mu}{(\psi-\theta)(\phi-\chi)}(N^2-L^2) + \frac{\nu}{(\theta-\phi)(\psi-\chi)}(L^2-M^2) = 0,$$

or $\sigma L^2 + \rho M^2 + \tau N^2 = 0$, where $\sigma + \rho + \tau = 0$.

The conics L, M, N are so related that the equations

$$M \pm N = 0, \quad N \pm L = 0, \quad L \pm M = 0$$

all represent pairs of straight lines; their equations must consequently be of the form

$$L \equiv BC + AD = 0, \quad M \equiv CA + BD = 0, \quad N \equiv AB + CD = 0,$$

where A, B, C, D are linear functions of the coordinates.

The equation of the quartic curve may now be written in the form

$$\sigma(B^2C^2 + A^2D^2) + \rho(C^2A^2 + B^2D^2) + \tau(A^2B^2 + C^2D^2) = 0,$$

which shows that it is a *desmic quartic, i. e.*, a plane section of the desmic surface studied by Stephanos,* Humbert,† Schroeter‡ and others.

It should be noticed that an equation of the form

$$p\,x_1 x_2 u_1 u_2 + q\,y_1 y_2 v_1 v_2 + r\,z_1 z_2 w_1 w_2 = 0$$

always represents a desmic quartic, for with the aid of the identical relation (1) we may reduce it to the form (2); we may also reduce it to the form

$$F\,x_1 x_2 u_1 u_2 = G\,y_1 y_2 v_1 v_2,$$

which indicates that the curve passes through the sixteen points of the configuration. Hence, there are ∞^1 desmic quartics through these sixteen points.

§ 10. *A Quartic Curve with* ∞^1 *Inscribed Configurations of Twenty-four Points.*

When the configuration of twelve lines touching the curve Γ is discussed with the aid of the parametric representation of this curve in terms of elliptic functions,§ it appears that the twelve lines can be divided into three tetrads such that the points of contact of the four lines of a tetrad lie on another tangent to Γ. The three new tangents obtained in this way meet in a point T, and so it appears that the configuration of points and lines is identical with one whose reciprocal has been studied by Hesse.‖ As the point T moves along a straight line, the configuration of sixteen points derived from it with the aid of the curve Γ, describes a desmic quartic.

Caporali has shown that Hesse's configuration of twelve points and sixteen lines is associated with a second configuration of the same type and that each configuration is derived from the other by the same construction. He has shown, moreover, that the two sets of twelve points lie on a quartic curve¶ which contains ∞^1 configurations of a similar type. This quartic is usually known as Caporali's quartic and depends on eleven arbitrary constants.

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* *Darboux's Bulletin*, sér. 2, t. 3 (1879), p. 424.

† *Liouville's Journal*, sér. 4, t. 7 (1891), p. 353.

‡ *Crelle's Journal*, Bd. 109 (1892), p. 341.

§ Humbert, *loc. cit.*

‖ *Crelle's Journal*, Bd. 36 (1848), p. 153. See also Durège, "Die ebene Curven dritter Ordnung," Leipzig (1871); Caporali, "Memorie di Geometria," p. 338; Schroeter, *Crelle's Journal*, Bd. 108 (1891), p. 269; J. de Vries, *Acta Math.*, t. 12 (1888), p. 63; Martinetti, *Atti dell' Accad. Catania*, ser. 4a, t. 3 (1891) p. 20; S. Nakagawa, *Proc. Tokyo Math. Phys. Soc.*, April, 1907.

¶ *Acc. di Napoli Rend.*, December, 1888, "Memorie di Geometria," pp. 336, 340, 349. *Ann. di Matematica* (2), t. 20 (1892), p. 274. See also Ciani, *Acc. di Napoli Rend.* (3), t. 2 (1896), p. 126. The curve was first studied by Hesse as the Jacobian of a line, a cubic curve and its Hessian.

The two conjugate sets of twelve points are such that any quartet of one set and any quartet of the other set lie on a pair of lines. Two sets of twelve points which possess this property generally depend on twelve arbitrary constants. In Hesse's configuration some other condition is satisfied, and the nine pairs of lines through the quartets of points do not generally belong to a net of conics.

We shall now obtain a type of quartic which contains ∞^1 conjugate pairs of sets of twelve points with the above property and for which the nine pairs of lines do belong to a net.

Going back to the leading ideas of § 8, we now consider the analogue of Darboux's theorem that if the sides of a triangle Δ intersect the sides of a second triangle Δ' in nine points lying on a given cubic curve H, and both triangles circumscribe a conic C_2, there are ∞^1 pairs of triangles circumscribing C_2 whose sides intersect on H. The quartic curve generated by the poles of tangents to C_2 with regard to a cubic curve C_3 having H as Hessian, will possess ∞^1 inscribed configurations of twenty-four points, viz., the poles of the sides of two triangles Δ, Δ', and these twenty-four points can be arranged into two sets of three quartets possessing the property mentioned above.* The lines containing quartets of different sets touch the Cayleyan Γ_3 of the cubic and belong to the net of conics which includes all the polar conics of C_3.

To obtain an equation for our quartic curve, let us consider the case when one vertex of the triangle Δ' lies on the conic; then two of the sides coalesce, and the sides of the triangle Δ touch H at points on a line. Now let $(ax^3)=0$ be the equation of the cubic curve, $\left(\frac{1}{ax}\right)=0$ the equation of its Hessian, the notation being the same as in § 2. The line $x_4=0$ meets the Hessian at points on the lines $x_1, x_2, x_3=0$, and the equations of the tangents at these points are respectively

$$a_1x_1+a_4x_4=0, \quad a_2x_2+a_4x_4=0, \quad a_3x_3+a_4x_4=0.$$

Now, these tangents meet the Hessian again at points on the line $(ax)=0$, and so may be taken as the sides of the triangle Δ. The four lines whose equations have just been given and the line $x_4=0$ all touch the conic C_2; also if (y_1, y_2, y_3, y_4) are the coordinates of any point P on the quartic, the polar line

* If we take two triangles whose sides intersect on H but do not touch a conic, the poles of their sides with regard to C_3 will form two sets of twelve points possessing the same property, but the twenty-four points do not generally lie on a quartic curve. This configuration clearly depends on twelve constants. I am inclined to think that there are two distinct types of configurations of twenty-four points with the above property.

of P with regard to the cubic, viz., $(a y^2 x) = 0$, must also touch C_2. Making use of the relation $(x) \equiv 0$ to get rid of x_1, we find that the six lines touch a conic if

$$\begin{aligned} & a_1^2 a_4 (a_2 - a_3)(a_2 a_3 a_4 + a_1 a_2 a_3 - a_3 a_1 a_4 - a_1 a_2 a_4) y_1^4 + a_2^2 a_4 (a_3 - a_1)(a_3 a_1 a_4 + a_1 a_2 a_3 \\ & \quad - a_1 a_2 a_4 - a_2 a_3 a_4) y_2^4 + a_3^2 a_4 (a_1 - a_2)(a_1 a_2 a_4 + a_1 a_2 a_3 - a_2 a_3 a_4 - a_3 a_1 a_4) y_3^4 \\ & \quad + a_1 a_2 a_3 (a_2 - a_3)(a_2 a_3 a_4 + a_3 a_1 a_4 + a_1 a_2 a_4 - a_1 a_2 a_3 - 2 a_1 a_4^2) y_2^2 y_3^2 \\ & \quad + a_1 a_2 a_3 (a_3 - a_1)(a_2 a_3 a_4 + a_3 a_1 a_4 + a_1 a_2 a_4 - a_1 a_2 a_3 - 2 a_2 a_4^2) y_3^2 y_1^2 \\ & \quad + a_1 a_2 a_3 (a_1 - a_2)(a_2 a_3 a_4 + a_3 a_1 a_4 + a_1 a_2 a_4 - a_1 a_2 a_3 - 2 a_3 a_4^2) y_1^2 y_2^2 \\ & = a_4 y_4^2 [a_1^2 y_1^2 (a_2 - a_3)(a_2 a_3 a_4 + a_1 a_2 a_3 - a_1 a_2 a_4 - a_3 a_1 a_4) \\ & \qquad + a_2^2 y_2^2 (a_3 - a_1)(a_3 a_1 a_4 + a_1 a_2 a_3 - a_1 a_2 a_4 - a_2 a_3 a_4) \\ & \qquad + a_3^2 y_3^2 (a_1 - a_2)(a_1 a_2 a_4 + a_1 a_2 a_3 - a_2 a_3 a_4 - a_3 a_1 a_4)]. \end{aligned}$$

It is clear from this equation that the curve passes through the vertices of the triangle $y_1 y_2 y_3 = 0$, through the four points given by $a_1 y_1^2 = a_2 y_2^2 = a_3 y_3^2$ and through a second set of four points given by equations of the type $c_1 y_1^2 = c_2 y_2^2 = c_3 y_3^2$, the two sets of four points being situated on a conic whose equation is obtained by equating to zero the expression within the square brackets. It appears from the equation that a quartic of the present type depends on eleven arbitrary constants.

On the Continuity of a Lebesgue Integral with Respect to a Parameter.

BY J. K. LAMOND.

Introduction.

Let $f(x, y)$ be a function of two variables defined over a limited, two-dimensional, measurable point set which may be denoted by $\mathfrak{A}$. Let $\mathfrak{B}$ be the projection of $\mathfrak{A}$ on the X-axis, and through each point x of $\mathfrak{B}$ let an ordinate be erected. Each ordinate cuts a section $\mathfrak{C}(x)$ out of $\mathfrak{A}$. When no confusion can arise, we denote this section simply by $\mathfrak{C}$. If each $\mathfrak{C}$ is measurable, and $f(x, y)$ is L-integrable in the sense of Professor Pierpont,* the resulting function

$$J(x) = \int_{\mathfrak{C}} f(x, y)\, dy$$

is a function of the parameter x. In the present paper I give sufficient conditions that

$$\lim_{x = \lambda} J(x) = \int_{\mathfrak{C}(\lambda)} f(\lambda, y)\, dy,$$

and consequently that $J(x)$ be a continuous function of x in $\mathfrak{B}$.

It is understood that the integral signs used denote L-integrals. In case a Riemann integral is used, the sign of integration will be prefixed by the letter R.

The upper measure of a point set $\mathfrak{A}$ will be denoted by the symbol $\overline{\text{meas.}}\,\mathfrak{A}$ or $\overline{\overline{\mathfrak{A}}}$. If $\mathfrak{A}$ is measurable, its measure will be denoted by meas. $\mathfrak{A}$ or $\hat{\overline{\mathfrak{A}}}$.†

§ 1. *Proper Integrals.*

THEOREM 1.‡ *Let* $0 \leq f \leq N$ *be L-integrable in measurable* $\mathfrak{A}$, *and let* $\mathfrak{A}_\kappa$ *be the points of* $\mathfrak{A}$ *at which* $f \geq \kappa$. *Then*

$$\int_{\mathfrak{A}} f = R \int_0^N \hat{\overline{\mathfrak{A}}}_\kappa \, d\kappa. \tag{1}$$

Effect a division of the interval $(0, N)$ of norm d by interpolating the points $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. For uniformity let $\kappa_0 = 0$ and $\kappa_n = N$. Corresponding

* "Lectures on the Theory of Functions of Real Variables," Vol. II, § 380. These will be referred to as "Lectures."

† This is the notation adopted in "Lectures," Vol. II.

‡ This theorem is similar to one given by W. H. Young in a paper "On the General Theory of Integration," *Phil. Trans.*, Vol. CCIV (1905).

to these points, we get the sets $\mathfrak{A}_{\kappa_0}, \mathfrak{A}_{\kappa_1}, \ldots, \mathfrak{A}_{\kappa_n}$ each of which is measurable, by "Lectures," Vol. II, §§ 424 and 408, 2, and each of which contains the following ones. The sets $(\mathfrak{A}_{\kappa_0}-\mathfrak{A}_{\kappa_1}), (\mathfrak{A}_{\kappa_1}-\mathfrak{A}_{\kappa_2}), \ldots, (\mathfrak{A}_{\kappa_{n-1}}-\mathfrak{A}_{\kappa_n})$ are all measurable; and meas. $(\mathfrak{A}_{\kappa_\iota}-\mathfrak{A}_{\kappa_{\iota+1}}) = \bar{\bar{\mathfrak{A}}}_{\kappa_\iota}-\bar{\bar{\mathfrak{A}}}_{\kappa_{\iota+1}}$, $(\iota = 0, 1, 2, \ldots, n-1)$, by "Lectures," Vol. II, § 352, 2.

But obviously any lower summation $\leq$ the L-integral of f over $\mathfrak{A} \leq$ any upper summation. Hence,

$$0 \cdot \text{meas.}\,(\mathfrak{A}_{\kappa_0}-\mathfrak{A}_{\kappa_1}) + x_1 \cdot \text{meas.}\,(\mathfrak{A}_{\kappa_1}-\mathfrak{A}_{\kappa_2}) + \ldots + x_{n-1} \cdot \text{meas.}\,(\mathfrak{A}_{\kappa_{n-1}}-\mathfrak{A}_{\kappa_n})$$
$$\leq \int_{\mathfrak{A}} f \leq$$
$$x_1 \cdot \text{meas.}\,(\mathfrak{A}_{\kappa_0}-\mathfrak{A}_{\kappa_1}) + x_2 \cdot \text{meas.}\,(\mathfrak{A}_{\kappa_1}-\mathfrak{A}_{\kappa_2}) + \ldots + x_n \cdot \text{meas.}\,(\mathfrak{A}_{\kappa_{n-1}}-\mathfrak{A}_{\kappa_n}).$$

Hence, removing the parentheses, we have

$$x_1 \bar{\bar{\mathfrak{A}}}_{\kappa_1} + (x_2 - x_1)\bar{\bar{\mathfrak{A}}}_{\kappa_2} + \ldots + (x_{n-1} - x_{n-2})\bar{\bar{\mathfrak{A}}}_{\kappa_{n-1}} - x_{n-1}\bar{\bar{\mathfrak{A}}}_{\kappa_n}$$
$$\leq \int_{\mathfrak{A}} f \leq$$
$$x_1 \bar{\bar{\mathfrak{A}}}_{\kappa_0} + (x_2 - x_1)\bar{\bar{\mathfrak{A}}}_{\kappa_1} + \ldots + (x_n - x_{n-1})\bar{\bar{\mathfrak{A}}}_{\kappa_{n-1}} - x_n\bar{\bar{\mathfrak{A}}}_{\kappa_n}.$$

Noticing that $\bar{\bar{\mathfrak{A}}}_{\kappa_n} = 0$, this may be written

$$\sum_0^{n-2} (x_{\iota+1} - x_\iota)\bar{\bar{\mathfrak{A}}}_{\kappa_{\iota+1}} \leq \int_{\mathfrak{A}} f \leq \sum_0^{n-1} (x_{\iota+1} - x_\iota)\bar{\bar{\mathfrak{A}}}_{\kappa_\iota}.$$

Let now $d \doteq 0$, and we have

$$R \underline{\int}_0^N \bar{\bar{\mathfrak{A}}}_\kappa \, d\kappa \leq \int_{\mathfrak{A}} f \leq R \overline{\int}_0^N \bar{\bar{\mathfrak{A}}}_\kappa \, d\kappa. \tag{2}$$

But since $\bar{\bar{\mathfrak{A}}}_\kappa$ is a limited, monotone function of κ in $(0, N)$, it is integrable. Hence, (2) gives (1).

COROLLARY 1. *Let $M < f \leq 0$ be L-integrable in measurable $\mathfrak{A}$. Let $\mathfrak{A}'_\kappa$ be the points of $\mathfrak{A}$ at which $f \leq \kappa$. Then*

$$\int_{\mathfrak{A}} f = R \int_M^0 \bar{\bar{\mathfrak{A}}}'_\kappa \, d\kappa.$$

COROLLARY 2. *Let f be an L-integrable function of both signs in measurable $\mathfrak{A}$, such that $M < f < N$. Let $\mathfrak{A}_\kappa$ be the points of $\mathfrak{A}$ at which $f \geq \kappa$, and $\mathfrak{A}'_\kappa$ be the points of $\mathfrak{A}$ at which $f \leq \kappa$. Then*

$$\int_{\mathfrak{A}} f = R \int_M^0 \bar{\bar{\mathfrak{A}}}'_\kappa \, d\kappa + R \int_0^N \bar{\bar{\mathfrak{A}}}_\kappa \, d\kappa, \tag{1}$$
$$= \int_M^0 \bar{\bar{\mathfrak{A}}}_\kappa \, d\kappa + \int_0^N \bar{\bar{\mathfrak{A}}}_\kappa \, d\kappa. \tag{2}$$

Let the points of $\mathfrak{A}$ for which $M < f \leq 0$ be A, for which $0 \leq f < N$ be B, and for which $f = 0$ be C. By "Lectures," Vol. II, §§ 428 and 408, 1 and 2, A, B and C are all measurable. Hence, $A - C$ is measurable. $\mathfrak{A} = (A - C) + B$ and meas. $\mathfrak{A} =$ meas. $(A - C)$ + meas. B. Therefore, by "Lectures," Vol. II, §§ 372 and 390, 2,

$$\int_{\mathfrak{A}} f = \int_{A-C} f + \int_B f.$$

But by "Lectures," Vol. II, § 401,

$$\int_{A-C} f = \int_A f.$$

These relations, with the results of theorem 1 and corollary 1, give (1). We get (2) from (1) by referring to "Lectures," Vol. II, § 381, 2.

THEOREM 2. *Let λ be a proper limiting point of $\mathfrak{B}$. Let $M<f<N$ be L-integrable in each $\mathfrak{C}$, $\mathfrak{C}$ measurable, in some $V_\delta(\lambda)$,* and let $g(y)$ be L-integrable in measurable $\mathfrak{C}(\lambda)$. Let the points of $\mathfrak{C}(\lambda)$ at which $g(y) \leq \kappa$ be ϕ'_κ, and at which $g(y) \geq \kappa$ be ϕ_κ. Let $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ converge uniformly† to $\widehat{\overline{\overline{\phi}}}{}'_\kappa$ in $(M, 0)$, and let $\widehat{\overline{\overline{\mathfrak{C}}}}_\kappa$ converge uniformly to $\widehat{\overline{\overline{\phi}}}_\kappa$ in $(0, N)$, along the section $x=\lambda$, except possibly for the points of a null set $\mathfrak{N}$ of values of κ. Then*

$$\lim_{x=\lambda} \int_{\mathfrak{C}(x)} f\, dy = \int_{\mathfrak{C}(\lambda)} g(y)\, dy. \tag{1}$$

For by theorem 1, corollary 2,

$$\int_{\mathfrak{C}} f\, dy = \int_M^0 \widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa\, d\kappa + \int_0^N \widehat{\overline{\overline{\mathfrak{C}}}}_\kappa\, d\kappa,$$

for x in $V_\delta(\lambda)$. Pass to the limit $x = \lambda$, and we have

$$\lim_{x=\lambda} \int_{\mathfrak{C}(x)} f\, dy = \lim_{x=\lambda} \left[\int_M^0 \widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa\, d\kappa + \int_0^N \widehat{\overline{\overline{\mathfrak{C}}}}_\kappa\, d\kappa\right]. \tag{2}$$

Let

$$D = \int_0^N \widehat{\overline{\overline{\mathfrak{C}}}}_\kappa\, d\kappa - \int_0^N \widehat{\overline{\overline{\phi}}}_\kappa\, d\kappa,$$

for x in $V_\delta(\lambda)$. For any x in $V_\delta(\lambda)$ and for any κ, $\widehat{\overline{\overline{\mathfrak{C}}}}_\kappa$ and $\widehat{\overline{\overline{\phi}}}_\kappa$ are both less than or equal to $\overline{\overline{\Delta}}$, where Δ is the projection of $\mathfrak{A}$ on the Y-axis. Choosing $\varepsilon > 0$, small at pleasure, we may enclose the points of $(0, N)$ belonging to $\mathfrak{N}$ in a set of non-overlapping intervals B, such that $\overline{\overline{B}} < \dfrac{\varepsilon}{4\overline{\overline{\Delta}}}$. Let the remaining

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* $V_\delta(\lambda)$ denotes those points of $\mathfrak{B}$ lying in the interval $\left(\lambda - \frac{\delta}{2}, \lambda + \frac{\delta}{0}\right)$, and is read *vicinity of* λ, *of norm* δ. This is the notation used in "Lectures." (See Vol. I, § 250.)

† See "Lectures," Vol. I, § 561. In this particular case, if, for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa - \widehat{\overline{\overline{\phi}}}{}'_\kappa| < \epsilon \tag{1}$$

for each x in $V_\delta(\lambda)$, and for every κ in $(M, 0)$, we say that $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ converges to $\widehat{\overline{\overline{\phi}}}{}'_\kappa$ *uniformly along the section* $x = \lambda$.

If $\mathfrak{N}$ is a null set of values of κ lying in $(M, 0)$, there exists a division of $(M, 0)$ into two sets of intervals, A and B, such that A contains no points of $\mathfrak{N}$ and the sum of the lengths of the intervals B is as small as we please. If for each $\epsilon > 0$, and for any A, there exists a $\delta > 0$, such that (1) holds for each x in $V_\delta(\lambda)$, and for every κ in A, we say that $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ converges to $\widehat{\overline{\overline{\phi}}}{}'_\kappa$ *uniformly along the section* $x = \lambda$ *except for the points of* $\mathfrak{N}$. If $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ converges to $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa(\lambda)$ uniformly along the section $x = \lambda$, except for the points of $\mathfrak{N}$, we say that $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ is a *uniformly continuous function of x along the section $x = \lambda$ except for the points of* $\mathfrak{N}$. If $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ is a uniformly continuous function of x along each section $x = \lambda$ of $\mathfrak{B}$, except for the points of $\mathfrak{N}$, we say that $\widehat{\overline{\overline{\mathfrak{C}}}}{}'_\kappa$ is a *uniformly continuous function of x in $\mathfrak{B}$ except for the points of* $\mathfrak{N}$.

49

parts of $(0, N)$ be A. Then

$$D = \int_A [\bar{\bar{\mathfrak{C}}}_\kappa - \bar{\bar{\phi}}_\kappa]\, d\varkappa + \int_B [\bar{\bar{\mathfrak{C}}}_\kappa - \bar{\bar{\phi}}_\kappa]\, d\varkappa.$$

Hence,

$$| D | \leq | \int_A | + | \int_B |. \tag{3}$$

But the second part of the right-hand member of (3) is less than $\frac{\varepsilon}{2}$. On the other hand, by hypothesis,

$$| \bar{\bar{\mathfrak{C}}}_\kappa - \bar{\bar{\phi}}_\kappa | < \frac{\varepsilon}{2N},$$

for x in $V_\delta(\lambda)$ and $\varkappa$ in A. Hence, the first part of the right-hand member of (3) is less than $\frac{\varepsilon}{2}$. Hence, $| D | < \varepsilon$, for x in $V_\delta(\lambda)$. Similarly,

$$| D' | = | \int_M^0 \bar{\bar{\mathfrak{C}}}'_\kappa\, d\varkappa - \int_M^0 \bar{\bar{\phi}}'_\kappa\, d\varkappa | < \varepsilon,$$

for x in $V_\delta(\lambda)$. These results, with (2), give

$$\lim_{x=\lambda} \int_{\mathfrak{C}(x)} f\, dy = \int_M^0 \bar{\bar{\phi}}'_\kappa\, d\varkappa + \int_0^N \bar{\bar{\phi}}_\kappa\, d\varkappa = \int_{\mathfrak{C}(\lambda)} g\,(y)\, dy,$$

by theorem 1.

COROLLARY 1. *Let f be L-integrable in each $\mathfrak{C}$, $\mathfrak{C}$ measurable, in $\mathfrak{B}$. Let $\bar{\bar{\mathfrak{C}}}'_\kappa$ and $\bar{\bar{\mathfrak{C}}}_\kappa$ be uniformly continuous functions of x in $\mathfrak{B}$, except possibly for a null set $\mathfrak{N}$ of values of $\varkappa$. Then*

$$J\,(x) = \int_{\mathfrak{C}} f\, dy$$

is continuous in $\mathfrak{B}$.

Example 1. Let f be defined over a set $\mathfrak{A}$, which lies in the rectangle $0, 0;\ 1, 0;\ 0, 2;\ 1, 2$, as follows:

$$f = \frac{1}{nq} \text{ for } x = \frac{m}{n},\ y = \frac{p}{q},^*$$

$$= xy \text{ for } x = \frac{m}{n},\ 0 \leq y \neq \frac{p}{q} \leq 1,$$

$$= x\,[y - 1) \text{ for } x \text{ irrational},\ 1 \leq y \neq \frac{p}{q} \leq 2.$$

Setting $\varkappa = xy$ and solving for y, we have $y = \frac{\varkappa}{x}$. Therefore, $\bar{\bar{\mathfrak{C}}}_\kappa$ equals $1 - \frac{\varkappa}{x}$, for $\varkappa \leq x$, and equals zero for $\varkappa > x$, where x is rational; that is, of the form $\frac{m}{n}$. Setting $\varkappa = x\,(y - 1)$ and solving for y, we have $y = \frac{\varkappa}{x} - 1$. Hence, $\bar{\bar{\mathfrak{C}}}_\kappa$

* All fractions, here and elsewhere, are supposed to be irreducible.

equals $2-\frac{\varkappa}{x}-1=1-\frac{\varkappa}{x}$, for $\varkappa \leq x$, and equals zero for $\varkappa > x$, where x is irrational. Thus $\overset{=}{\mathfrak{C}}_\kappa$ is a uniformly continuous function of x for any $\varkappa$, and hence, $J(x)$ is continuous in $\mathfrak{B}=(0,1)$.

Example 2. Let f be defined as in example 1 for x rational. For x irrational, and $\frac{x}{2} \leq y \leq \frac{x}{2}+1$, let $f=xy-\frac{x^2}{2}$. Let $\varkappa = xy-\frac{x^2}{2}$. Then $y=\frac{\varkappa}{x}+\frac{x}{2}$, and $\overset{=}{\mathfrak{C}}_\kappa$ equals $\frac{x}{2}+1-\left(\frac{\varkappa}{x}+\frac{x}{2}\right)=1-\frac{\varkappa}{x}$, for $\varkappa \leq x$, and equals zero for $\varkappa > x$. Hence, $J(x)$ is continuous in $\mathfrak{B}$, as in example 1.

§ 2. *Improper Integrals.*

Theorem 3. *Let λ be a proper limiting point of $\mathfrak{B}$. Let the improper L-integral* of f over each $\mathfrak{C}$, $\mathfrak{C}$ measurable, exist and converge uniformly† in some $V_\delta(\lambda)$, and let $g(y)$ be improperly L-integrable in measurable $\mathfrak{C}(\lambda)$. For any arbitrary but fixed values of α, β, such as a, b, let the points of $\mathfrak{C}_{ab}(\lambda)$ at which $g(y) \leq \varkappa$ be $\phi'_{ab,\kappa}$, and at which $g(y) \geq \varkappa$ be $\phi_{ab,\kappa}$. Let $\overset{=}{\mathfrak{C}}_{ab,\kappa}$ converge uniformly to $\overset{=}{\phi}_{ab,\kappa}$ in $(-a, 0)$, and let $\overset{=}{\mathfrak{C}}_{ab,\kappa}$ converge uniformly to $\overset{=}{\phi}_{ab,\kappa}$ in $(0, b)$, along the section $x=\lambda$, except possibly at the points of a null set $\mathfrak{N}$ of values of $\varkappa$. Then*

$$\lim_{x=\lambda} \int_{\mathfrak{C}(x)} f\,dy = \int_{\mathfrak{C}(\lambda)} g(y)\,dy.$$

For any x in $V_\delta(\lambda)$, $\mathfrak{C}_{ab}$ is measurable. Hence, by theorem 2,

$$\lim_{x=\lambda} \int_{\mathfrak{C}_{ab}(x)} f\,dy = \int_{\mathfrak{C}_{ab}(\lambda)} g(y)\,dy. \tag{2}$$

By "Lectures," Vol. II, § 146, passing to the limit $a, b=\infty$ in (2), we get (1).

Corollary 1. *Let the improper L-integral of f over measurable $\mathfrak{C}$ converge uniformly in $\mathfrak{B}$. For any arbitrary but fixed values of α, β, such as a, b, let $\overset{=}{\mathfrak{C}}'_{ab,\kappa}$ and $\overset{=}{\mathfrak{C}}_{ab,\kappa}$ be uniformly continuous functions of x in $\mathfrak{B}$, except possibly for a null set $\mathfrak{N}$ of values of $\varkappa$. Then*

$$J(x)=\int_{\mathfrak{C}} f\,dy$$

is continuous in $\mathfrak{B}$.

Wesleyan University, Middletown, Conn., *May*, 1913.

* The definition of an improper L-integral given by Professor Pierpont in "Lectures," Vol. II, § 425, is here used. The modification of the notation there employed to meet our present needs seems obvious.

† The definition of uniform convergence for Pierpont improper integrals as given by me in my paper "Improper Multiple Integrals over Iterable Fields," *Transactions of the American Mathematical Society*, Vol. XIII (1912), p. 436, applies equally well to improper L-integrals.

Geometry on Ruled Surfaces.

By S. LEFSCHETZ.

1. The object of this paper is to prove a formula given by W. E. Story for the number of common points of two curves C_a and C_b of order a and b lying on a scroll S_μ of order μ. He stated that

$$[C_a C_b] = b\alpha + a\beta - \mu\alpha\beta, \tag{1}$$

where $[C_a C_b]$ designates the number of intersections of the two curves C_a and C_b, α and β the number of intersections of C_a and C_b with an arbitrary generator S_μ. He gave the proof himself for $\mu=2$, 3, and F. B. Williams gave it for $\mu=4$, 5.* Of the two proofs here given, one is based upon a classical work of Severi,† the other is of a very elementary character.

2. Before proceeding with our first proof, let us recall a few definitions introduced by Severi. Two curves C and D traced on a surface S will be called *equivalent*, and we shall write $C \equiv D$, if there exists on S an algebraic system of curves containing them both totally. From this definition it is easy to pass to the meaning of $\lambda C \equiv \lambda' D$, λ, λ' being both positive integers, and finally to the meaning of $\Sigma \lambda_i C_i \equiv 0$, the λ's being integers. Severi proves that we can find on S, ρ curves such that between them and any other curve C on S there is always a relation of the type

$$\lambda C + \sum_1^{\rho} {}_i \lambda_i C_i \equiv 0,$$

The numbers $(\lambda, \lambda_1, \ldots., \lambda_\rho)$ are the *characters of C with respect to the base* $(C_1, C_2, \ldots., C_\rho)$. He also gives the so-called Bezout Theorem for a surface by showing that if for D we have

$$\mu D + \sum_1^{\rho} \mu_i C_i \equiv 0,$$

then

$$[CD] = \frac{1}{\lambda\mu} \sum_1^{\rho} {}_{i,k} \lambda_i \mu_k [C_i C_k].$$

In particular he states this for a scroll $\rho = 2$. We shall prove this here, by

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* "Curves on Quintic Scrolls," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 421. A complete bibliography is given there.

† "Sulla totalita delle curve algebriche, *Math. Ann.*, Vol. LXII, p. 124.

showing that, for S_μ, a base is found taking for C_1 a generator and for C_2 a plane section.*

3. Let $a=\mu\nu-h$, $h<\mu$, and let k be a positive integer. A surface of order $k\nu$ going through k curves infinitely near C_a, that is to say, through kC_a, has to satisfy at most $k(ak\nu+1)$ conditions; and if it has to go through kh arbitrary generators of S_μ, it must satisfy at most $kh(k\nu-k\alpha+1)$ conditions, so that the number of linearly independent surfaces of order $k\nu$ going through kC_a, kh generators, and not containing S_μ as a part, is not inferior to

$$N_{k\nu}=\binom{k\nu+3}{3}-\binom{k\nu-\mu+3}{3}-k(\alpha k\nu+1)-kh(k\nu-k\alpha+1)$$
$$=[\mu\nu^2\frac{(\mu-2)}{2}+\alpha h]k^2+\ldots,$$

the unwritten terms containing k to a power lower than 2. The coefficient of k^2 is positive if $\mu>2$. Hence, *when k is above a certain limit, kC_a has kh generators for residual intersection.* It is well known that for $\mu=2$, any curve is either a complete intersection or has a generator for residual intersection, so that the above is true without any restrictions. In the notation of paragraph 2, we have, therefore,

$$kC_a+khC_1-k\nu C_2\equiv 0.$$

As kC_a and khC_1 are cut out by a surface of order $k\nu$ and two arbitrary generators have no common points, it follows that the $k\nu$ intersections of a generator with the $S_{k\nu}$ will be on kC_a, so that

$$k\alpha=k\nu;\quad \therefore\ \alpha=\nu,$$

showing that the characters of C_a with respect to the base (C_1, C_2) are $(k, kh, -k\alpha)$. Similarly, C_b will have characters $(k', k'h', -k'\beta)$, with $a=\alpha\mu-h$, $b=\beta\mu-h'$. We also have

$$[C_1C_1]=0,\quad [C_1C_2]=1,\quad [C_2C_2]=\mu;$$

hence, by Bezout's theorem for S_μ,

$$[C_aC_b]=-h\beta-h'\alpha+\mu\alpha\beta=(a-\alpha\mu)\beta+(b-\beta\mu)\alpha+\mu\alpha\beta=a\beta+b\alpha-\mu\alpha\beta,$$

which was to be proved.

4. The following proof is elementary. The coordinates of any point on a generator can be expressed thus:

$$x_i=m_i-\lambda n_i,\qquad (i=1, 2, 3, 4),$$

where $m_i=m_i(\xi_1, \xi_2, \xi_3)$ and $n_i=n_i(\xi_1, \xi_2, \xi_3)$ are homogeneous functions of the ξ's of the same order ρ, λ is an arbitrary parameter, while the ξ's satisfy a relation $f(\xi_1, \xi_2, \xi_3)=0$, representing in their plane a curve of order m and same genus p as S_μ. If a curve C_a meets an arbitrary generator in α points, it will be represented by an equation of the type

$$\phi_0\lambda^\alpha+\phi_1\lambda^{\alpha-1}+\phi_2\lambda^{\alpha-2}+\ldots+\phi_\alpha=0,$$

*In the paper previously cited, Severi refers the reader, for a proof, to his paper, "Sulle corrispondenze tra i punti," *Torino Memorie*, Vol. LXIV. This being of difficult access, we give below our very simple proof for the sake of completeness.

the ϕ's being all homogeneous polynomials of same order a' in the ξ's. The curves $\phi_i=0$ will have in the ξ-plane a certain number α' of points in common with $f=0$. To find the order a of C_a, we have to find the number of its intersections with a variable plane section:

$$\sum_1^4 A_j m_j - \lambda \sum_1^4 A_j n_j = 0.$$

The result of the elimination of λ between the equations of the two curves is:

$$\sum_0^{\alpha} {}_i\, \phi_i \left(\sum A_j m_j\right)^{\alpha-i} \left(\sum A_j n_j\right)^i = 0,$$

of which we must find the number of variable intersections with $f=0$. If the curves $f=0$, $m_i=0$, $n_i=0$, $(i=1, 2, 3, 4)$, have γ common points, we have

$$a=(a'+\rho\alpha)m-\alpha\gamma.$$

If the equation

$$\psi_0\lambda^{\beta}+\psi_1\lambda^{\beta-1}+\ldots\ldots+\psi_{\beta}=0,$$

where the ψ's are of order b' in the ξ's, represents a second curve C_b of order b, we have

$$b=(b'+\rho\beta)m-\beta\gamma.$$

We obtain $[C_a C_b]$ by eliminating λ between the equations of the two curves. This gives:

$$R(\xi_1, \xi_2, \xi_3) = \left|\begin{matrix} \phi_0 & \phi_1 & \phi_2 & \ldots \\ 0 & \phi_0 & \phi_1 & \ldots \\ 0 & 0 & \phi_0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \psi_0 & \psi_1 & \psi_2 & \ldots \\ 0 & \psi_0 & \psi_1 & \ldots \\ 0 & 0 & \psi_0 & \ldots \\ 0 & 0 & 0 & \ldots \end{matrix}\right| \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}\beta \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}\alpha \end{matrix} = 0,$$

which represents in the ξ-plane a curve of order $(\alpha b'+\beta a')$ intersecting $f=0$ in $m(\alpha b'+\beta a')$ points. From these we must deduce the points common to all the ϕ's or the ψ's and f. A point of multiplicity 9 for all the ϕ's and multiplicity r for f will count for $9r$ points in the intersection of f and ϕ_i, and for $\beta 9r$ points in the intersection of f with R. A similar remark applies to the ψ's, so that finally:

$$\begin{aligned}[C_a C_b] &= m(\beta a'+\alpha b')-\alpha'\beta-\beta'\alpha \\ &= \beta(ma'-\alpha')+\alpha(mb'-\beta') \\ &= a\beta+b\alpha-2(m\rho-\gamma)\alpha\beta.\end{aligned}$$

In particular, if C_a and C_b are two plane sections of S_μ, we have $a=b=\mu$, $\alpha=\beta=1$.

$\therefore$ $[C_a C_b]=\mu=2\mu-2(m\rho-\gamma)$; $\therefore$ $2(m\rho-\gamma)=\mu$; $\therefore$ $[C_a C_b]=b\alpha+a\beta-\mu\alpha\beta$, and the proof is complete.

LINCOLN, NEBR., *October* 26, 1912.

Restricted Systems of Equations.

(Second Paper.)

By Arthur B. Coble.*

The following is a continuation of a previous paper in this Journal.† In § 3 incomplete restricted systems of defect one and two are considered, with the purpose of determining the index numbers of the residual M_1 and M_2. The results suggest the formulæ for the index numbers of a composite spread made up of an M_r and an M_s with a common M_t, where $s \overline{<} 2$, $t < s$.

In § 4 the "relative incidence numbers" of an $M_{r-k}(\gamma)$ on an $M_r(\alpha)$ are defined. They are utilized to determine a new type of index number attached to $M_r(\alpha)$, called the "residual index numbers." In case $M_r(\alpha)$ in S_n is regular or in case $M_r(\alpha)$ is an $M_{n-2}(\alpha)$, the residual index numbers can be expressed by means of the ordinary index numbers α. The new index numbers are employed also to obtain the solution of some particular cases of the more general problems of the theory.

Manifolds defined by matrices are considered in § 5. A simple proof of Salmon's formula for the order of a matrix is given; the formula for the genus of the curve defined by a matrix is derived; the third index number of the two-way defined by a matrix is determined, and all the index numbers of the manifold defined by the particular matrix with n rows and $n+1$ columns are obtained.

§ 3. *The Index Numbers of Residual Intersections and of Composite Manifolds.*

1. Given an $M_r(\alpha)$ in S_n, we have seen how the number, O_r, of points outside of $M_r(\alpha)$ and common to n spreads on $M_r(\alpha)$ can be determined in terms of the orders of the spreads and the index numbers α. This number is merely the order or first index number of the intersection residual to $M_r(\alpha)$. We are thus led to the following inquiry: Given $n-s$ spreads on $M_r(\alpha)$ in S_n, $s \overline{<} r$,

* Written under the auspices of the Carnegie Institution of Washington, D. C.

† Vol. XXXVI (1914), No. 2, pp. 167–186; referred to hereafter as "R. S.," I. The numbering of sections and theorems in this paper is consecutive with "R. S.," I.

which meet in a residual $M_s(\beta)$ which cuts $M_r(\alpha)$ in $M_{s-1}(\gamma)$, how far can the index numbers β and γ, as well as the relative index numbers $(\alpha\gamma)$ and $(\beta\gamma)$, be determined in terms of the $n-s$ given orders and the $r+1$ given index numbers α? For the particular case $s=1$ the answer to this is contained in the following theorem:

(42) *If $n-1$ spreads in S_n of orders $\lambda_1, \ldots, \lambda_{n-1}$ on $M_r(\alpha)$ meet in a residual $M_1(\beta)$ which cuts $M_r(\alpha)$ in $M_0(\gamma)$ points, then $B_0+A_{r-1}=\sigma_{n-1}$, $B_1=rC_0=rA_r$. The index numbers of a composite spread consisting of an $M_r(\alpha)$ and an $M_1(\beta)$ with $M_0(\gamma)$ common points are*

$$\alpha_0, \alpha_1, \ldots, \alpha_{r-2}, \alpha_{r-1}+\beta_0, \alpha_r+\beta_1-(r+1)\gamma_0.$$

Let us recall that the symbol A_j has been defined as follows:

$$A_j=\alpha_j+\alpha_{j-1}\sigma_1+\ldots+\alpha_1\sigma_{j-1}+\alpha_0\sigma_j,$$

the σ's being the elementary symmetric polynomials in the given orders λ. The B_j and C_j are similarly defined for the index numbers β and γ. The formulæ given above determine β_0, β_1, and $\gamma_0=(\alpha\gamma)_0=(\beta\gamma)_0$ in terms of the orders λ and the index numbers α.

According to [(24), "R. S.," I] the theorem is true for $r=1$. Let us assume it to be true for all values of the dimension up to the given r. An n-th spread of order λ on $M_r(\alpha)$ determines O_r points outside of $M_r(\alpha)$. This number O_r is $\lambda\beta_0-\gamma_0$ or, from [(7), "R. S.," I], is $\lambda\sigma_{n-1}-(A_r+\lambda A_{r-1})$. Equating coefficients of the arbitrary order λ, we find that $B_0+A_{r-1}=\sigma_{n-1}$ and $C_0=A_r$. Again, a spread of order μ on $M_1(\beta)$ cuts $M_r(\alpha)$ in $M_{r-1}(\alpha')$, which meets $M_1(\beta)$ in $M_0(\gamma)$ points. Then $M_{r-1}(\alpha')$ and $M_1(\beta)$ together constitute a composite manifold $M_{r-1}(\epsilon)$ which is a complete intersection, and $O_{r-1}=0$ $=\mu\sigma_{n-1}-E_{r-1}-\mu E_{r-2}$. But according to (42), for the dimension $r-1$, E_{r-2} $=A'_{r-2}+B_0$, and $E_{r-1}=A'_{r-1}+B_1-rC_0$. Furthermore, according to [(16), "R. S.," I], $A'_{r-1}+\mu A'_{r-2}=\mu A_{r-1}$, whence $0=\mu\sigma_{n-1}-\mu A_{r-1}-\mu B_0-B_1+rC_0$. Equating coefficients of μ, we find again that $\sigma_{n-1}=A_{r-1}+B_0$, and further that $B_1=rC_0$, which proves the first part of (42).

The $M_r(\alpha)$ and $M_1(\beta)$ constitute a complete $M_r(\delta)$ for which $D_r=0$. We know that $\delta_0=\alpha_0, \ldots, \delta_{r-2}, \alpha_{r-2}, \delta_{r-1}=\alpha_{r-1}+\beta_0$; let us assume that $\delta_r=\alpha_r+\beta_1-x$. Since $D_r=A_r+B_1-x=0$, we see that $x=A_r+B_1=C_0+rC_0=(r+1)\gamma_0$, which for this case at least proves the second part of (42).

A proof which applies generally can be formulated by the aid of the following lemma:

(43) *If $n-r+k$ spreads of orders $\lambda_1, \ldots, \lambda_{n-r+k}$ on $M_r(\alpha)$ in S_n meet in a residual $M_{r-k}(\beta)$ which cuts $M_r(\alpha)$ in an $M_{r-k-1}(\gamma)$, the relative index numbers of $M_{r-k-1}(\gamma)$ as to $M_{r-k}(\beta)$ are $(\beta\gamma)_i=A_{i+k+1}$, $(i=0, 1, \ldots, r-k-1)$.*

Let $r-k$ further spreads of orders $\tau(\mu_1,\ldots,\mu_{r-k})$* on $M_r(\alpha)$ meet in O_r points outside $M_r(\alpha)$, where

$$O_r=\sigma_{n-r+k}\tau_{r-k}-(A_r+A_{r-1}\tau_1+\ldots+A_{k+1}\tau_{r-k-1}+A_k\tau_{r-k}).$$

Since the spreads $\tau(\mu)$ cut $M_{r-k}(\beta)$ in the same number of points outside $M_{r-k-1}(\gamma)$,

$$O_r=\beta_0\tau_{r-k}-\{(\beta\gamma)_{r-k-1}(\beta\gamma)_{r-k-2}\tau_1+\ldots+(\beta\gamma)_1\tau_{r-k-2}+(\beta\gamma)_0\tau_{r-k-1}\}.$$

Noting that $\beta_0=\sigma_{n-r+k}-A_k$, the lemma is proved by equating coefficients of τ.

We need also the further fact:

(44) *The theorem* (42) *applies also to the case where* α, β, γ *are the relative index numbers of* $M_r(\alpha)$, $M_1(\beta)$, *and* $M_0(\gamma)$ *with respect to a manifold* M_n *containing them, provided* σ_{n-1} *is replaced by* $m\sigma_{n-1}$, *where* m *is the order of* M_n.

The proof of (44) parallels that of (42). Suppose, then, that the second part of (42) and of (44) also has been established for values of the dimension up to the value r. Let spreads $\sigma(\lambda_1,\ldots,\lambda_{n-1})$ on $M_r(\alpha)$ and $M_1(\beta)$ meet again in $M_r(\alpha')$, which contains $M_1(\beta)$ and which meets $M_r(\alpha)$ in $M_{r-1}(\alpha'')$. According to (43), $(\alpha'\alpha'')_i=A_{i+1}$. According to [(38), "R. S.," I], the relative index numbers of $M_1(\beta)$, on the composite spread $M_r(\varkappa)$ made up of $M_r(\alpha)$ and $M_r(\alpha')$, are $(\varkappa\beta)_0=\beta_0$ and $(\varkappa\beta)_1=\beta_1+\sigma_1\beta_0$. These relative index numbers can be determined by cutting $M_r(\varkappa)$ by r spreads $\rho(\nu)$ on $M_1(\beta)$ from the equation $O_1=\rho_r(\alpha_0+\alpha_0')-(\varkappa\beta)_0\rho_1-(\varkappa\beta)_1$. The O_1 points are made up of $\rho_r\alpha_0-\gamma_0$ points on $M_r(\alpha)$, and of $\rho_r\alpha_0'-(\alpha'\beta)_0\rho_1-(\alpha'\beta)_1$ points on $M_r(\alpha')$. Since $\beta_0=(\varkappa\beta)_0=(\alpha'\beta)_0$, we find that $(\alpha'\beta)_1=\beta_1+\beta_0\sigma_1-\gamma_0$. For the dimension $r-1$, we have assumed that the relative index numbers of $M_{r-1}(\alpha'')$ and $M_1(\beta)$ on $M_r(\alpha')$ are $(\alpha'\alpha'')_{r-1}+(\alpha'\beta)_1-r\gamma_0$, $(\alpha'\alpha'')_{r-2}+(\alpha'\beta)_0$, $(\alpha'\alpha'')_{r-3},\ldots,$ whence spreads $\tau(\mu_1,\ldots,\mu_r)$ on the two meet $M_r(\alpha')$ in

$$\begin{aligned}O'_{r-1}=\tau_r\alpha_0'-[(\alpha'\alpha'')_{r-1}+(\alpha'\beta)_1-r\gamma_0]\\-[(\alpha'\alpha'')_{r-2}+(\alpha'\beta)_0]\tau_1-(\alpha'\alpha'')_{r-3}\tau_2-\ldots-(\alpha'\alpha'')_0\tau_{r-1}\end{aligned}$$

further points. This can be written as

$$\begin{aligned}O'_{r-1}=\sigma_{n-r}\tau_r-\tau_r\alpha_0-[A_r+B_1-(r+1)\gamma_0]\\-[A_{r-1}+B_0]\tau_1-A_{r-2}\tau_2-\ldots-A_1\tau_{r-1}.\end{aligned}$$

Since this is also the number O_r of points outside of $M_r(\alpha)$ and $M_1(\beta)$ on the $\sigma(\lambda)$ and $\tau(\mu)$ spreads containing them, we see that the last index number of the composite spread must be $\alpha_r+\beta_1-(r+1)\gamma_0$, which completes the proof of (42). A similar argument completes the proof of (44).

* This notation indicates that τ_1, τ_2, ... are the elementary symmetric functions of the given orders.

Further theorems of the following type:

(45) *The index numbers of a composite curve composed of three curves* $M_1(\alpha)$, $M_1(\beta)$, $M_1(\gamma)$, *with respectively* ζ_0, η_0, ϑ_0 *points common to two and with points common to the three of which* δ_0 *have non-coplanar tangents and* δ_0' *have coplanar tangents, are* $\alpha_0+\beta_0+\gamma_0$ *and* $\alpha_1+\beta_1+\gamma_1-2\zeta_0-2\eta_0-2\vartheta_0-4\delta_0-6\delta_0'$,

might be given; but the number of particular cases increases rapidly with the dimension.

2. Let us next consider the case where $n-2$ spreads $\sigma(\lambda)$ on $M_2(\alpha)$ in S_n meet again in $M_2(\beta)$, which cuts $M_2(\alpha)$ in $M_1(\gamma)$. By taking a section and applying (42), we find that $B_0+A_0=\sigma_{n-2}$ and $B_1=C_0=A_1$. By applying the lemma (43), we find that $(\beta\gamma)_1=A_2$, and from the symmetry that $(\alpha\gamma)_1=B_2$. A further spread of order μ on $M_2(\alpha)$ meets $M_2(\beta)$ in $M_1(\gamma)$ and in $M_1(\delta)$, which has ε_0 points in common with $M_1(\gamma)$. Again applying (42), we find that $D_1+\mu D_0=2\varepsilon_0$. Considering $M_1(\gamma)$ and $M_1(\delta)$ on the one hand as a complete intersection of $M_2(\beta)$, and on the other as a composite curve, we have, according to [(17), "R. S.," I] and (42), the index numbers $\mu\beta_0=\gamma_0+\delta_0$, $\mu\beta_1-\mu^2\beta_0=\gamma_1+\delta_1-2\varepsilon_0$. By adding $(\sigma_1+\mu)$ times the first to the second, we get $\mu B_1=C_1+\mu C_0+D_1+\mu D_0-2\varepsilon_0$, whence $C_1=0$. Collecting the above equations, we have

$$(46)\qquad B_0+A_0=\sigma_{n-2},\quad B_1=C_0=A_1,\quad B_2=(\alpha\gamma)_1,\quad A_2=(\beta\gamma)_1,\quad C_1=0.$$

From these we derive

$$B_0+A_0=\sigma_{n-2},\quad B_1+A_1-2C_0=0,\quad B_2+A_2-2C_1-[(\beta\gamma)_1+(\alpha\gamma)_1]=0.$$

The composite spread $M_2(\alpha)$, $M_2(\beta)$ is regular, and these equations show, according to [(9), "R. S.," I], that the index numbers of the composite spread are

$$(47)\qquad \alpha_0+\beta_0,\ \alpha_1+\beta_1-2\gamma_0,\quad \alpha_2+\beta_2-2\gamma_1-[(\alpha\gamma)_1+(\beta\gamma)_1].$$

(48) *If* $n-2$ *spreads on* $M_2(\alpha)$ *in* S_n *meet again in* $M_2(\beta)$, *which cuts* $M_2(\alpha)$ *in* $M_1(\gamma)$, *the index numbers* β, γ, $(\alpha\gamma)$, $(\beta\gamma)$, *with the exception of either* β_2 *or* $(\alpha\gamma)_1$, *are determined in* (46). *The index numbers of the composite spread* $M_2(\alpha)$, $M_2(\beta)$ *with common* $M_1(\gamma)$ *are given in* (47).

We have obtained (47) in the particular case of a composite regular intersection. They are evaluated for the general case below. For the present they may be checked by thinking of the composite spread and of its parts as lying in an S_{n+1} containing S_n. According to [(13) and (34), "R. S.," I] the same formulæ hold, as of course they should.

Next let us consider the residual intersection $M_2(\beta)$ when $M_r(\alpha)$ is an $M_3(\alpha)$. Using the same method as above, we find that $B_0+A_1=\sigma_{n-2}$, $B_1=2C_0=2A_2$, $(\beta\gamma)_1=A_3$, $\varepsilon_0=A_3+\mu A_2$. But in this case $D_1+\mu D_0=3\varepsilon_0$, so that $C_1+A_3=0$. If a spread of order ν on $M_2(\beta)$ cuts $M_3(\alpha)$ in $M_2(\eta)$, where $\eta_0=\nu\alpha_0$, $\eta_1=\nu\alpha_1-\nu^2\alpha_0$, $\eta_2=\nu\alpha_2-\nu^2\alpha_1+\nu^3\alpha_0$, and where, according to [(39), "R. S.," I], $(\eta\gamma)_1=(\alpha\gamma)_1+\nu\gamma_0$, then $M_2(\eta)$ and $M_2(\beta)$ constitute an $M_2(\varkappa)$, which is a complete manifold. Therefore $K_2+\nu K_1=0$, where

$$\begin{aligned}
\varkappa_0&=\nu\alpha_0 &&+\beta_0\\
\varkappa_1&=\nu\alpha_1-\nu^2\alpha_0 &&+\beta_1-2\gamma_0\\
\varkappa_2&=\nu\alpha_2-\nu^2\alpha_1+\nu^3\alpha_0 &&+\beta_2-2\gamma_1-[(\alpha\gamma)_1+\nu\gamma_0+(\beta\gamma)_1].
\end{aligned}$$

Thus we find that $B_2-2C_1-[(\beta\gamma)_1+(\alpha\gamma)_1]=0$ or

$$(49)\qquad \begin{cases} B_0+A_1=\sigma_{n-2}, & B_1=2C_0=2A_2,\\ C_1=-A_3=-(\beta\gamma)_1, & B_2+A_3=(\alpha\gamma)_1. \end{cases}$$

These equations lead to

$$B_1+A_1=\sigma_{n-2},\quad B_1+A_2-3C_0=0,\quad B_2+A_3-3C_1-[(\alpha\gamma)_1+3(\beta\gamma)_1]=0,$$

which proves that the index numbers of the composite spread $M_3(\alpha)$, $M_2(\beta)$ are

$$(50)\qquad \alpha_0,\ \alpha_1+\beta_0,\ \alpha_2+\beta_1-3\gamma_0,\ \alpha_3+\beta_2-3\gamma_1-[(\alpha\gamma)_1+3(\beta\gamma)_1].$$

The above argument by which the result for $M_3(\alpha)$ is gotten from that for $M_2(\alpha)$ can be applied similarly to obtain analogous formulæ for an $M_r(\alpha)$ from those for an $M_{r-1}(\alpha)$. Thus a readily formulated deduction leads to the theorem:

(52) *In S_n, $n-2$ spreads on $M_r(\alpha)$ meet in a residual $M_2(\beta)$ which cuts $M_r(\alpha)$ in $M_1(\gamma)$. The index numbers β, γ, $(\alpha\gamma)$, and $(\beta\gamma)$, with the exception of either β_2 or $(\alpha\gamma)_1$, are determined in terms of α and the given orders by*

$$\begin{aligned}
B_0+A_{r-2}&=\sigma_{n-2}, & B_1&=(r-1)C_0=(r-1)A_{r-1},\\
-C_1&=(r-2)A_r=(r-2)(\beta\gamma), & B_2&+\binom{r-1}{2}A_r=(\alpha\gamma)_1.
\end{aligned}$$

The index numbers of the composite spread $M_r(\alpha)$, $M_2(\beta)$ with common $M_1(\gamma)$ are

$$\alpha_0,\ \alpha_1,\ldots,\alpha_{r-3},\ \alpha_{r-2}+\beta_0,\ \alpha_{r-1}+\beta_1-2\gamma_0,\ \alpha_r+\beta_2-2\gamma_1-\left[(\alpha\gamma)_1+\binom{r}{2}(\beta\gamma)_1\right].$$

Only in the particular case $r=n-2$ is the determination of β_2 made later (see the end of § 4). In fact, it seems probable that the orders and the index numbers α do not constitute, in general, sufficient data to determine β_2. Further index numbers of $M_r(\alpha)$ can be defined in terms of which β_2 can be expressed, but this is not done in this paper.

Let us obtain directly the index numbers of a composite spread $M_2(\alpha)$, $M_2(\beta)$ with common $M_1(\vartheta)$. Spreads $\sigma(\lambda)$ on the two meet in a residual $M_2(\gamma)$ which cuts $M_2(\alpha)$ in $M_1(\eta)$, and $M_2(\beta)$ in $M_1(\zeta)$, and $M_1(\vartheta)$ in $M_0(\delta)$. From the first equation of (46) we get

1°. $\alpha_0+\beta_0+\gamma_0=\sigma_{n-2}$.

From each of the other equations of (46) we deduce three equations according as $M_2(\alpha)$, $M_2(\beta)$, or $M_2(\gamma)$ is looked upon as the residual manifold. These equations are

$$2^\circ.\quad \begin{cases} \zeta_0+\eta_0+\vartheta_0=\zeta_0+\alpha_0\sigma_1+\alpha_1=(\beta_0+\gamma_0)\sigma_1+\beta_1+\gamma_1-\zeta_0, \\ \zeta_0+\eta_0+\vartheta_0=\eta_0+\beta_0\sigma_1+\beta_1=(\gamma_0+\alpha_0)\sigma_1+\gamma_1+\alpha_1-\eta_0, \\ \zeta_0+\eta_0+\vartheta_0=\vartheta_0+\gamma_0\sigma_1+\gamma_1=(\alpha_0+\beta_0)\sigma_1+\alpha_1+\beta_1-\vartheta_0, \end{cases}$$

$$3^\circ.\quad \left.\begin{cases} \sigma_1(\eta_0+\vartheta_0)+\eta_1+\vartheta_1-2\delta_0=0, \\ \sigma_1(\vartheta_0+\zeta_0)+\vartheta_1+\zeta_1-2\delta_0=0, \\ \sigma_1(\zeta_0+\eta_0)=\zeta_1+\eta_1-2\delta_0=0, \end{cases}\right\} \text{ or } \begin{cases} \sigma_1\zeta_0+\zeta_1=\delta_0, \\ \sigma_1\eta_0+\eta_1=\delta_0, \\ \sigma_1\vartheta_0+\vartheta_1=\delta_0, \end{cases}$$

$$4^\circ.\quad \begin{cases} \sigma_2\alpha_0+\sigma_1\alpha_1+\alpha_2=(\gamma\eta)_1+(\beta\vartheta)_1, \\ \sigma_2\beta_0+\sigma_1\beta_1+\beta_2=(\alpha\vartheta)_1+(\gamma\zeta)_1, \\ \sigma_2\gamma_0+\sigma_1\gamma_1+\gamma_2=(\beta\zeta)_1+(\alpha\eta)_1, \end{cases}$$

$$5^\circ.\quad \begin{cases} \sigma_2(\beta_0+\gamma_0)+\sigma_1(\beta_1+\gamma_1-2\zeta_0)+x_{\beta\gamma}=(\alpha\eta)_1+(\alpha\vartheta)_1-2\delta_0, \\ \sigma_2(\gamma_0+\alpha_0)+\sigma_1(\gamma_1+\alpha_1-2\eta_0)+x_{\gamma\alpha}=(\beta\vartheta)_1+(\beta\zeta)_1-2\delta_0, \\ \sigma_2(\alpha_0+\beta_0)+\sigma_1(\alpha_1+\beta_1-2\vartheta_0)+x_{\alpha\beta}=(\gamma\zeta)_1+(\gamma\eta)_1-2\delta_0, \end{cases}$$

where $x_{\beta\gamma}$, etc., are the unknown third index numbers of the composite spread, $M_2(\beta)$, $M_2(\gamma)$, etc. From 1° γ_0 is obtained. Equations 2° reduce to three which determine γ_1, η_0, ζ_0 in terms of α_0, α_1, β_0, β_1, ϑ_0, γ_0. Then from 3° δ_0, ζ_1, η_1 are obtained in terms of ζ_0, η_0, ϑ_0. From 4° $(\gamma\eta)_1+(\gamma\zeta)_1$ is determined in terms of α_i, β_i, $(\alpha\vartheta)_1$, and $(\beta\vartheta)_1$; and, finally, from 5° we get $x_{\alpha\beta}$, which turns out to be $\alpha_2+\beta_2-2\vartheta_1-[(\alpha\vartheta)_1+(\beta\vartheta)_1]$.

By adding 2° we find that

$$\sigma_1(\alpha_0+\beta_0+\gamma_0)+\alpha_1+\beta_1+\gamma_1-2\zeta_0-2\eta_0-2\vartheta_2=0;$$

and by adding 3° and 4°, that

$$6^\circ.\quad \sigma_2(\alpha_0+\beta_0+\gamma_0)+\sigma_1(\alpha_1+\beta_1+\gamma_1-2\zeta_0-2\eta_0-2\vartheta_0) + [\alpha_2+\beta_2+\gamma_2-2\zeta_1-2\eta_1-2\vartheta_1-\{(\alpha\eta)_1+(\alpha\vartheta)_1+(\beta\vartheta)_1 + (\beta\zeta)_1+(\gamma\zeta)_1+(\gamma\eta)_1\}+6\delta_0]=0.$$

Since $M_2(\alpha)$, $M_2(\beta)$, $M_2(\gamma)$ constitute a regular intersection, this shows that

(52) *The index numbers of $M_2(\alpha)$, $M_2(\beta)$, $M_2(\gamma)$ with curves $M_1(\zeta)$, $M_1(\eta)$, $M_1(\vartheta)$ common to two respectively, and points $M_0(\delta)$ common to the three, are the coefficients of σ in 6°.*

We get the same result by using the above formula for $x_{\alpha\beta}$, taking $M_2(\alpha)$, $M_2(\beta)$ as a manifold $M_2(x)$, and $M_2(\gamma)$ as the residual spread meeting $M_2(x)$ in $M_1(\lambda)=M_1(\zeta), M_1(\eta)$. For $\gamma_2=\gamma_2$, $x_2=\alpha_2+\beta_2-2\vartheta_1-[(\alpha\vartheta_1)+(\beta\vartheta)_1]$, $\lambda_1=\zeta_1+\eta_1-2\delta_0$, $(\gamma\lambda)_1=(\gamma\eta)_1+(\gamma\zeta)_1-2\delta_0$, and $(x\lambda)_1=(\alpha\eta)_1+(\beta\zeta)_1$. Hence,

$$x_2+\gamma_2-2\lambda_1-[(x\lambda)_1+(\gamma\lambda)_1]=\alpha_2+\beta_2+\gamma_2-2(\zeta_1+\eta_1+\vartheta_1)$$
$$-[(\alpha\eta)_1+(\alpha\vartheta)_1+(\beta\vartheta)_1+(\beta\zeta)_1+(\gamma\zeta)_1+(\gamma\eta)_1]+6\delta_0.$$

The theorems above can be generalized, as in (44), to apply to the relative index numbers of the $M_2(\beta)$ residual to an $M_r(\alpha)$ on an M_n.

§ 4. *Residual Index Numbers.*

1. Given an $M_{r-k}(\varepsilon)$ on an $M_r(\alpha)$ in S_n; then $n-k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_{k+1}(\beta)$ which has an $M_k(\gamma)$ in common with $M_r(\alpha)$. This $M_k(\gamma)$ meets $M_{r-k}(\varepsilon)$ in I_{r-k} points, and we define the *relative incidence number*, $[\alpha\varepsilon]_{r-k}$, *of* $M_{r-k}(\varepsilon)$ *as to* $M_r(\alpha)$ by means of the equation

$$(53)\qquad I_{r-k}=\sigma_{r-k}[\alpha\varepsilon]_0+\sigma_{r-k-1}[\alpha\varepsilon]_1+\sigma_{r-k-2}[\alpha\varepsilon]_2$$
$$+\ldots.+\sigma_1[\alpha\varepsilon]_{r-k-1}+[\alpha\varepsilon]_{r-k},$$

in terms of the orders λ, the number I_{r-k}, and the earlier incidence numbers, $[\alpha\varepsilon]_{r-k-1},\ldots.,[\alpha\varepsilon]_0$, which are similarly defined for successive sections, in particular $[\alpha\varepsilon]_0$ being ε_0. We shall prove that

(54) *The relative incidence numbers are independent of the orders of the spreads used to define them, and they depend on the underlying dimension* S_n *just as do the ordinary index numbers.*

For if λ_1 increase by one, its spread by an S_{n-1} which cuts $M_r(\alpha)$ in $M_{r-1}(\alpha)$, then the $n-k-2$ spreads $\sigma'(\lambda_2,\ldots.,\lambda_{n-k-1})$ on $M_{r-1}(\alpha)$ meet in a residual $M'_{k+1}(\beta)$ in S_{n-1} which meets $M_{r-1}(\alpha)$ in $M'_k(\gamma)$. This $M'_k(\gamma)$ meets $M_{r-k-1}(\varepsilon)$ in

$$I'_{r-k-1}=\sigma'_{r-k-1}[\alpha\varepsilon]_0+\sigma'_{r-k-2}[\alpha\varepsilon]_1+\ldots.+\sigma'_1[\alpha\varepsilon]_{r-k-2}+[\alpha\varepsilon]_{r-k-1}$$

points. Hence, I_{r-k} is increased by I'_{r-k-1}, which is precisely the increase in I_{r-k} of (54) due alone to the change in λ_1; *i. e.*, $[\alpha\varepsilon]_{r-k}$ is unaltered, and it is independent of the order λ_1. If, however, $M_r(\alpha)$ be supposed to lie in an S_{n+1} containing S_n, we must use $\sigma(1,\lambda_1,\ldots.,\lambda_{n-k-1})$ for the same I_{r-k}. If $[\alpha\varepsilon]'_i+[\alpha\varepsilon]'_{i-1}=[\alpha\varepsilon]_i$ be assumed true for $i=1,\ldots.,r-k-1$, as it is for $i=0$, then $[\alpha\varepsilon]'_{r-k}+[\alpha\varepsilon]'_{r-k-1}=[\alpha\varepsilon]_{r-k}$. Here the $[\alpha\varepsilon]'_i$ refer to the relative incidence numbers in S_{n+1}. Comparing these relations with [(13) and (14), "R. S.," I], we see that the dependence of the relative incidence numbers upon the underlying dimension is the same as that of the ordinary index numbers. The generalization of this result analogous to [(17) and (36), "R. S.," I] is:

(55) *If* $[\alpha\epsilon]_i$ *are the relative incidence numbers of* $M_{r-k}(\epsilon)$ *as to* $M_r(\alpha)$, *the relative incidence numbers* $[\alpha\epsilon]'_i$ *of* $M_{r-k-1}(\epsilon)$ *as to* $M_{r-1}(\alpha)$, *the meets of* $M_{r-k}(\epsilon)$ *and of* $M_r(\alpha)$ *with a spread of order* q, *are given by*

$$[\alpha\epsilon]'_i=q[\alpha\epsilon]_i-q^2[\alpha\epsilon]_{i-1}+q^3[\alpha\epsilon]_{i-2}-\ldots\ldots+(-1)^i q^{i+1}[\alpha\epsilon]_0,$$
$$(i=0,\ldots\ldots,r-k-1).$$

We might expect the relative incidence numbers to behave like the ordinary index numbers, since the latter are a special case of the former. For if $M_{r-k}(\epsilon)$ coincides with $M_r(\alpha)$, k is zero and I_r becomes the $C_0=A_r$ of (42). Comparing A_r with the right-hand member of (53) and noting that, in this case, $[\alpha\epsilon]_0=\epsilon_0=\alpha_0$, we have $[\alpha\epsilon]_i=\alpha_i$.

(56) *The relative incidence numbers of* $M_r(\alpha)$ *as to* $M_r(\alpha)$ *itself are the ordinary index numbers of* $M_r(\alpha)$.

We see from the definition that the relative incidence numbers of $M_a(\epsilon)$ and $M_{a'}(\epsilon')$ on $M_r(\alpha)$ are the sums of the respective relative incidence numbers of $M_a(\epsilon)$ and of $M_{a'}(\epsilon')$ if $a=a'$; otherwise they are the same as those of the manifold of greater dimension.

(57) *If* $M_{r-k}(\epsilon)$ *is the regular intersection of* $M_r(\alpha)$ *and an* M_{n-k} *of order* q, *then* $[\alpha\epsilon]_i=q\alpha_i$, $(i=0,1,\ldots\ldots,r-k)$.

Since $[\alpha\epsilon]_0=\epsilon_0=q\alpha_0$, let us assume that $[\alpha\epsilon]_i=q\alpha_i$ for $i-0,\ldots\ldots,r-k-1$ and determine $[\alpha\epsilon]_{r-k}$. Let $n-k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet again in $M_{k+1}(\beta)$, which cuts $M_r(\alpha)$ in $M_k(\gamma)$. If $M_k(\gamma)$ meets $M_{r-k}(\epsilon)$ in I_{r-k} points, these points are the meets of $M_k(\gamma)$ and M_{n-k}, and

$$I_{r-k}=q\gamma_0=\sigma_{r-k}[\alpha\epsilon]_0+\sigma_{r-k-1}[\alpha\epsilon]_1+\ldots\ldots+\sigma_1[\alpha\epsilon]_{r-k-1}+[\alpha\epsilon]_{r-k}.$$

By applying (42) to a proper section, we find that

$$\gamma_0=\sigma_{r-k}\alpha_0+\sigma_{r-k-1}\alpha_1+\ldots\ldots+\sigma_1\alpha_{r-k-1}+\alpha_{r-k}.$$

Multiplying by q and subtracting, we have $[\alpha\epsilon]_{r-k}=q\alpha_{r-k}$.

(58) *If* $M_{r-k}(\epsilon)$ *on* $M_r(\alpha)$ *be cut regularly by an* M_{n-l} *of order* q *in an* $M_{r-k-l}(\epsilon')$, *then the relative incidence numbers of* $M_{r-k-l}(\epsilon')$ *on* $M_r(\alpha)$ *are* $[\alpha\epsilon']_i=q[\alpha\epsilon]_i$, $(i=0,1,\ldots\ldots,r-k-l)$.

This being true for $[\alpha\epsilon']_0$, let us assume it true for $i=1,\ldots\ldots,r-k-l-1$. If $n-k-l-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_{k+l+1}(\beta)$ which cuts $M_r(\alpha)$ in $M_{k+l}(\gamma)$, the $M_{k+l}(\gamma)$ will meet $M_{r-k}(\epsilon)$ in an $M_l(\delta)$ and $M_{r-k-l}(\epsilon')$ in I'_{r-k-l} points, where $I'_{r-k-l}=q\delta_0$. From a proper section,

$$\delta_0=\sigma_{r-k-l}[\alpha\epsilon]_0+\sigma_{r-k-l-1}[\alpha\epsilon]_1+\ldots\ldots+\sigma_1[\alpha\epsilon]_{r-k-l-1}+[\alpha\epsilon]_{r-k-l}.$$

Multiplying by q and using the assumed relations, we have

$$q\delta_0=I'_{r-k-l}=\sigma_{r-k-l}[\alpha\varepsilon']_0+\sigma_{r-k-l-1}[\alpha\varepsilon']_1+\ldots+\sigma_1[\alpha\varepsilon']_{r-k-l-1}+q[\alpha\varepsilon]_{r-k-l}.$$

Hence, $[\alpha\varepsilon']_{r-k-l}=q[\alpha\varepsilon]_{r-k-l}$.

The theorem (58) for $k=0$ becomes, by the use of (56), the theorem (57). Either will serve as a basis for the following definitions of the residual index numbers of $M_r(\alpha)$.

2. If $n-r-1$ spreads be passed through $M_r(\alpha)$ in S_n, they meet in an $M_{r+1}(\beta)$, which has in common with $M_r(\alpha)$ the manifold $M_r(\alpha)$ itself. The relative incidence numbers of $M_r(\alpha)$ with regard to itself, $[\alpha\alpha]_i$, according to (56), are the ordinary index numbers of $M_r(\alpha)$, α_i, which in this connection we will denote by $\alpha_{0,i}$. Thus $[\alpha\alpha]_i=\alpha_{0,i}$, $(i=0, 1, \ldots, r)$.

If $n-r$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_r(\beta)$ which cuts $M_r(\alpha)$ in $M_{r-1}(\gamma)$, let $[\alpha\gamma]_i$, $(i=0, \ldots, r-1)$, be the relative incidence numbers of $M_{r-1}(\gamma)$ on $M_r(\alpha)$. If one of the spreads, say that of order λ_1, be multiplied by a spread F of order q, then $M_{r-1}(\gamma)$ is increased by $M'_{r-1}(\gamma')$, the meet of F and $M_r(\alpha)$, and $[\alpha\gamma]_i$ is increased by $[\alpha\gamma']_i$, which, according to (57), is $q\alpha_i=q\alpha_{0,i}$. But $\sigma_1\alpha_{0,i}$ is increased by $q\alpha_{0,i}$, whence $\alpha_{1,i}$, defined by

$$(59)\qquad [\alpha\gamma]_i=\alpha_{1,i}+\sigma_1\alpha_{0,i}+\alpha_{0,i+1},\qquad (i=0, 1, \ldots, r-1),$$

is independent of the orders λ and can be regarded as as index number attached to the manifold $M_r(\alpha)$. The change in these index numbers due to a change from S_n to S_{n+1} can be obtained from the corresponding change in $[\alpha\gamma]_i$ [see (54)], in σ, and in $\alpha_{0,i}$.

(60) *The residual index numbers of the second rank defined by* $[\alpha\gamma]_i=\alpha_{1,i}+\sigma_1\alpha_{0,i}+\alpha_{0,i+1}$, $(i=0, \ldots, r-1)$, *in terms of* $M_{r-1}(\gamma)$ *are independent of the orders of the spreads which determine* $M_{r-1}(\gamma)$. *The change in them due to a change from* S_n *to* S_{n+1} *is expressed by*

$$\alpha'_{0,i}+\alpha'_{0,i-1}=\alpha_{0,i},\quad \alpha'_{1,i}+\alpha'_{1,i-1}=\alpha_{1,i}-\alpha_{0,i-1}.$$

In particular, according to (42), $\alpha_{1,0}=0$. For example, let $M_2(\alpha)$ in S_4 be the regular intersection of $u^l=0$, $u^m=0$. Then $\alpha_{0,0}=lm$, $\alpha_{0,1}=-lm(l+m)$, $\alpha_{0,2}=lm(l^2m+lm+m^2)$. Let $u^lf^{\lambda-l}+u^mf^{\lambda-m}=0$, $u^lf^{\mu-l}+u^mf^{\mu-m}=0$ be spreads of orders λ, μ on $M_2(\alpha)$ which meet in $M_2(\beta)$, which cuts $M_2(\alpha)$ in $M_1(\gamma)$. Then $M_1(\gamma)$ is the regular intersection of $u^l=0, u^m=0$, and $\begin{vmatrix} f^{\lambda-l} & f^{\lambda-m} \\ f^{\mu-l} & f^{\mu-m}\end{vmatrix}=0$. According to (57), $[\alpha\gamma]_0=lm(\lambda+\mu-l-m)$ and $[\alpha\gamma]_1=-lm(l+m)(\lambda+\mu-l-m)$. Thus, $\alpha_{1,0}=[\alpha\gamma]_0-(\lambda+\mu)\alpha_{0,0}-\alpha_{0,1}=0$ and $\alpha_{1,1}=[\alpha\gamma]_1-(\lambda+\mu)\alpha_{0,1}-\alpha_{0,2}=l^2m^2$; *i. e.*, $\alpha_{1,0}$ and $\alpha_{1,1}$ are independent of λ, μ.

The residual index numbers of the third rank of $M_r(\alpha)$ are defined as follows: Let $n-r+1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in the residual $M_{r-1}(\beta^{(1)})$

which cuts $M_r(\alpha)$ in $M_{r-2}(\gamma^{(1)})$ with residual incidence numbers $[\alpha\gamma^{(1)}]_i$. Then, if $\alpha_{2,i}$ be defined by

$$(61) \qquad [\alpha\gamma^{(1)}]_i=\alpha_{2,i}+\sigma_1\alpha_{1,i}+\alpha_{1,i+1}+\sigma_2\alpha_{0,i}+\sigma_1\alpha_{0,i+1}+\alpha_{0,i+2}, \quad (i=0, 1, \ldots, r-2),$$

it is independent of the orders and is an index number of $M_r(\alpha)$. For if λ_1 be increased by 1 by adding a linear factor, $M_{r-2}(\gamma^{(1)})$ is increased by a section of the $M_{r-1}(\gamma')$ obtained from $\sigma'(\lambda_2, \ldots, \lambda_{n-r+1})$, $[\alpha\gamma^{(1)}]_i$ is increased by $[\alpha\gamma']_i$, which by the definition of the numbers of the second rank is $\alpha_{1,i}+\sigma'_1\alpha_{0,i}+\alpha_{0,i+1}$. This change in $[\alpha\gamma^{(1)}]_i$ is balanced by the change in the σ's on the right of (61), whence $\alpha_{2,i}$ is unaltered.

The residual numbers of the $(k+1)$-th rank are defined in terms of those of earlier ranks as follows: If $n-r+k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_{r-k+1}(\beta^{(k-1)})$ which cuts $M_r(\alpha)$ in $M_{r-k}(\gamma^{(k-1)})$, whose residual incidence numbers are $[\alpha\gamma^{(k-1)}]$, then $\alpha_{k,i}$ is defined by

$$\begin{aligned}(62) \qquad [\alpha\gamma^{(k-1)}]_i&=\alpha_{k,i}+(\sigma_1\alpha_{k-1,i}+\alpha_{k-1,i+1})+(\sigma_2\alpha_{k-2,i}+\sigma_1\alpha_{k-2,i+1}+\alpha_{k-2,i+2})\\ &\quad+\ldots+(\sigma_k\alpha_{0,i}+\sigma_{k-1}\alpha_{0,i+1}+\ldots+\alpha_{0,i+k})\\ &=(\alpha_{k,i}+\alpha_{k-1,i+1}+\ldots+\alpha_{0,i+k})+\sigma_1(\alpha_{k-1,i}+\alpha_{k-2,i+1}+\ldots\\ &\quad+\alpha_{0,i+k-1})+\ldots+\sigma_{k-1}(\alpha_{1,i}+\alpha_{0,i+1})+\sigma_k(\alpha_{0,i}),\\ &\qquad (k=0, 1, \ldots, r;\ i=0, 1, \ldots, r-k).\end{aligned}$$

It can be shown, as above, that they are independent of the orders λ, and therefore also of the manifold $M_{r-k}(\gamma^{(k-1)})$. A somewhat more convenient form of the definition can be obtained by introducing the abbreviations

$$(63) \qquad \begin{cases} A_{c,d}=\alpha_{c,d}+\sigma_1\alpha_{c,d-1}+\sigma_2\alpha_{c,d-2}+\ldots+\sigma_d\alpha_{c,0}, \text{ and} \\ \overline{A}_{c,d}=\alpha_{c,d}+\sigma_1\alpha_{c-1,d}+\sigma_2\alpha_{c-2,d}+\ldots+\sigma_c\alpha_{0,d}. \end{cases}$$

(64) *The residual index numbers of $M_r(\alpha)$ of the $(k+1)$-th rank, $\alpha_{k,i}$, $(k=0, \ldots, r;\ i=0, \ldots, r-k)$, are defined by*

$$[\alpha\gamma^{(k-1)}]_i=\sum_{l=0}^{i+k}A_{l,i+k-l}-\sum_{l=k+1}^{i+k}\overline{A}_{l,i+k-l},$$

in terms of the index numbers of lower ranks. They are independent of the orders λ and the manifold $M_{r-k}(\gamma^{(k-1)})$.

To identify the two definitions, note that the coefficient of σ_j on the right in (64) is $\sum_{l=0}^{i+k}\alpha_{l,i+k-l-j}-\sum_{l=1+k}^{i+k}\alpha_{l-j,i+k-l}$, while in the original definition it is $\alpha_{k-j,i}+\ldots+\alpha_{0,i+k-j}$. The difference between the two is

$$\sum_{l=i+k-j+1}^{i+k}\alpha_{l,i+k-l-j}-\sum_{l=k+1}^{i+k}\alpha_{l-j,i+k-l}=\sum_{l=i+k-j+1}^{i+k}\alpha_{l,i+k-j-l}.$$

This last sum vanishes, since in each term the second subscript of the α is neg-

ative. Note that in the definition (64) index numbers of rank greater than k formally appear, but they cancel in the expanded formula.

3. (65) *The complete set of residual index numbers of a regular $M_r(\alpha)$ in S_n, determined by spreads of orders $\lambda_1, \ldots, \lambda_{n-r}$, is given by*

$$\alpha_{k,i} = (-1)^{k+i}\pi_{n-r}[\Sigma_k \Sigma_r - \Sigma_{k-1}\Sigma_{i+1}],$$

where π_{n-r} is the product, and Σ_j the complete symmetric polynomial of degree j, formed from the given orders.

Let $n-r+k-1$ spreads $\sigma(l)$ on $M_r(\alpha)$ meet in a residual $M_{r-k+1}(\beta)$ which cuts $M_r(\alpha)$ in $M_{r-k}(\gamma)$. These spreads can be taken in the form

$$\begin{array}{l} u^{\lambda_1} v^{l_1-\lambda_1} + \ldots \quad + u^{\lambda_{n-r}} v^{l_1-\lambda_{n-r}} \quad = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ u^{\lambda_1} v^{l_{n-r+k-1}-\lambda_1} + \ldots + u^{\lambda_{r-s}} v^{l_{n-r+k-1}-\lambda_{n-r}} = 0, \end{array}$$

where $u^{\lambda_1}=0, \ldots, u^{\lambda_{n-r}}=0$ cut out $M_r(\alpha)$. Then $M_{r-k}(\gamma)$ is the regular intersection of $M_r(\alpha)$ by the spread defined by the matrix

$$\left\| \begin{array}{l} v^{l_1-\lambda_1}, \ldots, \quad v^{l_1-\lambda_{n-r}} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ v^{l_{n-r+k-1}-\lambda_1}, \ldots, v^{l_{n-r+k-1}-\lambda_{n-r}} \end{array} \right\| = 0,$$

whose order, according to Salmon's formula,* is

$$(\sigma_k - \sigma_{k-1}\Sigma_1 + \sigma_{k-2}\Sigma_2 - \ldots + (-1)^{k-1}\sigma_1\Sigma_{k-1} + (-1)^k\Sigma_k).$$

From (57) and (64) we have, respectively,

$$[\alpha\gamma^{(k-1)}]_i = (\sigma_k - \sigma_{k-1}\Sigma_1 + \ldots + (-1)^k\Sigma_k)\alpha_i, \text{ and}$$

$$[\alpha\gamma^{(k-1)}]_i = \sum_{l=0}^{i+k} A_{l,\,i+k-l} - \sum_{l=k+1}^{k+i} \overline{A}_{l,\,i+k-l}.$$

Equating the terms in σ_{k-j}, we find that

$$(-1)^j \Sigma_j \alpha_i = \sum_{l=0}^{i+k} \alpha_{l,\,i-l+j} - \sum_{l=k+1}^{k+i} \alpha_{l-k+j,\,i+k-l}, \quad (j=0, 1, \ldots, k).$$

Writing this equality for $j=k$, $i=i$, and again for $j=k-1$, $i=i+1$, we get, by subtraction, $(-1)^k(\Sigma_k\alpha_i + \Sigma_{k-1}\alpha_{i+1}) = \alpha_{k,i}$. According to [(11), "R. S.," I],

$$\alpha_i = (-1)^i \pi_{n-r}\Sigma_i, \quad \alpha_{i+1} = (-1)^{i+1}\pi_{n-r}\Sigma_{i+1},$$

whence $\alpha_{k,i} = (-1)^{i+k}\pi_{n-r}(\Sigma_k\Sigma_i - \Sigma_{k-1}\Sigma_{i+1})$.

4. In the above particular case of a regular $M_r(\alpha)$ the relations $\alpha_{k,i} + \alpha_{i+1,k-1} = 0$, $\alpha_{k+1,k} = 0$ exist. Let us prove generally that

* A simple proof of this formula is given in § 5.

(66) *The $\frac{1}{2}r(r+1)$ residual index numbers of $M_r(\alpha)$ satisfy the following relations: $\alpha_{k,i}+\alpha_{i+1,k-1}=0$, and $\alpha_{k,k-1}=0$.*

Let $n-r+k-1$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ determine $M_{r-k}(\gamma^{(k-1)})$, and $n-k-1$ spreads $\tau(\mu)$ on $M_r(\alpha)$ determine $M_k(\gamma^{(r-k-1)})$. If these two manifolds on $M_r(\alpha)$ meet in I_{r-k} points, then we have from (53), if we use first $M_{r-k}(\gamma^{(k-1)})$ as $M(\epsilon)$ and second $M_k(\gamma^{(r-k-1)})$ as $M(\epsilon)$, the equations:

$$I_{r-k}=[\alpha\gamma^{(k-1)}]_{r-k}+\tau_1[\alpha\gamma^{(k-1)}]_{r-k-1}+\ldots\ldots+\tau_{r+k}[\alpha\gamma^{(k-1)}]_0,$$
$$=[\alpha\gamma^{(r-k-1)}]_k+\sigma_1[\alpha\gamma^{(r-k-1)}]_{k-1}+\ldots\ldots+\sigma_k[\alpha\gamma^{(r-k-1)}]_0.$$

Here the $[\alpha\gamma^{(k-1)}]_j$ are given by (64) in terms of the σ's, and the $[\alpha\gamma^{(r-k-1)}]_j$ are given in terms of the τ's. Equating coefficients of $\sigma_s\tau_t$, we get

$$\sum_{l=0}^{r-t}\alpha_{l,\,r-l-s-t}-\sum_{l=k+1}^{r-t}\alpha_{l-s,\,r-l-t}=\sum_{l=0}^{r-s}\alpha_{l,\,r-l-s-t}-\sum_{l=r-k+1}^{r-s}\alpha_{l-t,\,r-l-s},$$

where $0\overline{<}t\overline{<}r-k$ and $0\overline{<}s\overline{<}k$. If in this equality t be increased by 1 and s be diminished by 1, and the result be subtracted from the given equality, we get $\alpha_{r-t,-s}-\alpha_{k+1-s,\,r-t-k-1}=-\alpha_{r-s+1,\,-t-1}+\alpha_{r-k-t,\,k-s}$, where now $0\overline{<}t\overline{<}r-k-1$ and $1\overline{<}s\overline{<}k$. Thus two of the α's have a negative subscript and vanish, and $\alpha_{j,m}+\alpha_{m+1,j-1}=0$, $(j=k,\ldots,1;\ m=r-k-1,\ldots,0)$.

5. There is one case in which the residual index numbers all can be expressed in terms of the $r+1$ ordinary index numbers.

(67) *The residual index numbers of an $M_{n-2}(\alpha)$ in S_n are given in terms of the ordinary index numbers by the formula $\alpha_{h,l}=\alpha_{h-1}\alpha_{l-1}-\alpha_{h-2}\alpha_l$, where $h=1, 2,\ldots, n-2$ and $l=0, 1,\ldots, n-h-2$, while $\alpha_{-1}=0$.*

Using the above notation for $n-r=2$, the spreads $\sigma(\lambda)$ meet in a residual $M_{n+k-1}(\beta)$ and the spreads $\tau(\mu)$ in a residual $M_{k+1}(\beta')$. The two residual spreads meet in $\beta_0\beta'_0$ points, where

$$\beta_0=\sigma_{k+1}-A_{k-1},\text{ and}$$
$$\beta'_0=\tau_{n-k-1}-\alpha_0\tau_{n-k-3}-\alpha_1\tau_{n-k-4}-\ldots\ldots-\alpha_{n-k-4}\tau_1-\alpha_{n-k-3}.$$

These common points consist of the O points common to the spreads $\sigma(\lambda)$ and $\tau(\mu)$ outside of $M_{n-2}(\alpha)$, and of the I points common to the $M_{n-k-2}(\gamma^{(k-1)})$ and the $M_k(\gamma^{(n-k-3)})$ on $M_{n-2}(\alpha)$. Here

$$O=\sigma_{k+1}\tau_{n-k-1}-\tau_{n-k-1}A_{k-1}-\tau_{n-k-2}A_k-\ldots\ldots-\tau_1A_{n-3}-A_{n-2},\text{ and}$$
$$I=[\alpha\gamma^{(k-1)}]_0\tau_{n-k-2}+[\alpha\gamma^{(k-1)}]_1\tau_{n-k-3}+\ldots\ldots+[\alpha\gamma^{(k-1)}]_{n-k-3}\tau_1+[\alpha\gamma^{(k-1)}]_{n-k-2}.$$

By equating the coefficients of τ in $\beta_0\beta'_0=O+I$, we get

$$\alpha_{l-1}(A_{k-1}-\sigma_{k+1})=-A_{k+l}+[\alpha\gamma^{(k-1)}]_l,\quad (l=0, 1,\ldots, n-k-2).$$

In this, after dropping the obvious equalities found from the coefficients of σ_k and σ_{k+1}, we find from the coefficient of σ_j that

$$\alpha_{l-1}\cdot\alpha_{k-j-1}=-\alpha_{k+l-j}+\alpha_{k-j,\,l}+\alpha_{k-j-1,\,l+1}+\ldots\ldots+\alpha_{0,\,k-j+l},$$

where $j=0, 1, \ldots\ldots, k-1$. Putting $k-j=h$, we have

$$\alpha_{h-1}\cdot\alpha_{l-1}=-\alpha_{h+l}+\alpha_{h,\,l}+\alpha_{h-1,\,l+1}+\ldots\ldots+\alpha_{0,\,h+l},$$
$$(l=0, \ldots\ldots, n-k-2;\ h=1, 2, \ldots\ldots, k).$$

Allowing h to diminish by 1 and l to increase by 1 and subtracting, we get $\alpha_{h,\,l}=\alpha_{h-1}\alpha_{l-1}-\alpha_{h-2}\alpha_l$, which is also true when $h=1$ if $\alpha_{-1}=0$.

As a verification let $M_{n-2}(\alpha)$ be regular, being cut out by spreads of orders λ_1, λ_2. Then $\alpha_{h,\,l}=(-1)^{h+l}\lambda_1^2\lambda_2^2(\Sigma_{h-1}\Sigma_{l-1}-\Sigma_{h-2}\Sigma_l)$. But for this case we had found in (65) that $\alpha_{h,\,l}=(-1)^{h+l}\lambda_1\lambda_2(\Sigma_h\Sigma_l-\Sigma_{h-1}\Sigma_{l+1})$. To identify these, we notice by direct multiplication that $\Sigma_h\Sigma_l-\Sigma_{h-1}\Sigma_{l+1}=(\lambda_1\lambda_2)^{h-1}\Sigma_{l-h}$ if $h \overline{<} l$. Hence, $\Sigma_{h-1}\Sigma_{l-1}-\Sigma_{h-2}\Sigma_l=(\lambda_1\lambda_2)^{h-2}\Sigma_{l-h}$, and the two expressions are reconciled.

6. The generalized problem in restricted systems has, in the case of a common curve, the following solution:

(68) *If in* S_n *an* $M_{r_1}(\alpha^{(1)}), \ldots\ldots, M_{r_i}(\alpha^{(i)})$, *where* $r_1+\ldots\ldots+r_i=(i-1)n$, *have in common an* $M_1(\gamma)$, *they meet outside* $M_1(\gamma)$ *in*

$$O_1=\alpha_0^{(1)}\cdot\alpha_0^{(2)}\cdot\ldots\ldots\cdot\alpha_0^{(i)}-\gamma_0\sigma_1\Big(\frac{-[\alpha^{(k)}\gamma]_1}{\gamma_0}\Big)-\gamma_1 \textit{ points.}$$

Let spreads $\sigma^{(k)}(\lambda_1^{(k)}, \ldots\ldots, \lambda_{n-r_k}^{(k)})$ on $M_{r_k}(\alpha^{(k)})$ meet again in $M_{r_k}(\beta^{(k)})$, which meets $M_1(\gamma)$ in $I_1^{(k)}=\sigma_1^{(k)}\gamma_0+[\alpha^{(k)}\gamma]_1$ points, where $\beta_0^{(k)}=\pi(\lambda^{(k)})-\alpha_0^{(k)}=\pi_k-\alpha_0^{(k)}$. All n of the spreads on $M_1(\gamma)$ meet outside $M_1(\gamma)$ in

$$\Omega_1=\pi_1\pi_2\ldots\pi_i-\gamma_0(\sigma_1^{(1)}+\sigma_2^{(1)}+\ldots\ldots+\sigma_i^{(1)})-\gamma_1 \text{ points.}$$

The Ω_1 points are made up of the O_1 points common to $M_{r_1}(\alpha^{(1)}), \ldots\ldots, M_{r_i}(\alpha^{(i)})$; of the $(\pi_1-\alpha_0^{(1)})\alpha_0^{(2)}\ldots\alpha_0^{(i)}-I_1^{(1)}$ points common to $M_{r_1}(\beta^{(1)})$, $M_{r_2}(\alpha^{(2)}), \ldots\ldots$, $M_{r_i}(\alpha^{(i)})$, etc.; of the $(\pi_1-\alpha_0^{(1)})(\pi_1-\alpha_0^{(2)})\alpha_0^{(3)}\ldots\alpha_0^{(i)}$ points common to $M_{r_1}(\beta^{(1)})$, $M_{r_2}(\beta^{(2)})$, $M_{r_3}(\alpha^{(3)}), \ldots\ldots, M_{r_i}(\alpha^{(i)})$, etc.; ....; finally of the $(\pi_1-\alpha_0^{(1)})(\pi_2-\alpha_0^{(2)})$ .... $(\pi_i-\alpha_0^{(i)})$ points common to $M_{r_1}(\beta^{(1)}), \ldots\ldots, M_{r_i}(\beta^{(i)})$. Thus

$$\Omega_1=O_1+\Sigma\{(\pi_1-\alpha_0^{(1)})\alpha_0^{(2)}\ldots\alpha_0^{(i)}-I_1\}$$
$$+\Sigma\{(\pi_1-\alpha_0^{(1)})(\pi_1-\alpha_0^{(2)})\alpha_0^{(3)}\ldots\alpha_0^{(i)}\}+\ldots\ldots+\pi(\pi_1-\alpha_0^{(1)}).$$

By using the identity

$$x_1x_2\ldots x_i=[(x_1-y_1)+y_1][(x_2-y_2)+y_2]\ldots[(x_i-y_i)+y_i]$$
$$=y_1y_2\ldots y_i+\Sigma(x_1-y_1)y_2\ldots y_i$$
$$+\Sigma(x_1-y_2)(x_2-y_2)y_3\ldots y_i+\ldots\ldots+\pi(x_1-y_1),$$

we find that

$$\Omega_1=O_1-\Sigma I_1^{(1)}+\pi_1\pi_2\ldots\pi_i-\alpha_0^{(1)}\alpha_0^{(2)}\ldots\alpha_0^{(i)}$$
$$=\pi_1\pi_2\ldots\pi_i-\gamma_0[\sigma_1^{(1)}+\ldots\ldots+\sigma_1^{(i)}]-\{[\alpha^{(1)}\gamma]_1+\ldots\ldots$$
$$+[\alpha^{(i)}\gamma]_1\}-\alpha_0^{(1)}\alpha_0^{(2)}\ldots\alpha_0^{(i)}+O_1.$$

Therefore $O_1=\alpha_0^{(1)}\alpha_0^{(2)}\ldots\ldots\alpha_0^{(i)}-\gamma_0\left\{-\frac{[\alpha^{(1)}\gamma]_1}{\gamma_0}-\ldots\ldots-\frac{[\alpha^{(i)}\gamma]_1}{\gamma_0}\right\}-\gamma_1.$

This formula for O_1 is like the usual formula, except that in forming σ_1 the given orders are replaced by $-\frac{[\alpha^{(k)}\gamma]_1}{\gamma_0}=-\frac{[\alpha^{(k)}\gamma]_1}{[\alpha^{(k)}\gamma]_0}$.

The relative incidence numbers $[\alpha^k\gamma]_1$ can be replaced by relative index numbers according to the following theorem, but the formula for O_1 then loses its resemblance to the original formula.

(69) *For an* $M_1(\gamma)$ *on* $M_r(\alpha)$, $\gamma_0=(\alpha\gamma)_0=[\alpha\gamma]_0$ *and* $\gamma_1=(\alpha\gamma)_1+[\alpha\gamma]_1$.

Only the last equality needs proof, the others being a matter of definition. Let $n-r$ spreads $\sigma(\lambda)$ on $M_r(\alpha)$ meet in a residual $M_r(\beta)$ which cuts $M_r(\alpha)$ in $M_{r-1}(\varepsilon)$, which in turn meets $M_1(\gamma)$ in $I_1=\sigma_1\gamma_0+[\alpha\gamma]_1$ points. Then r further spreads $\tau(\mu)$ on $M_1(\gamma)$ determine an $M_{n-r}(\zeta)$ which meets $M_r(\alpha)$ and $M_r(\beta)$ in O_1 points outside of $M_1(\gamma)$. Here

$$O_1=\sigma_{n-r}\tau_r-\gamma_0(\sigma_1+\tau_1)-\gamma_1=\{\tau_r\alpha_0-\gamma_0\tau_1-(\alpha\gamma)_1\}+\{\tau_r\beta_0-I_1\}.$$

Since $\alpha_0+\beta_0=\sigma_{n-r}$, this leads to $\gamma_1=(\alpha\gamma)_1+[\alpha\gamma]_1$.

7. From (68) and (69) the undetermined index number β_2 or $(\alpha\gamma)_1$ of (51) can be obtained when $r=n-2$. For then $M_{n-2}(\alpha)$ and $M_2(\beta)$ meet in $M_1(\gamma)$ and no further points, whence, from (68),

$$O_1=0=\alpha_0\beta_0-\gamma_0\left(-\frac{[\alpha\gamma]_1}{\gamma_0}-\frac{[\beta\gamma]_1}{\gamma_0}\right)-\gamma_1.$$

Replacing $[\alpha\gamma]_1$ and $[\beta\gamma]_1$ according to (69), we have

$$0=\alpha_0\beta_0+\gamma_1-(\alpha\gamma)_1-(\beta\gamma)_1.$$

(70) *The undetermined index number* β_2 *or* $(\alpha\gamma)_1$ *of* (51) *is found in the case of an* $M_{n-2}(\alpha)$ *from either*

$$\alpha_0\beta_0=(\alpha\gamma)_1+(\beta\gamma)_1-\gamma_1 \text{ or } B_2+\left\{\binom{n-3}{2}+1\right\}A_{n-2}=\alpha_0\beta_0+\gamma_1.$$

§ 5. *Manifolds Defined by Matrices.*

1. In order that the various terms in the expansion of a determinant or a subdeterminant of a matrix, $M_{n+k,n}$, with n rows and $n+k$ columns, whose elements are forms in $d+1$ variables, may be homogeneous, it is necessary that the orders of the elements be taken as indicated in the array

$$M_{n+k,n}=\left\|\begin{matrix} l_1+\lambda_1 & l_2+\lambda_1 & \ldots\ldots & l_{n+k}+\lambda_1 \\ l_1+\lambda_2 & l_2+\lambda_2 & \ldots\ldots & l_{n+k}+\lambda_2 \\ \ldots & \ldots & \ldots & \ldots \\ l_1+\lambda_n & l_2+\lambda_n & \ldots\ldots & l_{n+k}+\lambda_n \end{matrix}\right\|.$$

If the $d+1$ variables are linear in the $c+1$ coordinates of an S_c, the vanishing of the matrix defines a manifold of dimension $c-(k+1)$ in S_c. For if the matrix of the first $n-1$ columns vanishes on a certain manifold M', the $k+1$ spreads obtained by adding each of the remaining columns to form a determinant all contain M' and meet in a residual $M_{c-(k+1)}$ which cuts M' in an $M_{c-(k+2)}$. Since for the general point of $M_{c-(k+1)}$ M' does not vanish, the vanishing of the $k+1$ determinants entails the vanishing of the matrix. In this section the first three index numbers of $M_{c-(k+1)}$ in S_c or of $M_{n+k,n}$ are derived, all the index numbers of $M_{n+1,n}$ are obtained, and a tentative formula for the general index number of $M_{n+k,n}$ which holds for the first $k+3$ numbers is given. A formula for the number of linear spaces which meet a prescribed number of given linear spaces is obtained in terms of the index numbers of $M_{n+k,n}$. Owing to the limitation of the tentative formula, this number is evaluated only for the case when the given spaces are lines and the case when the given spaces are nine planes in S_5.

2. Throughout this section we denote respectively by μ_i and $\bar{\mu}_i$ the elementary and the complete symmetric functions of degree i formed from $l_1, \ldots, l_{n+k}$; and by ν_i and $\bar{\nu}_i$, respectively, the complete and the elementary symmetric functions of $\lambda_1, \ldots, \lambda_n$. Further, let

$$(71) \qquad \begin{cases} H_j = \mu_j + \mu_{j-1}\nu_1 + \mu_{j-2}\nu_2 + \ldots + \nu_j, \text{ and} \\ \bar{H}_j = \bar{\mu}_j + \bar{\mu}_{j-1}\bar{\nu}_1 + \bar{\mu}_{j-2}\bar{\nu}_2 + \ldots + \bar{\nu}_j; \end{cases}$$

and let $m_{n+k,n;j}$, $(j=0, 1, \ldots)$, be the index numbers of $M_{n+k,n}$. From the well-known relations

$$m_j = \mu_j - \mu_{j-1}\bar{\mu}_1 + \mu_{j-2}\bar{\mu}_2 - \ldots + (-1)^j\bar{\mu}_j = 0, \text{ and}$$
$$n_j = \nu_j - \nu_{j-1}\bar{\nu}_1 + \nu_{j-2}\bar{\nu}_2 - \ldots + (-1)^j\bar{\nu}_j = 0, \text{ there follows}$$
$$h_j = H_j - H_{j-1}\bar{H}_1 + H_{j-2}\bar{H}_2 - \ldots + (-1)^j\bar{H}_j = 0.$$

For if we compare h_j with $m_j + m_{j-1}n_1 + m_{j-2}n_2 + \ldots + n_j$, which is evidently zero, we find that $h_j = \sum_{r=0}^{j}(-1)^r H_{j-r}\bar{H}_r = \sum_{r,s,t}(-1)^r\mu_{j-r-s}\bar{\mu}_{r-t}\nu_s\bar{\nu}_t$, while

$$\sum_{r=0}^{j} m_{j-r}n_r = \sum_{r=0}^{j}\Big(\sum_{s=0}^{j-r}(-1)^s\mu_{j-r-s}\bar{\mu}_s\Big)\Big(\sum_{t=0}^{r}(-1)^t\nu_{r-t}\bar{\nu}_t\Big) = \sum_{r,s,t}(-1)^{s+t}\mu_{j-r-s}\bar{\mu}_s\nu_{r-t}\bar{\nu}_t.$$

If we set $r-t=s'$, $s=r'-t'$, $t=t'$, the first sum reduces to the second.

The following equations are fairly obvious:

$$(72) \qquad H_j^{0,0} = H_j^{1,0} + l_1 H_{j-1}^{1,0}, \quad H_j^{0,0} = \lambda_1 H_{j-1}^{0,0} + H_j^{0,1}, \quad H_j^{1,0} = \lambda_1 H_{j-1}^{1,0} + H_j^{1,1},$$

where the first superscript 1 or 0 refers to the omission or retention of l_1 in the formation of H, and the second superscript indicates similarly the omission or retention of λ_1.

3. With the aid of these formulæ an immediate proof* of Salmon's formula for the order of a matrix can be given.

(73) *The order of a matrix* $M_{n+k,n}$ *is* H_{k+1}.

The theorem is obviously true for the matrix $M_{k,1}$; let us assume it to be true for all matrices $M_{r,s}$ such that $r+s<2n+k$. Let $M_{n+k-1,n}$ be the matrix of order $m_{n+k-1,n}$ obtained by dropping the first column of $M_{n+k,n}$; and let $M_{n+k-1,n-1}$ of order $m_{n+k-1,n-1}$ be that obtained by dropping the first row and column. Consider the section of the manifold $M_{n+k,n}=0$ by the spreads of orders $l_1+\lambda_2, \ldots, l_1+\lambda_n$ in the first column, $\pi=(l_1+\lambda_2)\cdot(l_1+\lambda_3)\cdot\ldots\cdot(l_1+\lambda_n)$. This section is regular and of order $\pi m_{n+k,n}$. It breaks up into two partial sections. For the one partial section the spread of order $l_1+\lambda_1$ and $M_{n+k-1,n}$ vanish, whence its order is $\pi\cdot(l_1+\lambda_1)\cdot m_{n+k-1,n}$. For the other partial section the spread of order $l_1+\lambda_1$ does not vanish, while $M_{n+k-1,n-1}$ does vanish, whence its order is $\pi\cdot m_{n+k-1,n-1}$. Therefore $m_{n+k,n}=(l_1+\lambda_1)m_{n+k-1,n}+m_{n+k-1,n-1}$. Both orders on the right are known from the assumed formula, whence $m_{n+k,n}=(l_1+\lambda_1)H_k^{1,0}+H_{k+1}^{1,1}$. By using (72) we find that $m_{n+k,n}=l_1H_k^{1,0}+H_{k+1}^{1,0}$ $=H_{k+1}^{0,0}=H_{k+1}$, which proves the theorem.

4. Similar considerations readily lead to the following theorem:

(74) *The curve* (*in* S_{k+2}) *of order* H_{k+1} *defined by the matrix* $M_{n+k,n}=0$ *has the second index number* $m_{n+k,n;1}=-kH_{k+2}-H_1H_{k+1}$ *and the genus* p *determined by* $2p-2=kH_{k+2}+H_1H_{k+1}-(k+3)H_{k+1}$.

The value of the genus is obtained at once from that of $m_{n+k,n;1}$ by using [(27), "R. S.," I]. The first and second index numbers of the matrix

$$M_{1+k,1}=\|l_1+\lambda_1, \ldots, l_{1+k}+\lambda_1\|=0,$$

a regular spread, are

$$m_{1+k,1;0}=(l_1+\lambda_1)\cdot\ldots\cdot(l_{1+k}+\lambda_1)$$

and

$$m_{1+k,1;1}=-(l_1+\ldots+l_{1+k}+(k+1)\lambda_1)m_{1+k,1;0}=-(k\lambda_1+H_1)H_{k+1}.$$

Since $H_{k+2}^{0,0}=\lambda_1H_{k+1}^{0,0}+H_{k+2}^{0,1}$ and $H_{k+2}^{0,1}=\mu_{k+2}=0$, the value of $m_{1+k,1;1}$ given in (74) also reduces to $-(k\lambda_1+H_1)H_{k+1}$. We assume, then, that (74) is true for matrices $M_{r,s}$ such that $r+s<2n+k$, and proceed as above. If Σ_1 denote $(l_1+\lambda_2)+\ldots+(l_1+\lambda_n)$, the second index number of the section of $M_{n+k,n}$, according to [(18), "R. S.," I], is $\pi\{m_{n+k,n;1}-\Sigma_1m_{n+k,n;0}\}$; of the one partial section is $\pi(l_1+\lambda_1)\{m_{n+k-1,n;1}-(\Sigma_1+l_1+\lambda_1)m_{n+k-1,n;0}\}$; and of the other par-

* This formula, inferred by Salmon ("Higher Algebra") from a number of special cases, was proved for the first time by Prof. F. F. Decker. This proof will appear in a subsequent number of this JOURNAL.

tial section is $\pi\{m_{n+k-1,\,n-1\,;\,1}-\Sigma_1\, m_{n+k-1,\,n\,;\,0}\}$. Furthermore, the two partial sections meet in $(l_1+\lambda_1)\pi m_{n+k-1,\,n-1\,;\,0}$ points. Using the formula (42) for the second index number of the composite curve, we have, after factoring out π,

$$\begin{aligned} m_{n+k,\,n\,;\,1}-\Sigma_1\, m_{n+k,\,n\,;\,0} &= (l_1+\lambda_1)\{m_{n+k-1,\,n\,;\,1}-(\Sigma_1+l_1+\lambda_1)\,m_{n+k-1,\,n\,;\,0}\} \\ &\quad +\{m_{n+k-1,\,n-1\,;\,0}-\Sigma_1 m_{n+k-1,\,n-1\,;\,0}\}-2(l_1+\lambda_1)\,m_{n+k-1,\,n-1\,;\,0}. \end{aligned}$$

The terms in Σ_1 cancel, due to the equation connecting the orders. By means of the same equation the term containing $(l_1+\lambda_1)^2$ can be eliminated. Then

$$m_{n+k,\,n\,;\,1}=(l_1+\lambda_1)\{m_{n+k-1,\,n\,;\,1}-m_{n+k,\,n\,;\,0}-m_{n+k-1,\,n-1\,;\,0}\}+m_{n+k-1,\,n-1\,;\,1}.$$

Thus, in order to prove (74) we have only to verify that

$$\begin{aligned} kH^{0,0}_{k+2}+H^{0,0}_1 H^{0,0}_{k+1} &= (l_1+\lambda_1)\{(k-1)H^{1,0}_{k+1}+H^{1,0}_1 H^{1,0}_k+H^{0,0}_{k+1}+H^{1,1}_{k+1}\} \\ &\quad +\{kH^{1,1}_{k+2}+H^{1,1}_1 H^{1,1}_{k+1}\} \end{aligned}$$

is true. The terms containing k as a factor vanish, due to the relation

$$H^{0,0}_j=(l_1+\lambda_1)H^{1,0}_{j-1}+H^{1,1}_j$$

used in the proof of Salmon's formula. The remaining terms are

$$H^{0,0}_1 H^{0,0}_{k+1}=(l_1+\lambda_1)[-H^{1,0}_{k+1}+H^{0,0}_{k+1}+H^{1,1}_{k+1}+H^{1,0}_1 H^{1,0}_k]+H^{1,1}_1 H^{1,1}_{k+1}.$$

In the bracket, $-H^{1,0}_{k+1}+H^{0,0}_{k+1}=l_1 H^{1,0}_k$ and $l_1 H^{1,0}_k+H^{1,0}_1 H^{1,0}_k=H^{0,0}_1 H^{1,0}_k$, whence $H^{0,0}_1 H^{0,0}_{k+1}=(l_1+\lambda_1)[H^{0,0}_1 H^{1,0}_k+H^{1,1}_{k+1}]+H^{1,1}_1 H^{1,1}_{k+1}$. The two terms in $H^{1,1}_{k+1}$ reduce to $H^{0,0}_1 H^{1,1}_{k+1}$, so that $H^{0,0}_1$ factors out, leaving $H^{0,0}_{k+1}=(l_1+\lambda_1)H^{1,0}_k+H^{1,1}_{k+1}$, which is true and completes the proof of (74).

5. In order to obtain the third index number of $M_{n+k,\,n}$, we first derive all the index numbers $m_{n-1,\,n\,;\,j}$ of the matrix $M_{n-1,\,n}$ obtained by using the first $n-1$ columns of $M_{n+k,\,n}$. The fact that the number of columns of $M_{n-1,\,n}$ is less than the number of rows requires, according to the conventions in paragraph 2, that the elementary and complete symmetric polynomials be interchanged (so far as rows and columns are concerned); and with this in mind we apply the μ, ν, H notation to $M_{n-1,\,n}$. It is an M_{k-1} in S_{k+1}, and on it we have $k+1$ spreads $\sigma(r)$ of orders $r_0=H_1+l_n, \ldots, r_k=H_1+l_{n+k}$, which meet outside of $M_{n-1,\,n}=0$ in the points of S_{k+1} determined by $M_{n+k,\,n}=0$. This number is given by Salmon's formula (73), which, with our present notation and the use of τ for the elementary symmetric polynomials in $l_n, \ldots, l_{n+k}$, takes the form

$$\begin{aligned} &(\bar\nu_{k+1}+\bar\nu_k\tau_1+\ldots\ldots+\tau_{k+1})+\bar\mu_1(\bar\nu_k+\bar\nu_{k-1}\tau_1+\ldots\ldots+\tau_k)+\ldots\ldots+\bar\mu_k(\bar\nu_1+\tau_1)+\bar\mu_{k+1} \\ &\qquad =\overline{\overline{H}}_{k+1}+\overline{\overline{H}}_k\tau_1+\overline{\overline{H}}_{k-1}\tau_2+\ldots\ldots+\tau_{k+1}. \end{aligned}$$

On the other hand, this number is given by

$$\sigma_{k+1}-m_{n-1,\,n\,;\,0}\,\sigma_{k-1}-m_{n-1,\,n\,;\,1}\,\sigma_{k-2}-\ldots\ldots-m_{n-1,\,n\,;\,k-1}.$$

Equating coefficients of τ_{k+1-j}, we get

$$(75)\quad \overline{H}_j = H_1^j - \binom{j}{2} H_1^{j-2} m_{n-1,\,n;\,0} - \binom{j}{3} H_1^{j-3} m_{n-1,\,n;\,0} - \ldots - \binom{j}{j} m_{n-1,\,n;\,j-2}.$$

If we begin with the obvious equations $1=1$ and $\overline{H}_1 = H_1$, and take also the $j-1$ equations (75) for $j=2, 3, \ldots, j$, and if we multiply this system of $j+1$ equations in order, beginning with the last, by

$$\binom{j}{0},\ -\binom{j}{1}H_1,\ \binom{j}{2}H_1^2,\ \ldots,\ (-1)^j H_1^j,$$

respectively, and add, we find that

$$(76)\quad m_{n-1,\,n;\,j-2} = (-1)^{j-1}\Big[H_1^j - \binom{j}{1}H_1^{j-1}\overline{H}_1 + \binom{j}{2}H_1^{j-2}\overline{H}_2 \ldots (-1)^j \binom{j}{j}\overline{H}_j\Big],$$

due to the binomial formula

$$(77)\quad S = \binom{a}{0}\binom{a+c}{b} - \binom{a}{1}\binom{a+c-1}{b} + \binom{a}{2}\binom{a+c-2}{b}$$
$$- \ldots + (-1)^{a+c-b}\binom{a}{a+c-b} = \binom{c}{b-a},$$

with the particular cases: $S=1$ if $b=a$, or if $b=a+c$; $S=0$ if $b=0$, or if $b<a$, or if $b>a+c$, or if $c=0$, unless at the same time $b=a$. Hence,

(78) *The index numbers of*

$$M_{n-1,\,n} = \left\| \begin{matrix} l_1+\lambda_1, & \ldots, & l_{n-1}+\lambda_1 \\ \cdots & \cdots & \cdots \\ l_1+\lambda_n, & \ldots, & l_{n-1}+\lambda_n \end{matrix} \right\|$$

are given by (76), *where* $H_1 = \overline{H}_1$, $\overline{H}_i = \bar{\mu}_i + \bar{\mu}_{i-1}\bar{\nu}_1 + \ldots + \bar{\nu}_i$, *and* $\bar{\mu}_i(\bar{\nu}_i)$ *are the complete (elementary) symmetric functions of* $\lambda_1, \ldots, \lambda_n$ $(l_1, \ldots, l_{n-1})$.

6. This result will be applied now to the determination of $m_{n+k,\,n;\,2}$. In S_{k+3}, $M_{n+k,\,n}=0$ defines an M_2, the residual intersection of the above $\sigma(r)$ spreads on $M_{n-1,\,n}=0$. From (51) and (70) we get the following equations:

$$1^\circ.\quad \begin{cases} m_{n+k,\,n;\,0} = \sigma_{k+1} - A_{k-1}, \\ m_{n+k,\,n;\,1} = -\sigma_1\, m_{n+k,\,n;\,0} + kA_k, \\ m_{n+k,\,n;\,2} = (m_{n-1,\,n;\,0} - \sigma_2)\, m_{n+k,\,n;\,0} - \sigma_1(m_{n+k,\,n;\,1} + A_k) - \binom{k+1}{2}A_{k+1}. \end{cases}$$

Here the index numbers in A_j are obtained from (76), in which hereafter the H's will be primed to distinguish them from the H's formed for $M_{n+k,\,n}$. Let us first derive the value of A_j. From the system of equations

$$\sigma_0=\binom{k+1}{0},$$

$$\sigma_1=\binom{k+1}{1}H_1'+\binom{k}{0}\tau_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\sigma_j=\binom{k+1}{j}H_1'^j+\binom{k}{j-1}\tau_1 H_1'^{j-1}+\ldots+\binom{k-j+2}{1}\tau_{j-1}H_1'+\binom{k-j+1}{0}\tau_j;$$

and the following system obtained from (76) together with two obvious equations adjoined for convenience in summation,

$$m_{n-1,n;j}=(-1)^{j+1}\{H_1'^{j+2}-\binom{j+2}{1}H_1'^{j+1}\overline{H}_1'+\binom{j+2}{2}H_1'^j\overline{H}_2'-\ldots(-1)^j\binom{j+2}{j+2}\overline{H}_{j+2}'\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$m_{n-1,n;0}=(-1)^1\{H_1'^2-\binom{2}{1}H_1'\overline{H}_1'+\binom{2}{2}\overline{H}_2'\},$$

$$m_{n-1,n;-1}=(-1)^0\{H_1'-\binom{1}{1}\overline{H}_1'\}=0,$$

$$m_{n-1,n;-2}=(-1)^{-1}\{H_1'^0\}+1=0;$$

we find by applying (77) that

$$A_j=\sum_{i=0}^{j+2}\sigma_i\,m_{n-1,n;j-i}=\sigma_{j+2}-\Sigma(-1)^{j-r-s}\binom{j+1-k}{r+s-k-1}H_1'^{j+2-r-s}\overline{H}_s',$$

where in Σ $r=0,\ldots,j+2$ and $s=0,\ldots,j+2$, while $r+s \overline{<} j+2$. Note that in Σ r, s occur only in the combination $r+s$, except with $\tau_r\overline{H}_s'$, and that $\sum_{r+s=c}\tau_r\overline{H}_s'=H_{r+s}$. If, then, $r+s$ be replaced by t, we get

$$\sigma_{j+2}-A_j=\sum_{t=0}^{j+2}(-1)^{j-t}\binom{j+1-k}{t-k-1}H_tH_1'^{k+1-t}.$$

We are interested in the values $j=k-1$, k, $k+1$ only. When $j=k-1$,

$$\sigma_{k+1}-A_{k-1}=\sum_{t=0}^{k+1}(-1)^{k-1-t}\binom{0}{t-k-1}H_tH_1'^{k+1-t}.$$

The binomial coefficient is 1 when $t=k+1$, otherwise it is zero, whence

2°. $\sigma_{k+1}-A_{k-1}=H_{k+1}$.

Since σ_{k+2}, σ_{k+3}, ... all vanish, we find for $j=k$ that

$$-A_k=\sum_{t=0}^{k+2}(-1)^{k-t}\binom{1}{t-k-1}H_tH_1'^{k+2-t},$$

whence

3°. $-A_k=-H_{k+1}H_1'+H_{k+2}$.

Similarly,

4°. $-A_{k+1}=H_{k+1}H'^2-2H_{k+2}H'+H_{k+3}$.

52

From 1° and 2° we again obtain (73), or

5°. $m_{n+k,\,n\,;\,0}=H_{k+1}$.

From 1° and 3° we find that $m_{n+k,\,n\,;\,1}=-\sigma_1 H_{k+1}+kH'_1 H_{k+1}-kH_{k+2}$. Since $-\sigma_1+kH'_1=-(H'_1+l_n+\ldots+l_{n+k})=-H_1$, we verify (74), or

6°. $m_{n+k,\,n\,;\,1}=-\{kH_{k+2}+H_1 H_{k+1}\}$.

From 1° and 4° we find that

$$m_{n+k,\,n\,;\,2}=\binom{k+1}{2}H_{k+3}+\binom{k+1}{1}H_1 H_{k+2}+H_{k+1}\{\sigma_1(H_1-H'_1)-kH'_1\tau_1-\tau_2 \\ -H_1'^2+2H'_1\overline{H}'_1-\overline{H}'_2\}.$$

In the coefficient of H_{k+1} the terms in k vanish, due to $\sigma_1=H_1+kH'_1$ and $H_1=\tau_1+\overline{H}'_1=\tau_1+H'_1$; the coefficient then reduces to $H_1(H_1-H'_1)-\tau_2+\overline{H}_1'^2-\overline{H}'_2$. This becomes $H_1^2-H_2$, due to $H_2=\tau_2+\tau_1\overline{H}'_1+\overline{H}'_2$ and $\overline{H}'_1=H'_1$. Hence,

7°. $m_{n+k,\,n\,;\,2}=\binom{k+1}{2}H_{k+3}+\binom{k+1}{1}H_1 H_{k+2}+(H_1^2-H_2)H_{k+1}$.

(79) *The third index number of* $M_{n+k,\,n}$ *is given in* 7°.

The method used to determine the first two index numbers in (73) and (74) failed for the third, because for the composite two-way there was required the second relative index number of $M_{n-k-1,\,n-1}$ as to $M_{n-k-1,\,n}$ (the $(\alpha\gamma)_1$ of (51)). This can now be found by using $m_{n+k,\,n;\,2}$, and as the result of a calculation similar to those made above we have

(80) *The second relative index number of* $M_{n+k-1,\,n-1}$ *as to* $M_{n+k-1,\,n}$ *is* $-H^{1,1}_{k+2}+\lambda_1 H^{1,1}_{k+1}$.

7. The index numbers of $M_{n+k,\,n}$ found thus far can be written

$$m_{n+k,\,n\,;\,0}=\quad\left\{\binom{k-1}{0}H_{k+1}\right\},$$
$$m_{n+k,\,n\,;\,1}=-\left\{\binom{k}{1}H_{k+2}+\binom{k}{0}\overline{H}_1 H_{k+1}\right\},$$
$$m_{n+k,\,n\,;\,2}=\quad\left\{\binom{k+1}{2}H_{k+3}+\binom{k+1}{1}\overline{H}_1 H_{k+2}+\binom{k+1}{0}\overline{H}_2 H_{k+1}\right\}.$$

On this somewhat slender basis let us generalize the formulæ and assume as a tentative formula for the general index number of $M_{n+k,\,n}$

$$(81)\quad m_{n+k,\,n\,;\,j}=(-1)^j\left\{\binom{k+j-1}{j}H_{k+j+1}+\binom{k+j-1}{j-1}\overline{H}_1 H_{k+j} \\ +\binom{k+j-1}{j-2}\overline{H}_2 H_{k+j-1}+\ldots \\ +\binom{k+j-1}{1}\overline{H}_{j-1}H_{k+2}+\binom{k+j-1}{0}\overline{H}_j H_{k+1}\right\}.$$

In order to check this assumed formula, consider the matrix $M_{1+k,1}$. In this case

$$H_i=\mu_i+\lambda_1\mu_{i-1}+\lambda_1^2\mu_{i-2}+\ldots\ldots+\lambda_1^i\,; \quad i.\ e.,\quad H_i-\lambda_1 H_{i-1}=\mu_i.$$

Since $\mu_{k+2}=\mu_{k+3}=\ldots\ldots=0$,

$$H_{k+1}=(l_1+\lambda_1)\cdot\ldots\ldots\cdot(l_{1+k}+\lambda_1),\quad H_{k+1+r}=\lambda_1^r H_{k+1}.$$

The $M_{1+k,1}=0$ is regular, and its index numbers are formed from the complete symmetric functions of the orders; *i. e.*,

$$1^\circ.\quad m_{1+k,1;\,j}=(-1)^j H_{k+1}\left\{\bar\mu_j+\binom{k+j}{1}\bar\mu_{j-1}\lambda_1+\binom{k+j}{2}\bar\mu_{j-2}\lambda_1^2+\ldots\ldots+\binom{k+j}{j}\lambda_1^j\right\}.$$

The values of $\bar\mu_i$ in terms of the H's are

$$2^\circ.\quad \bar\mu_j=\bar{\bar H}_j-\lambda_1\bar{\bar H}_{j-1}+\lambda_1^2\bar{\bar H}_{j-2}-\ldots\ldots+(-1)^j\lambda_1^j.$$

To prove this, we note that it is true for $j=1$; let us assume it true up to $j=j$, and prove it for $j=j$. From the assumed formula, $\bar\mu_i+\lambda_1\bar\mu_{i-1}=\bar{\bar H}_i$ for $i=1, 2, \ldots\ldots, j-1$. Since

$$\begin{aligned}\bar\mu_j&=\mu_1\bar\mu_{j-1}-\mu_2\bar\mu_{j-2}+\mu_3\bar\mu_{j-3}-\ldots\ldots+(-1)^j\mu_j\\&=(H_1-\lambda_1)\bar\mu_{j-1}-(H_2-\lambda_1 H_1)\bar\mu_{j-2}+(H_3-\lambda_1 H_2)\bar\mu_{j-3}\\&\qquad-\ldots\ldots+(-1)^{j-1}(H_j-\lambda_1 H_{j-1}),\\ \bar\mu_j+\lambda_1\bar\mu_{j-1}&=H_1(\bar\mu_{j-1}+\lambda_1\bar\mu_{j-2})-H_2(\bar\mu_{j-2}+\lambda_1\bar\mu_{j-3})+\ldots\ldots+(-1)^j H_j\\&=H_1\bar{\bar H}_{j-1}-H_2\bar{\bar H}_{j-2}+\ldots\ldots+(-1)^{j-1}H_j=\bar{\bar H}_j.\end{aligned}$$

This proves the above formula for $i=j$, and therefore completes the proof of 2°. Substituting the values 2° in 1°, we get

$$\begin{aligned}m_{1+k,1;\,j}&=(-1)^j\sum_{i=0}^{j}H_{k+1}\left[\binom{k+j}{i}-\binom{k+j}{i-1}+\binom{k+j}{i-2}-\ldots\ldots+(-1)^i\binom{k+j}{0}\right]\bar{\bar H}_{j-i}\lambda_1^i\\&=(-1)^j\sum_{i=0}^{j}\binom{k+j-1}{i}\bar{\bar H}_{j-i}\cdot\lambda_1^i H_{k+1}=(-1)^j\sum_{i=0}^{j}\binom{k+j-1}{i}\bar{\bar H}_{j-i}\cdot H_{k+1+i},\end{aligned}$$

which is the same result as is given by (81). In this particular case the H's, up to and including H_{k+1}, are independent quantities. The further H's are connected with the earlier ones by the equations $H_{k+1+i}=\lambda_1^i H_{k+1}$. The first homogeneous relation which is a consequence of these equations is $H_{k+1}H_{k+3}-H_{k+2}^2$, which can occur first in $m_{n+k,n;\,k+3}$.

(82) *On the assumption that the index numbers of $M_{n+k,n}$ can be given as polynomials in H_i, the formula* (81) *is correct for the first $k+3$ index numbers, and for these only except in the above case of an $M_{1+k,1}$.*

That this assumption is correct can hardly be doubted in view of the above values for the first three index numbers of $M_{n+k,\,n}$, and for all the index numbers of $M_{1+k,\,1}$, $M_{n-1,\,n}$ and $M_{n,\,n}$. That (81) is *not* correct for values $j \gtrless k+3$ is clear from the cases $M_{n,\,n}$ and $M_{n+1,\,n}$. The index numbers of the determinant $M_{n,\,n}$, according to [(11), "R. S.," I], are $m_{n,\,n;\,j} = (-1)^j H_1^j$, so that there must be added to the right side of (81) the following corrections for successive values of j: 0, 0, 0, $2(H_1H_3-H_2^2)$, $-5H_1(H_1H_3-H_2^2)$, $9H_1^2(H_1H_3-H_2^2)+2(H_1^2H_4-3H_1H_2H_3+2H_2^3)+2(H_1H_5-4H_2H_4+3H_3^2)$, $-14H_1^3(H_1H_3-H_2^2)-7H_1(H_1^2H_4-3H_1H_2H_3+2H_2^3)-7H_1(H_1H_5-4H_2H_4+3H_3^2)$, ... In the case of an $M_{n+1,\,n}$ we find from (76) that the following corrections must be added to the right side of (81): 0, 0, 0, 0, $5(H_2H_4-H_3^2)$, ...

The only apparent law followed by these corrections is that they are seminvariants of the binary form with coefficients H_{k+1}, H_{k+2}, H_{k+3}, H_{k+4}, etc.

Let us make, finally, an application of the index numbers of $M_{n+k,\,n}$ to a geometric enumeration.

In S_{n+k-1} an S_{n-1} is determined by nk conditions, and it is one condition that an S_{n-1} meet a given S_{k-1} in S_{n+k-1}. We ask, then, for the number of S_{n-1}'s which meet nk given S_{k-1}'s in S_{n+k-1}. The S_{n-1} is given by n linearly independent points within it; *i. e.*, by

$$M_{n+k,\,n} = \left\| \begin{matrix} x_{1,\,1} & x_{1,\,2} & \cdots & x_{1,\,n+k} \\ x_{2,\,1} & x_{2,\,2} & \cdots & x_{2,\,n+k} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n,\,1} & x_{n,\,2} & \cdots & x_{n,\,n+k} \end{matrix} \right\|.$$

If the variables x_{ij} be point coordinates in a space $\Sigma_{n(n+k)-1}$, the n points determine a point in $\Sigma_{n(n+k)-1}$. Since any other n linearly independent points will serve the same purpose, the S_{n-1} itself is represented by a Σ_{n^2-1} in $\Sigma_{n(n+k)-1}$. Take a section of $\Sigma_{n(n+k)-1}$ by a Σ_{nk}, and in Σ_{nk} the S_{n-1} is represented by a point. Conversely, a point in Σ_{nk} is given by values x_{ij} and determines an S_{n-1} in S_{n+k-1} unless the point of Σ_{nk} is on the spread defined by the vanishing of $M_{n+k,\,n}$. The condition that S_{n-1} meet a given S_{k-1} is linear in the determinants of $M_{n+k,\,n}$. It is therefore represented by a spread of order n in Σ_{nk} on the manifold $M_{n+k,\,n}=0$, whose dimension is $k(n-1)-1$. The number required is the number of points of Σ_{nk} outside of $M_{n+k,\,n}=0$, and on nk given spreads of order n containing $M_{n+k,\,n}=0$. This number is

$$(83) \qquad O = n^{nk} - \sum_{j=0}^{k(n-1)-1} \binom{nk}{nk-n-1-j} n^{nk-n-1-j}\, m_{n+k,\,n;\,j}.$$

(84) *The number of S_{n-1}'s which meet nk given S_{k-1}'s in S_{n+k-1} is given by O in* (83), *where $m_{n+k,\,n;\,j}$ is the $(j+1)$-th index number of a matrix $M_{n+k,\,n}$ whose elements are linear forms.*

9. Since, according to (82), we are limited to values $j<k+3$, *i. e.*, $k(n-1)-1<k+3$, we can obtain the explicit number only for $n=2$, $k=$ any integer, and for $n=3$, $k<4$. For a matrix $M_{n+k,\,n}$ with linear elements,

$$H_i=\binom{n+k}{i}, \quad \text{and} \quad \overline{H}_i=\binom{n+k-1+i}{i}.$$

Putting these values for $n=2$ in (81) and substituting in (83), we find that

$$\begin{aligned}O=2^{2k}-(k+2)\Big\{&\binom{k+1}{0}\binom{2k}{k-1}2^{k-1}-\binom{k+2}{1}\binom{2k}{k-2}2^{k-2}\\&+\ldots\ldots+(-1)^k\binom{2k}{k-1}\binom{2k}{0}2^0\Big\}\\+\Big\{&\binom{k}{1}\binom{k+1}{0}\binom{2k}{k-2}2^{k-2}-\binom{k+1}{1}\binom{k+2}{1}\binom{2k}{k-3}2^{k-3}\\&+\binom{k+2}{1}\binom{k+3}{2}\binom{2k}{k-4}2^{k-4}-\ldots\ldots+(-1)^{k-1}\binom{2k-2}{1}\binom{2k-1}{k-2}\binom{2k}{0}2^0\Big\}.\end{aligned}$$

In the first brace note that

$$\binom{2k}{k-r}\binom{k+r}{r-1}=\binom{2k}{k-1}\binom{k-1}{r-1},$$

whence it becomes

$$\begin{aligned}-(k+2)\binom{2k}{k-1}&\Big\{\binom{k-1}{0}2^{k-1}-\binom{k-1}{1}2^{k-2}+\ldots\ldots+(-1)^k\binom{k-1}{k-1}2^0\Big\}\\&=-(k+2)\binom{2k}{k-1}(2-1)^{k-1}=-(k+2)\binom{2k}{k-1}.\end{aligned}$$

The second brace is

$$\begin{aligned}\sum_{r=0}^{k-2}(-1)^r\binom{k+r}{1}\binom{k+1+r}{r}\binom{2k}{k-2-r}2^{k-2-r}&=k\sum_{r=0}^{k-2}(-1)^r\binom{k+1+r}{r}\binom{2k}{k-2-r}2^{k-2-r}\\&+(k+2)\sum_{r=1}^{k-2}(-1)^r\binom{k+1+r}{r-1}\binom{2k}{k-2-r}2^{k-2-r}.\end{aligned}$$

The first part of this, according to [11°, p. 179, "R. S.," I], is $k\sum_{r=2}^{k}\binom{2k}{k-r}$, and the second is $-(k+2)\sum_{r=3}^{k}\binom{2k}{k-r}$, whence the sum of the two is

$$k\binom{2k}{k-2}-2\sum_{r=3}^{k}\binom{2k}{k-r}=k\binom{2k}{k-2}-2^{2k}+2\binom{2k}{k-2}+2\binom{2k}{k-1}+\binom{2k}{k}.$$

Hence,

$$O=\binom{2k}{k}-k\binom{2k}{k-1}+(k+2)\binom{2k}{k-2}=\frac{(2k)\,!}{k\,!\,(k+1)\,!}=\frac{1}{k+1}\binom{2k}{k}.$$

(85) *The number of lines which meet* $2k$ *given* S_{k-1}*'s in* S_{k+1} *is*

$$\frac{(2k)\,!}{k\,!\,(k+1)\,!}=\binom{2k}{k}\frac{1}{k+1}.$$

This well-known fact can be regarded as an excellent numerical check on the previous theorems. The case $n=3$, $k=2$ of (84) is the dual of the case $n=2$, $k=3$ of (85), so that the case $n=3$, $k=3$ of (84) remains. From (81) we find that the index numbers of an $M_{6,3}$ in S_9 with linear elements are

$$15,\ -108,\ 465,\ -30\cdot 51,\ 21\cdot 210,\ -42\cdot 244,$$

whence

$$O=3^9-\binom{9}{5}3^5\cdot 15+\binom{9}{4}3^4\cdot 108-\binom{9}{3}3^3\cdot 465$$
$$+\binom{9}{2}3^2\cdot 30\cdot 51-\binom{9}{1}3\cdot 21\cdot 210+42\cdot 244=42.$$

(86) *There are forty-two planes which meet nine given planes in* S_5.

This again is checked by Schubert's formula,

$$\frac{1!2!3!\ldots\ldots r![(n-r)(r+1)]!}{(n-r)!(n-r+1)!\ldots\ldots(n-1)!\,n!},$$

for the number of S_r's which meet $(r+1)(n-r)$ given S_{n-r-1}'s in S .

BALTIMORE, *February* 1, 1914.

*Character of the Solutions of Certain Functional Equations.**

By Thomas E. Mason.

Introduction.

By means of a transformation of the form $z = (mx + n)/(rx + s)$, the equation †

$$F\left(\frac{az+b}{cz+d}\right) = \frac{A'(z)F(z)+B'(z)}{C'(z)F(z)+D'(z)}, \qquad ad - bc \neq 0,$$

in which $A'(z)$, $B'(z)$, $C'(z)$, $D'(z)$ are rational, can be transformed to the ordinary difference equation

$$\psi(x+1) = \frac{A(x)\,\psi(x) + B(x)}{C(x)\,\psi(x) + D(x)}, \tag{M}$$

or to the q-difference equation

$$\psi(qx) = \frac{A(x)\,\psi(x) + B(x)}{C(x)\,\psi(x) + D(x)}, \tag{N}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ are rational, according as the substitution $z' = (az+b)/(cz+d)$ has one or two double points. In order to do this it is sufficient to choose the transforming substitution so that in the first case the single double point in the z-plane is carried to the point infinity in the x-plane, and in the second case the two double points in the z-plane are carried to the points zero and infinity in the x-plane. The determination of the character of the solutions of equations (M) and (N) obviously carries with it the determination of the character of the solutions of the more general equation from which they were derived.

Tietze ‡ has investigated the solutions of equation (M) from the point of view of their transcendentally transcendental character, and Stridsberg § has investigated the solutions of equations (M) and (N) from the point of view of their algebraically transcendental character. Tietze normalized to the equation

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* Read before the American Mathematical Society, September, 1913.

† This equation is linear or non-linear according as $C(x) \equiv 0$ or $C(x) \not\equiv 0$.

‡ Tietze, *Monatshefte für Mathematik und Physik*, XVI (1905), pp. 329–364.

§ Stridsberg, *Arkiv för Matematik, Astronomi och Fysik*, VI (1910), Nos. 15 and 18. The linear equation is considered in No. 15, and the non-linear in No. 18.

$\phi(x)\{1+\phi(x+1)\}=r(x)$ and showed that if $\lim_{x=\infty} r(x)=0$ and the equation has no rational solution, then it has only transcendentally transcendental solutions. Stridsberg showed that if the equations (M) and (N) have no algebraic solutions and $C(x)\not\equiv 0$, then their algebraically transcendental solutions, if any exist, also satisfy certain Riccati equations. I had proved a theorem* essentially equivalent to this important theorem of Stridsberg before the work of that investigator came to my attention.

The investigations of these men were incomplete. Neither gave a method of determining the algebraic solutions, and each left unanswered an important question—the question of whether there ever exist algebraically transcendental solutions in case the equation has no algebraic solution. Tietze says concerning the restriction of his theorem to the case when $\lim_{x=\infty} r(x)=0$: "Es mag dahingestellt bleiben, ob der Satz, dass eine Gleichung (2) $[\phi(x)\{1+\phi(x+1)\}=r(x)]$, die keine rationalen Lösungen hat, auch keine algebraische-transcendental Lösungen besitzt, ohne Einschränkungen über $r(x)$ gilt." Stridsberg quotes these words of Tietze and follows them with the remark: "Je suis tout d' accord avec M. Tietze sur ce dernier point."

This conjecture of these men was wrong. In this paper I have demonstrated the existence of algebraically transcendental solutions in certain cases where the original equations have no algebraic solutions. The proof of existence is made to depend upon the known properties of the second-order linear difference equation.

The methods of this paper will suffice to completely characterize the solutions of any given equation of the forms (M) and (N), except where the theory concerning the solutions of the corresponding second order linear equation is incomplete.

The normal equations,

$$\phi(x)\,\phi(x+1)=R(x) \quad \text{and} \quad \phi(x)\,\phi(qx)=R(x),$$

to which certain equations of types (M) and (N) can be transformed, are of special interest in that I give simple necessary and sufficient conditions for rational and algebraic non-rational solutions; and in case there are no algebraic solutions, I prove that there are no algebraically transcendental solutions.

In § 1 the linear and non-linear equations are both normalized to type-forms, and explicit formulas are given for writing down the normal form of any given equation. It is shown that a non-linear equation can be transformed to a linear equation if a rational solution of the non-linear equation is known.

* See § 5 of this paper.

In § 2 methods are given for determining all the rational solutions of any equation of the forms considered. The characterization of the solutions of the linear equation is completed in this section.

In § 3 algebraic solutions other than rational solutions of the normal forms of equation (N) are shown to exist in some cases, and the method of determining them is made to depend on finding the rational solutions of certain auxiliary equations.

In § 4 are given the conditions which must be satisfied in order that a function can at the same time be a solution of the functional equation (M) or (N) and of a Riccati equation. Equations are exhibited which have no algebraic solutions but which do have algebraically transcendental solutions.

In § 5 I state a condition that an equation have only transcendentally transcendental solutions. A list of several classes of equations which have only transcendentally transcendental solutions is included in this section.

§ 1. *Transformation to Normal Forms.*

Instead of writing $\chi(x+n)$ or $\chi(q^n x)$ we shall write $\chi_n(x)$ for $\chi(x+n)$ or for $\chi(q^n x)$, according as we have the ordinary difference or q-difference equation. We shall also write $\chi_{-n}(x)$ for $\chi(x-n)$ or for $\chi(q^{-n}x)$.

In the equation

$$\psi_1(x)=\frac{A(x)\psi(x)+B(x)}{C(x)\psi(x)+D(x)},$$

make the substitution

$$\psi(x)=\frac{\alpha(x)\Omega(x)+\beta(x)}{\gamma(x)\Omega(x)+\delta(x)}, \qquad \alpha(x)\delta(x)-\beta(x)\gamma(x)\neq 0,$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\delta(x)$ are rational. Clearing the resulting equation of fractions, we have

$$\begin{aligned}
&\Omega_1(x)\Omega(x)\{\alpha_1(x)[\alpha(x)C(x)+\gamma(x)D(x)]-\gamma_1(x)[\alpha(x)A(x)+\gamma(x)B(x)]\}\\
&\quad+\Omega_1(x)\{\alpha_1(x)[\beta(x)C(x)+\delta(x)D(x)]-\gamma_1(x)[\beta(x)A(x)+\delta(x)B(x)]\}\\
&=\Omega(x)\{\delta_1(x)[\alpha(x)A(x)+\gamma(x)B(x)]-\beta_1(x)[\alpha(x)C(x)+\gamma(x)D(x)]\}\\
&\quad+\{\delta_1(x)[\beta(x)A(x)+\delta(x)B(x)]-\beta_1(x)[\beta(x)C(x)+\delta(x)D(x)]\}. \quad (1)
\end{aligned}$$

This equation is linear provided that the coefficient of $\Omega_1(x)\Omega(x)$ is zero or that the coefficient of $\Omega_1(x)$ and the term not containing $\Omega(x)$ are zero. If the coefficient of $\Omega_1(x)\Omega(x)$ is zero, the original equation has the rational solution $\alpha(x)/\gamma(x)$; and if the term without $\Omega(x)$ is zero, it has the rational solution $\beta(x)/\delta(x)$. If $A(x)+B(x)=C(x)+D(x)$, there is the obvious

solution $\psi(x)=1$, and the equation can be transformed to a linear equation and studied in that form. Any other rational solution will effect this transformation to a linear equation. If there is no obvious rational solution, we choose $\alpha(x)=\gamma(x)=1$, $\beta(x)=B(x)-D(x)$ and $\delta(x)=C(x)-A(x)$. These values of $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\delta(x)$ reduce the coefficient of $\Omega_1(x)$ to zero, but do not make the coefficient of $\Omega_1(x)\Omega(x)$ or the term without $\Omega(x)$ zero, unless the original equation has the solution $\psi(x)=1$. Making these substitutions in (1) and reducing, we have

$$\Omega_1(x)=\frac{g(x)}{\Omega(x)}-h(x),$$

where

$$g(x)=\frac{(B(x)C(x)-A(x)D(x))(A_1(x)+B_1(x)-C_1(x)-D_1(x))}{A(x)+B(x)-C(x)-D(x)}$$

and

$$h(x)=-\frac{(A(x)+B(x))(A_1(x)-C_1(x))+(C(x)+D(x))(B_1(x)-D_1(x))}{A(x)+B(x)-C(x)-D(x)}.$$

If $h(x)\not\equiv 0$, the substitution

$$\Omega(x)=h_{-1}(x)\,\phi(x)$$

will transform the equation to

$$\phi_1(x)=\frac{R(x)}{\phi(x)}-1, \tag{2}$$

where

$$R(x)=\frac{g(x)}{h_{-1}(x)\,h(x)}.$$

If $h(x)\equiv 0$, the substitution

$$\Omega(x)=\frac{(g_{-1}(x)-g(x))\phi(x)-g(x)(1-g_{-1}(x))}{(g_{-1}(x)-g(x))\phi(x)-(1-g_{-1}(x))}$$

will give an equation of the form (2), where

$$R(x)=\frac{g(x)(1-g_1(x))(1-g_{-1}(x))}{(g(x)-g_{-1}(x))(g_1(x)-g(x))}.$$

The determinant of the transformation and the denominator of the expression for $R(x)$ can be zero only when $g(x)=g$, a constant, in which case the equation $\Omega_1(x)=g/\Omega(x)$ has the obvious rational solution $\Omega(x)=\sqrt{g}$ and can be transformed to a linear equation.

Thus, we have shown that any non-linear equation of the forms considered can be transformed to the normal form (2) or to a linear equation. It is obvious that the transformations here carried out have not affected the solutions as to their rational, algebraic non-rational, algebraically transcendental,

or transcendentally transcendental character. If a linear equation is transformed to the normal form (2), it is obvious that the normal form has a rational solution; for the transformation is reversible, and a necessary and sufficient condition that a non-linear equation of this form transform to a linear equation by means of a linear fractional transformation, is that the non-linear equation have a rational solution.

The equation in $\psi(x)$ can be transformed to the special normal form

$$\phi_1(x) = \frac{R(x)}{\phi(x)} \tag{3}$$

in certain cases; that is, when the transformation can be chosen so as to make the coefficients of $\Omega_1(x)$ and $\Omega(x)$ in (1) equal to zero.

In the non-homogeneous linear equation

$$\psi_1(x) = A(x)\,\psi(x) + B(x),$$

make the substitution

$$\psi(x) = \alpha(x)\,\phi(x) + \beta(x).$$

Thus we have

$$\phi_1(x) = \frac{A(x)\,\alpha(x)}{\alpha_1(x)}\,\phi(x) + \frac{A(x)\,\beta(x) + B(x) - \beta_1(x)}{\alpha_1(x)}.$$

This becomes homogeneous if the original equation has a rational solution and $\beta(x)$ is chosen as that solution. In any event we can choose $\beta(x) = 0$ and $\alpha(x) = B_{-1}(x)$, and the equation takes the form

$$\phi_1(x) = R(x)\,\phi(x) + 1. \tag{4}$$

§ 2. *Rational Solutions of the Normal Equations and Other Solutions of the Linear Equations.*

The equation

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \tag{2}$$

has two very simple cases. If $R(x)$ is a constant, there is always a rational solution $\phi(x)$ = a constant. In the case of the q-difference equation, if q is an n-th root of unity, the solutions can be obtained by eliminating from the following equations the functions ϕ with arguments other than x, and solving the resulting algebraic equation of second degree for $\phi(x)$:

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1,$$

$$\phi(q^2x) = \frac{R(qx)}{\phi(qx)} - 1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\phi(q^{n-1}x) = \frac{R(q^{n-2}x)}{\phi(q^{n-2}x)} - 1,$$

$$\phi(x) = \frac{R(q^{n-1}x)}{\phi(q^{n-1}x)} - 1.$$

For the case of the ordinary difference equation and of the q-difference equation when q is not a root of unity, we seek to determine an upper bound to the number of rational solutions of equation (2) by substituting a power series for $\phi(x)$. This method gives at most two determinations for the first coefficient of $\phi(x)$. Suppose that two solutions $\phi(x)$ and $\psi(x)$ have the same first coefficient; we desire to know whether they are identical. We have

$$\phi(x)\{1+\phi_1(x)\} = R(x) = \psi(x)\{1+\psi_1(x)\},$$

or

$$\frac{1+\phi_1(x)}{1+\psi_1(x)} = \frac{\psi(x)}{\phi(x)}.$$

Now suppose

$$\phi(x) = a_n x^n + \ldots + a_{n-t+1}x^{n-t+1} + a_{n-t}x^{n-t} + \ldots,$$

$$\psi(x) = a_n x^n + \ldots + a_{n-t+1}x^{n-t+1} + b_{n-t}x^{n-t} + \ldots, \qquad b_{n-t} \neq 0,$$

where $a_{n-t} \neq b_{n-t}$, but the corresponding preceding coefficients are equal.

In the ordinary difference equation the expansions for $\phi_1(x)$ and $\psi_1(x)$ when rearranged as descending power series, will have each term after the first modified by the addition of elements from the terms preceding. Since the expansions for $\phi(x)$ and $\psi(x)$ are alike up to the term containing x^{n-t}, the coefficients of x^{n-t} in the expansions of $\phi_1(x)$ and $\psi_1(x)$ will be of the form $a_{n-t}+c$ and $b_{n-t}+c$, respectively. We shall thus have

$$\frac{\psi(x)}{\phi(x)} = 1 + \frac{b_{n-t}-a_{n-t}}{a_n}x^{-t} + \ldots, \tag{5}$$

and

$$\frac{1+\phi_1(x)}{1+\psi_1(x)} = 1 + \frac{a_{n-t}-b_{n-t}}{a_n+\delta_{0n}}x^{-t} + \ldots, \qquad \delta_{0n} = \begin{cases} 0 \text{ if } n \neq 0, \\ 1 \text{ if } n = 0, \end{cases} \tag{6}$$

provided that we do not have simultaneously $n=0$ and $a_0 = -1$.

When $n=0$ and $a_0=-1$, the first term of the expansion for $1+\psi_1(x)$ is $a_{-k}x^{-k}$, $k \geqq 1$. In this event, the second quotient above is either

$$1 + \frac{a_{-t}-b_{-t}}{a_{-k}}x^{-t+k} + \ldots, \quad \text{or} \quad \frac{a_{-t}}{b_{-t}} + \ldots,$$

according as $k<t$ or $k=t$. In the first case there is no term in the quotient $\psi(x)/\phi(x)$ to compare with the term in x^{-t+k}, and hence $a_{-t}=b_{-t}$; in the second case $a_{-t}/b_{-t}=1$; that is, $a_{-t}=b_{-t}$.

Comparing the expansions (5) and (6), we have, for $n \neq 0$, $b_{n-t} = a_{n-t}$. Comparing for $n = 0$, we have either $a_{-t} = b_{-t}$ or $a_0 = -1/2$. When $n = 0$ and $a_0 = -1/2$, the expansion for $R(x)$ in descending powers of x obviously starts with the constant $-1/4$.

Since there are at most two ways for the expansion for $\phi(x)$ to start, the work above shows that there are at most two formal power-series solutions, except when the expansion for $R(x)$ in descending powers of x starts with the constant $-1/4$. The number of rational solutions is not greater than the number of power-series solutions.

We have thus demonstrated the following theorem:

The equation

$$\phi(x+1) = \frac{R(x)}{\phi(x)} - 1$$

*can have at most two rational solutions, except possibly when the expansion for $R(x)$ in descending powers of x starts with the constant** $-1/4$.

For the q-difference equation we have

$$\frac{\psi(x)}{\phi(x)} = 1 + \frac{b_{n-t} - a_{n-t}}{a_n} x^{-t} + \dots.$$

and

$$\frac{1 + \phi_1(x)}{1 + \psi_1(x)} = 1 + \frac{a_{n-t} - b_{n-t}}{a_n + \delta_{0n}} q^{n-t} x^{-t} + \dots, \qquad \delta_{0n} = \begin{cases} 0 \text{ if } n \neq 0, \\ 1 \text{ if } n = 0, \end{cases}$$

provided that we do not have simultaneously $n = 0$ and $a_0 = -1$.

When $n = 0$ and $a_0 = -1$, the same argument as that given for the ordinary difference equation will show that we must have $a_{-t} = b_{-t}$. When this case does not arise, by comparing coefficients we readily have either

$$a_{n-t} = b_{n-t} \qquad \text{or} \qquad a_n(q^n + q^{n-t}) + \delta_{0n} = 0.$$

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* That there is an actual exception in this case is shown by the following examples: If $R(x)$ is taken equal to $-1/4$, equation (2) has the solutions

$$\phi(x) = -1/2 \qquad \text{and} \qquad \phi(x) = -\frac{x+a}{2(x+a-1)},$$

where a is arbitrary. Another example can be obtained from the equation given by Tietze (*loc. cit.*, foot-note, p. 330) to show the existence of rational and algebraically transcendental solutions at the same time. The equation

$$\phi(x)\{1 + \phi(x+1)\} = -\frac{(r(x) - r(x-1))\,(r(x+2) - r(x+1))}{(r(x+1) - r(x-1))\,(r(x+2) - r(x))},$$

where $r(x)$ is any rational function, has the solutions

$$\phi(x) = -\frac{r(x) - r(x-1)}{r(x+1) - r(x-1)} \qquad \text{and} \qquad \phi(x) = -\frac{(r(x) - r(x-1))\,(u(x) + r(x+1))}{(r(x+1) - r(x-1))\,(u(x) + r(x))},$$

where $u(x)$ satisfies the equation $u(x+1) = u(x)$. If $u(x)$ is chosen a constant, we have a rational solution with an arbitrary constant.

This latter equation can be satisfied only when $n \neq 0$ and q is a root of unity, or when $n=0$ and $a_0=-q^t/(1+q^t)$. There are, therefore, for the q-difference-equation case at most two rational solutions, except possibly when the expansion for $R(x)$ in the neighborhood of infinity starts with the constant $-q^t/(1+q^t)^2$. If we use expansions valid in the neighborhood of the origin, we find that there are at most two rational solutions, except possibly when the expansion for $R(x)$ starts with the same constant as for the expansion at infinity determined above.

Consequently, we are led to the following theorem:

The equation

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1$$

has at most two rational solutions, except possibly when the expansions for $R(x)$ in the neighborhood of the origin and of infinity start with the constant term $-q^t/(1+q^t)^2$, where t is an integer.*

The power-series expansions, while of value in determining an upper bound to the number of rational solutions, are not effective for the actual construction of these solutions. The following method, however, is practicable and suffices to determine *all* the rational solutions of any given equation of the types under consideration.

If the equation has a rational solution $\phi(x)$, we can write

$$\phi(x) = aP(x)/Q(x) \quad \text{and} \quad R(x) = r\rho(x)/\pi(x),$$

where a and r are constants and $P(x)$, $Q(x)$, $\rho(x)$, $\pi(x)$ are polynomials with the leading coefficients unity, and the fractions $P(x)/Q(x)$ and $\rho(x)/\pi(x)$ are in their lowest terms. We have

$$a\frac{P(x)}{Q(x)}\left(\frac{aP_1(x)+Q_1(x)}{Q_1(x)}\right) = r\frac{\rho(x)}{\pi(x)}.$$

Now, $P(x)$ and $Q_1(x)$ may have a common divisor, as may also $aP_1(x)+Q_1(x)$ and $Q(x)$. Write

$$\frac{P(x)}{Q_1(x)} = \frac{\alpha(x)}{\beta(x)} \quad \text{and} \quad a\left(\frac{aP_1(x)+Q_1(x)}{Q(x)}\right) = r\frac{\gamma(x)}{\delta(x)}, \tag{7}$$

---

* An exception will be shown if we choose $R(x)=-2/9$, where $q=2$ and $t=1$. With this value for $R(x)$, equation (2) will have the solutions

$$\phi(x)=-1/3, \qquad \phi(x)=-2/3, \qquad \phi(x)=\frac{-2x^3-mx^2-m^2x+1}{3(x^3-1)}, \qquad \phi(x)=\frac{-2(x+a)}{3(x+2a)},$$

where m takes each of the values of the cube roots of unity, and a is an arbitrary constant. Another example could be obtained if, in the example of Tietze already given, we replace the ordinary difference by the q-difference.

where $\alpha(x)/\beta(x)$ and $\gamma(x)/\delta(x)$ are in their lowest terms. Then $\alpha(x)\gamma(x) = \rho(x)$ and $\beta(x)\delta(x) = \pi(x)$. Combining the two equations (7) to eliminate $P(x)$, we have

$$a\,\alpha_1(x)\,\delta(x)\,Q_2(x) + a\,\beta_1(x)\,\delta(x)\,Q_1(x) - r\,\beta_1(x)\,\gamma(x)\,Q(x) = 0. \qquad (8)$$

Hence, if there is a rational $\phi(x)$, it must be possible to separate $\rho(x)$ into factors $\alpha(x)$ and $\gamma(x)$, and $\pi(x)$ into factors $\beta(x)$ and $\delta(x)$ in such a way that this equation in $Q(x)$ has a solution which is a polynomial with leading coefficient unity. This value of $Q(x)$ must also, when substituted in the first equation of (7), give $P(x)$ a polynomial with leading coefficient unity. Direct substitution in equation (8) of a polynomial $Q(x)$ of degree n will determine a, n and the coefficients of this polynomial, if any exists. Since there are only a finite number of ways of breaking $\rho(x)$ and $\pi(x)$ up into polynomial factors, a finite number of trials will give all the rational solutions of the original equation.

Turning now to the special normal form (3), we shall prove the following theorem:

A necessary and sufficient condition that the equation

$$\phi_1(x) = \frac{R(x)}{\phi(x)}$$

have a rational solution, is as follows:

1) *To every finite zero* [pole] of $R(x)$, except at the point $x=0$ for the q-difference equation, there is a congruent zero [pole] at an odd number of steps away,† or a congruent pole [zero] at an even number of steps away.*

2) *In the case of the q-difference equation, if $x=0$ is a zero or pole of the m-th order, then m is even.*

In the case $R(x)$ = a constant, there are no zeros or poles, and the conditions of the theorem are satisfied.

If $R(x)$ has a zero [pole] at a, then, from the equation, we see that either $\phi(x)$ has a zero [pole] at a or $\phi_1(x)$ has a zero [pole] at that point. We shall consider the consequences of the two alternatives.

(1) $\phi(x)$ has a zero [pole] at a; then $\phi_1(x)$ has a zero [pole] at a_{-1}. If $R(x)$ has no zero [pole] at a_{-1}, then $\phi(x)$ must have a pole [zero] at a_{-1},

* A zero [pole] of multiplicity m is to be counted as m simple zeros [poles], and the zeros or poles which balance it need not be at the same point. Thus, a zero of order 5 may be balanced by a congruent zero of order 3 an odd number of steps away (see next foot-note) and a pole of order 2 at an even number of steps away.

† If there is a zero or pole at a, then a zero or pole at $a+n$ $[aq^n]$ will be said to be congruent on the right at n steps away from a, and a zero or pole at $a-n$ $[aq^{-n}]$ will be said to be congruent on the left at n steps away from a. The first will be denoted by a_n and the second by a_{-n}.

in which case $\phi_1(x)$ has a pole [zero] at a_{-2}. Then either $R(x)$ has a pole [zero] at a_{-2} or $\phi(x)$ has a zero [pole] at a_{-2}, and consequently $\phi_1(x)$ a zero [pole] at a_{-3}. Then $R(x)$ has a zero [pole] at a_{-3} or $\phi(x)$ has a pole [zero] at a_{-3}, etc.

(2) $\phi_1(x)$ has a zero [pole] at a; then $\phi(x)$ has a zero [pole] at a_1, in which case $R(x)$ has a zero [pole] at a_1 or $\phi_1(x)$ has a pole [zero] at a_1. If $\phi_1(x)$ has a pole [zero] at a_1, then $\phi(x)$ has a pole [zero] at a_2 and $R(x)$ must either have a pole [zero] at a_2 or $\phi_1(x)$ a zero [pole] at that point. If $\phi_1(x)$ has a zero [pole] at a_2, then $\phi(x)$ has a zero [pole] at a_3 and $R(x)$ has a zero [pole] or $\phi_1(x)$ has a pole [zero] at a_3, etc.

It is thus seen that, if the zeros and poles are not arranged as stated in the theorem, the function $\phi(x)$ will have an infinite number of congruent zeros and poles. The condition is therefore necessary to the existence of a rational solution.

That the condition is sufficient is seen in the fact that, if the zeros and poles of $R(x)$ occur as stated in the theorem, we can at once write down a rational solution.

We have shown that a non-linear equation of type (2) can be transformed to a linear equation, provided that the non-linear equation has a rational solution; and we shall now consider the solutions of the linear form. Thus we shall complete the theory of the non-linear equation when it has a rational solution. In § 1 we have shown that the non-homogeneous equation can be reduced to the homogeneous equation, if a rational solution is known, and in any case it can be transformed to the normal form

$$\phi_1(x) = R(x)\,\phi(x) + 1. \tag{4}$$

We shall now show how all the rational solutions of equation (4) can be found.

If, in any set of congruent poles, $\phi(x)$ [$\phi_1(x)$] has a pole to the left [right] of the leftmost [rightmost] pole of $R(x)$, then $\phi_1(x)$ [$\phi(x)$] has a pole at one step (see foot-note, p. 000) to the left [right], and to balance the poles would require that $\phi(x)$ have an infinite number of poles. The order of any pole of $\phi(x)$ can not be greater than the sum of the orders of the congruent poles and zeros of $R(x)$. If $\phi(x)$ has a pole to which there is no congruent pole or zero of $R(x)$, it must have an infinite number. By substituting a power series in descending powers of x for $\phi(x)$ and finding the degree of its first term, we can determine the relative number of zeros and poles of $\phi(x)$. Knowing the possible location of the poles and the relative number of poles and zeros, we have reduced the finding of the rational solutions of this normal form to the solving of a finite number of simple algebraic equations.

For the homogeneous linear equation we have the following theorem:

A necessary and sufficient condition that the equation

$$\phi_1(x) = R(x)\,\phi(x)$$

have a rational solution (other than zero and infinity) is that the congruent zeros and poles of $R(x)$ *can be paired, a zero and a pole together. If* $R(x)$ *is a constant, it must be* 1 *for the ordinary difference equation or a power of* q *for the* q*-difference equation.*

That it is a necessary condition is shown as follows: If $R(x)$ has its left-most pole [zero] at a, then either:

(1) $\phi(x)$ has a zero [pole] at a and $\phi_1(x)$ a zero [pole] at a_{-1}; to balance this, either $R(x)$ has a zero [pole] or $\phi(x)$ must have a zero [pole] at a_{-1}, and $\phi_1(x)$ therefore has a zero [pole] at a_{-2}, etc.

Or, (2) $\phi_1(x)$ has a pole [zero] at a and $\phi(x)$ a pole [zero] at a_1; then either $\phi_1(x)$ has a pole [zero] at a_1 or $R(x)$ has a zero [pole] at a_1. If $\phi_1(x)$ has a pole [zero] at a_1, then $\phi(x)$ has a pole [zero] at a_2, and either $\phi_1(x)$ has a pole [zero] at a_2 or $R(x)$ has a zero [pole] at a_2, etc.

Thus, to every zero [pole] of $R(x)$ there must be a congruent pole [zero], or $\phi(x)$ will have an infinite number of zeros [poles].

That this is a sufficient condition can be seen from the fact that, if the zeros and poles occur as stated in the theorem, a rational solution can immediately be written out.

By methods similar to those employed in the next section for the study of the non-linear equation, it can be shown that there may be algebraic solutions other than rational solutions of certain linear q-difference equations, provided that the expansions of $R(x)$ in the neighborhood of infinity and of the origin start with a constant which is a fractional power of q. The method of finding such solutions is given in § 3.

Stridsberg has shown* that if the equation

$$\phi(x+1) = R(x)\,\phi(x) + a, \qquad a = 1 \text{ or } 0,$$

has an algebraically transcendental solution $\phi(x)$, that solution can be written in the form

$$\phi(x) = P(x)\,G(x) + Q(x),$$

where $Q(x)$ is a rational solution of the original equation and $P(x)$ is a rational solution of

$$P(x+1) = \frac{R(x)}{c}\,P(x), \qquad \lim_{x=\infty} R(x) = c,\ c \neq 0,\ \neq \infty,$$

* *Loc. cit.*, p. 2, No. 15.

and where $G(x)$ satisfies the functional equation

$$G(x+1) = c\,G(x),$$

and also satisfies a differential equation with constant coefficients.

From the discussion above concerning rational solutions of linear equations, it is easy to find, for any particular equation, whether it has algebraically transcendental solutions.

For the q-difference equation Stridsberg has shown that if the equation

$$\phi(qx) = R(x)\,\phi(x) + a, \qquad a = 1 \text{ or } 0,$$

has an algebraically transcendental solution $\phi(x)$, that solution can be written in the form

$$\phi(x) = P(x)\left(G(x) + \mu\frac{\log x}{\log q}\right) + Q(x), \qquad \mu \text{ a constant},$$

where $G(x)$ satisfies a differential equation with constant coefficients and also satisfies the functional equation

$$G(qx) = c\,x^l G(x),$$

where c is a constant and l a rational number. If $\mu \neq 0$, then $c\,x^l = 1$.

Thus, either the equation

$$P(qx) = \frac{R(x)}{c\,x^l}\,P(x)$$

and the original equation have algebraic solutions or the equations

$$P(qx) = R(x)\,P(x) \quad \text{and} \quad Q(qx) = R(x)\,Q(x) - \mu P(qx) + a$$

have algebraic solutions, according as μ is or is not zero. Hence, there are algebraically transcendental solutions of the linear q-difference equation of the forms considered, only when one of the sets of equations above has algebraic solutions.

From the results of the preceding paragraphs we conclude that the linear ordinary difference equation of the forms considered has only transcendentally transcendental solutions, unless it has a rational solution; and that the linear q-difference equation of the forms considered has only transcendentally transcendental solutions, unless one of the sets of equations above has algebraic solutions.

We can now readily see, by use of the theorem concerning rational solutions of the homogeneous equation, that the function $\Gamma(x)$ is transcendentally transcendental, as was proved by Hölder.* Likewise, it is easy to show that the solutions of the equation $\phi(qx) = (1+x)\,\phi(x)$, discussed by Stridsberg, are transcendentally transcendental.

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* *Mathematische Annalen*, Vol. XXVIII (1887), pp. 1–13.

§ 3. *Algebraic Solutions Other than Rational Functions.*

From the equations (M) and (N) it is evident that if there are singularities of the solutions, other than poles, in the finite plane, the origin excepted for the q-difference equation, there is an infinite number of such singularities. The only algebraic solutions for the ordinary difference equation are therefore rational functions. The origin may be a branch point in the q-difference equation, and its algebraic solutions are therefore rational in $x^{1/j}$, $j \geqq 1$. We have in this section to study only the q-difference equation.

In order to find what values j can have, make substitution of a descending power series in $x^{1/j}$ in equation (2), and we have

$$(a_n x^{n/j} + a_{n-1} x^{(n-1)/j} + \ldots\ldots)(1 + a_n q^{n/j} x^{n/j} + a_{n-1} q^{(n-1)/j} x^{(n-1)/j} + \ldots\ldots)$$
$$= r_m x^m + r_{m-1} x^{m-1} + \ldots\ldots.$$

Equating coefficients, we have $n/j = m/2$ and

$$a_n^2 q^{n/j} = r_m,$$

for $m > 0$. If m is even, the first exponent is an integer; and if m is odd, it is a multiple of $1/2$. If it is an integer, no fractional exponent can enter the expansion; for, if a fractional exponent l/j enters, the equation for the determination of its coefficient will be

$$a_n a_l (q^{m/2} + q^{l/j}) = 0.$$

If q is not a root of unity, we shall have $a_l = 0$. If m is odd, there can be no other exponents than multiples of $1/2$. Suppose, an exponent not a multiple of $1/2$ enters, say k/j. The equation for the determination of the corresponding coefficient will be

$$a_n a_k (q^{n/2} + q^{k/j}) = 0,$$

since $m/2 + k/j$ is not an integer. Hence, if q is not a root of unity, $a_k = 0$.

If $m < 0$, the first exponent will be an integer, and the same argument as that given above will show that no fractional exponent can enter unless q is a root of unity.

If $m = 0$, the first coefficient will be determined by the equation

$$a_0^2 + a_0 = r_0.$$

If a fractional exponent $-l/j$ enters, the equation for determining the corresponding coefficient will be

$$a_{-l}(a_0 + 1 + a_0 q^{-l/j}) = 0.$$

Either $a_{-l} = 0$ or $a_0 = -q^{l/j}/(1 + q^{l/j})$; in the latter case, $r_0 = -q^{l/j}/(1 + q^{l/j})^2$.

Consequently, j is either 1 or 2, except possibly when the expansion for $R(x)$ in the neighborhood of infinity starts with the constant r_0 above deter-

mined. Similarly, if we used expansions valid in the neighborhood of the origin, we should find that j is either 1 or 2, except possibly when the expansion for $R(x)$ starts with the same constant r_0.

We thus have the following theorem:

If $R(x)$ is infinite of odd order both at zero and at infinity, the equation

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1,$$

where q is not a root of unity, can have no algebraic solutions other than those which are rational in $x^{1/2}$ but not in x.

If the expansions for $R(x)$ in the neighborhood of zero and of infinity start with the constant $-q^{l/j}/(1+q^{l/j})^2$, where l and j are positive integers, there may be solutions rational in $x^{1/j}$.*

In all other cases any algebraic solution is rational.

If $\phi(x)$ is a solution rational in $x^{1/j}$, we can write it in the form

$$\phi(x) = \frac{P(x) + \varepsilon x^{1/j} Q(x) + \varepsilon^2 x^{2/j} S(x) + \ldots + \varepsilon^{j-1} x^{(j-1)/j} T(x)}{P'(x) + \varepsilon x^{1/j} Q'(x) + \varepsilon^2 x^{2/j} S'(x) + \ldots + \varepsilon^{j-1} x^{(j-1)/j} T'(x)},$$

where ε is 1 and $P(x)$, $P'(x)$, $Q(x)$, etc., are rational in x. Then the above expression is a solution when ε is any j-th root of unity.

If there are solutions of the normal form (2) which are rational in $x^{1/j}$, then the equation

$$\phi(q^{1/j} x) = \frac{R(x^j)}{\phi(x)} - 1$$

has corresponding solutions rational in x. The methods of the section on rational solutions will enable us to determine those solutions. Hence, we can find all the algebraic non-rational solutions of a given equation. A similar process will enable us to find all the algebraic non-rational solutions of the linear equation.

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* That solutions rational in fractional powers of x do exist is shown by the following examples:

For $q=4$ and $R(x) = 2(x - 1/9)$ the normal equation (2) has the solutions

$$\phi(x) = \pm x^{1/2} - 1/3.$$

For $q=8$ and $R(x) = -2/9$ equation (2) has the solutions

$$\phi(x) = -1/3, \quad \phi(x) = -2/3, \quad \phi(x) = -\frac{2x + m x^{2/3} + m^2 x^{1/3} - 1}{3(x-1)}, \quad \phi(x) = \frac{-2(x^{1/3}+a)}{3(x^{1/3}+2a)},$$

where m takes the values of the three cube roots of unity and a is arbitrary.

For $q=8$ and $R(x) = -\frac{2(x^2 - 9x + 1)}{9(x^2 + 9x + 1)}$ equation (2) has the solutions

$$\phi(x) = -\frac{2x + m x^{2/3} + m^2 x^{1/3} - 1}{3(x+1)},$$

where m takes the values of the three cube roots of unity.

Concerning the normal form (3), we have the following theorem:

If $R(x)$ is infinite of odd order both at zero and at infinity, the equation

$$\phi(qx) = \frac{R(x)}{\phi(x)},$$

where q is not a root of unity, can have no other algebraic solutions than those rational in $x^{1/2}$ but not in x. In all other cases the only algebraic solutions are rational.

These algebraic non-rational solutions can be found by finding the rational solutions of

$$\phi(q^{1/2}x) = \frac{R(x^2)}{\phi(x)}.$$

The proof of the above theorem follows the argument for the other normal form so closely that it will not be given here.

§ 4. *Algebraically Transcendental Solutions.*

Stridsberg proved that, if the normal equation (2) has no algebraic solution, its algebraically transcendental solutions, if any exist, also satisfy a Riccati equation with coefficients rational in x.

While proving this theorem, Stridsberg seems to have doubted the existence of such functions, as has been noted in the Introduction.

Assuming that $\phi(x)$ satisfies the two equations

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \tag{2}$$

and

$$\phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x) = 0, \tag{8}$$

where $L(x)$, $M(x)$, $N(x)$ are rational, we shall find some conditions which enable us to put the theorem in a different form.

We have from (8), by changing the argument,

$$\phi_1'(x) + L_1(x)\{\phi_1(x)\}^2 + M_1(x)\phi_1(x) + N_1(x) = 0. \tag{9}$$

Substituting in (9) from (2) so as to have only the argument x in the function ϕ, we have

$$\frac{\phi(x)R'(x) - R(x)\phi'(x)}{\alpha\{\phi(x)\}^2} + L_1(x)\left\{\frac{R(x)}{\phi(x)} - 1\right\}^2 + M_1(x)\left\{\frac{R(x)}{\phi(x)} - 1\right\} + N_1(x) = 0.$$

In this equation and what follows, α is to be considered 1 or q, according as we have the ordinary difference or the q-difference equation.

Comparing this equation with (8), after clearing of fractions and dividing by the coefficient of $\phi'(x)$, we have the following equations which $L(x)$, $M(x)$, $N(x)$ must satisfy:

$$\left.\begin{aligned} N(x) + \alpha L_1(x) R(x) = 0, \\ M(x) R(x) - 2\alpha L_1(x) R(x) + \alpha M_1(x) R(x) + R'(x) = 0, \\ L(x) R(x) + \alpha L_1(x) - \alpha M_1(x) + \alpha N_1(x) = 0; \end{aligned}\right\} \tag{10}$$

or, by elimination of $M(x)$ and $N(x)$, we have the equation

$$\alpha^3 R_2(x) L_3(x) + \alpha^2 \{R_1(x) + 1\} L_2(x) - \alpha \{R_1(x) + 1\} L_1(x) \\ - R(x) L(x) - \alpha R_1'(x)/R_1(x) = 0. \tag{11}$$

This equation puts all the restriction on $R(x)$ that the system of three equations above does. That is, in order that equation (2) have a solution $\phi(x)$ which is also a solution of (8), it is necessary that (11) have a rational solution $L(x)$. This last sentence is simply another statement of the theorem.

We want to prove that this necessary condition for algebraically transcendental solutions is also sufficient. In order to do this, set

$$D(x) = \phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x),$$

where $L(x), M(x), N(x)$ satisfy the relations (10). A necessary and sufficient condition that a solution $\phi(x)$ of the equation (2) also satisfy the Riccati equation, is that $D(x) \equiv 0$. Now,

$$D_1(x) = \phi_1'(x) + L_1(x)\{\phi_1(x)\}^2 + M_1(x)\phi_1(x) + N_1(x).$$

Multiply the equation in $D(x)$ by $R(x)/\{\phi(x)\}^2$, the equation in $D_1(x)$ by α, and add, making use of the equations (10). Then we have readily

$$\frac{D_1(x)}{D(x)} = -\frac{R(x)}{\alpha\{\phi(x)\}^2}.$$

This equation shows that, if there is a solution $\phi(x)$ of equation (2), in the case of the ordinary difference equation, which is asymptotic to a power of x times a descending power series in any fractional power of x, the only possible value of $D(x)$ is zero, except when the expansion of $R(x)$ in the neighborhood of infinity starts with the constant $-1/4$. In the case of the q-difference equation, q not a root of unity, the exception is when the expansions for $R(x)$ in the neighborhood of zero and of infinity each start with the constant $-q/(1+q)^2$.

We shall now determine when equation (2) has solutions which are asymptotic to a power of x times a power series in $1/x$ or in $1/\sqrt{x}$. The substitution

$\phi(x)=\psi_1(x)/\psi(x)$ will transform the normal form (2) to the second-order linear homogeneous equation

$$\psi_2(x)+\psi_1(x)-R(x)\psi(x)=0. \tag{12}$$

We prove in the next section that if $R(x)$ has a zero at infinity, then equation (11) has no rational solution, so that we shall not consider such an $R(x)$ here.

For the ordinary difference equation, if $R(x)$ has a pole of even order at infinity or approaches a constant different from $-1/4$ as x approaches infinity, then the works of Carmichael* and of Birkhoff† show that there is a solution of equation (12) such that, when we form the quotient $\psi_1(x)/\psi(x)$, it has the property of being asymptotic to a power of x times a power series in $1/x$. When $R(x)$ has a pole of odd order at infinity, it can be shown‡ that the linear equation (12) has a solution $\psi(x)$ such that, when we form the quotient $\psi_1(x)/\psi(x)$, it has the property of being asymptotic to a power of x times a power series in $1/\sqrt{x}$.

For the q-difference equation, if $R(x)$ has a pole of even order at infinity or approaches a constant different from $-q^t/(1+q^t)^2$, where t is an integer, as x approaches infinity, then the results of a paper by Carmichael§ show that the quotient $\psi_1(x)/\psi(x)$ has the property of being asymptotic to a power of x times a power series in $1/x$. If $R(x)$ has a pole of odd order at infinity, we can replace $R(x)$ by $R(x^2)$ and q by $\sqrt{q}$, and have an equation to which the results of the paper mentioned are applicable. Making the reverse substitution in the solution of the new equation, we shall have for the quotient $\psi_1(x)/\psi(x)$ a power of x times a power series in $1/\sqrt{x}$.

From these considerations it follows that in each case there will be two solutions of equation (12) asymptotic to such formal expansions that, if we form the quotient $\phi(x)=\psi_1(x)/\psi(x)$, it will be asymptotic to a power of x times a power series in $1/x$ or in $1/\sqrt{x}$.

From the results thus obtained we have the theorem:

If, as x approaches infinity, $R(x)$ does not approach the constant $-\alpha/(1+\alpha)^2$, where α is 1 or q according as we have the ordinary difference or the q-difference equation, and the equation

$$\phi_1(x)=\frac{R(x)}{\phi(x)}-1 \tag{2}$$

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* *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 99–134.

† *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 243–284.

‡ Lectures of Dr. Carmichael at Indiana University, 1913–14.

§ AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 147–168.

has no algebraic solution, a necessary and sufficient condition that equation (2) *have a solution which is algebraically transcendental is that the equation*

$$\alpha^3 R_2(x) L_3(x) + \alpha^2 \{R_1(x)+1\} L_2(x) - \alpha \{R_1(x)+1\} L_1(x) \\ - R(x) L(x) - \alpha R_1'(x)/R_1(x) = 0 \qquad (11)$$

have a rational solution $L(x)$. *Any solution of equation* (2) *which is asymptotic to one of its formal solutions in powers of* $1/x$ *or of* $1/\sqrt{x}$ *will be algebraically transcendental. It will satisfy the Riccati equation*

$$\phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x) = 0,$$

where $M(x)$ *and* $N(x)$ *are determined from equations* (10).

For the ordinary difference equation $R(x) = k\dfrac{x(x+3)}{(x+1)(x+2)}$, $k \neq -1/4$, and for the q-difference equation $R(x) = k\dfrac{(x+q)(x+q^4)}{(x+q^2)(x+q^3)}$, $k \neq -q^t/(1+q^t)^2$, where t is a rational number, the normal form (2) has no algebraic solution; while equation (11) does have a rational solution. The normal equations, therefore, have as solutions transcendental functions which also satisfy Riccati equations. I expect to develop the theory of these functions in detail.

§ 5. *Equations Having only Transcendentally Transcendental Solutions.*

A result obtained in the preceding section may be restated in the following form:*

If the equation

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \qquad (2)$$

has no algebraic solution, then all its solutions are transcendentally transcendental, provided that the equation

$$\alpha^3 R_2(x) L_3(x) + \alpha^2 \{R_1(x)+1\} L_2(x) - \alpha \{R_1(x)+1\} L_1(x) \\ - R(x) L(x) - \alpha R_1'(x)/R_1(x) = 0 \qquad (11)$$

has no rational solution.

We shall next consider rational solutions of equation (11). In order to do this, we shall first investigate the relative number of poles and zeros of $L(x)$ by means of series. Make the substitution

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* This theorem is here derived by means of a theorem of Stridsberg quoted in the preceding section. I had, however, proved the theorem in essentially the present form before the work of Stridsberg came to my attention. In view of the results in § 4, it is seen that the two theorems are equivalent.

$$L(x) = l_\lambda x^\lambda + l_{\lambda-1} x^{\lambda-1} + \ldots, \qquad l_\lambda \neq 0,$$
$$R(x) = r_m x^m + r_{m-1} x^{m-1} + \ldots, \qquad r_m \neq 0,$$

in equation (11). Equating the coefficients of the highest powers of x to zero, we shall have for the ordinary difference equation when $m > 0$,

$$r_m l_\lambda(0) - \{m\} = 0,$$

the quantity $\{m\}$ entering only if $m + \lambda = -1$;

$$r_m l_{\lambda-1}(0) + r_{m-1} l_\lambda(0) + r_m l_\lambda(2m + 3\lambda + m + 2\lambda - m - \lambda) - \{m\} = 0,$$

the quantity $\{m\}$ entering only if $m + \lambda = 0$.

It is obvious that these equations can be satisfied only when $\lambda = -m/2$ or $\lambda = -m$.

If $m = 0$, our equations are

$$r_0 l_\lambda(0) + l_\lambda(0) - \{s\} = 0,$$

where $\{s\}$ is a constant $\neq 0$ and enters only if $\lambda \leqq -2$;

$$r_{-1} l_\lambda(0) + r_0 l_{\lambda-1}(0) + l_{\lambda-1}(0) + l_\lambda(2\lambda - \lambda) + r_0 l_\lambda(4\lambda) - \{s\} = 0,$$

where $\{s\}$ enters only if $\lambda \leqq -2$.

These equations can be satisfied by $\lambda = 0$, $\lambda = -k + 1$, where $r_{-k} x^{-k}$ is the second term in the expansion of $R(x)$, or by $r_0 = -1/4$.

If $m < 0$, the equations are

$$l_\lambda(0) + \{m\} = 0,$$

where $\{m\}$ enters only if $\lambda = -1$;

$$l_{\lambda-1}(0) + l_\lambda(2\lambda - \lambda) + \{m\} = 0,$$

where $\{m\}$ enters only if $\lambda = 0$.

These equations can be satisfied for no value of λ; that is, *if $R(x)$ is zero at infinity,* there is no rational $L(x)$ satisfying equation* (11).

For the q-difference equation the same method gives the following results:

$m > 0$, $\lambda = -m/2 - 1$ or $\lambda = -m - 1$;

$m = 0$, there can exist a rational $L(x)$ satisfying the equation (11) only when $r_0 = -q^{1+\lambda}/(1 + q^{1+\lambda})^2$, where λ is an integer;

$m < 0$, *there exists no rational $L(x)$ satisfying the equation.*

Expansions in the neighborhood of the origin will give analogous results for the q-difference equation.

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* This was the case proved by Tietze.

We shall next consider the possible number and location of the poles of the rational function $L(x)$ satisfying equation (11). If the leftmost zero or pole of $R(x)$ in any congruent set is a and the rightmost is a_k, then the leftmost pole of $R_1'(x)/R_1(x)$ in the same set is a_{-1} and the rightmost is a_{k-1}. If $L(x)$ has a congruent pole to the left of a_1, then $L_3(x)$ has a pole to the left of a_{-2}; and since $R_2(x)$ does not have a zero to the left of that point, $L(x)$, $L_1(x)$, or $L_2(x)$ must have a pole at the point at which $L_3(x)$ has a pole. Then $L_3(x)$ has a pole farther to the left, and it must again be balanced and there must be an infinite number of congruent poles. Similarly, one sees that the rightmost pole of $L(x)$ can not be to the right of a_k without introducing an infinite number of congruent poles. Thus, it follows that the congruent poles of a rational solution $L(x)$ in any set can be only at the points $a_1, a_2, \ldots, a_k$, where a and a_k are the leftmost and rightmost congruent zeros or poles of $R(x)$ in that set. Clearly, $L(x)$ can not have a pole which is not congruent to a zero or a pole of $R(x)$. Likewise it can be shown that there can be no rational $L(x)$ for an $R(x)$ which has a zero or pole to which there is no congruent zero or pole, the origin excepted for the q-difference equation. The order of any pole can not be greater than the sum of the orders of the congruent zeros and poles of $R(x)$.

We have found an upper bound to the number and order, and have found the possible locations of the poles of $L(x)$, and we know the relative number of poles and zeros, except when the expansion for $R(x)$ starts with the particular constants noted above; we may therefore replace $L(x)$ by $P(x)/Q(x)$, where $Q(x)$ is a fully determined polynomial and $P(x)$ is a polynomial of known degree, with the exceptions noted. We can substitute a polynomial of proper degree for $P(x)$ and determine its coefficients by simple algebraic processes if a rational $L(x)$ exists. In the exceptional cases we can substitute a general polynomial and reckon out its degree as well as its coefficients.

In addition to furnishing methods for determining the character of the solutions of any given equation, the results obtained in this paper enable us to name a number of classes of equations which have only transcendentally transcendental solutions. We shall define the more comprehensive classes in the following theorems:

The equation

$$\phi(x+1) = \frac{R(x)}{\phi(x)} - 1$$

has only transcendentally transcendental solutions, provided that $R(x)$ *has a*

pole of odd order at infinity and has a zero or a pole to which there is no congruent zero or pole.

The equation

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1$$

has only transcendentally transcendental solutions, provided that $R(x)$ satisfies any one of the following sets of conditions:

1) *$R(x)$ has a zero at infinity [zero] and a pole of odd order at zero [infinity].*

2) *$R(x)$ has a pole of odd order at infinity [zero] and a pole of even order at zero [infinity], and has a zero or pole to which there is no congruent zero or pole.*

3) *$R(x)$ has a pole of odd order at infinity [zero], and at zero [infinity] is a constant different from zero, infinity and $-q^t/(1+q^t)^2$, where t is a rational number.*

We return to a consideration of the normal form (3), $\phi_1(x)\phi(x) = R(x)$. This equation transforms by a linear fractional substitution (see § 1) to the normal form (2); and hence, by § 4, if the normal form (3) has no algebraic solutions, its algebraically transcendental solutions, if any exist, satisfy a Riccati equation. If, as in the preceding section, we assume that a solution $\phi(x)$ of equation (3) also satisfies the Riccati equation

$$\phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x) = 0,$$

we are led in the same way to the following equations which must be satisfied by rational functions $L(x)$, $M(x)$, $N(x)$:

$$\begin{aligned} L(x)R(x) + \alpha N_1(x) &= 0, \\ \alpha M_1(x) + M(x) + R'(x)/R(x) &= 0, \\ \alpha L_1(x)R(x) + N(x) &= 0. \end{aligned}$$

Consider the second of these equations. The poles of $R'(x)/R(x)$ are all of first order, and the residues at those poles are all integers. Consequently, we are led to see that, in order for $M(x)$ to be rational, it must have all its poles of the first order, and the residues at the poles must be integers. These properties of $M(x)$ are sufficient to make the expressions $e^{\int M(x)dx}$ and $e^{\int \alpha M_1(x)dx}$ represent rational functions. If we integrate the equation and write in exponential form, we have

$$e^{\int \alpha M_1(x)dx}\, e^{\int M(x)dx}\, e^{\int R'(x)/R(x)dx} = 1.$$

Write $1/D(x)$ for $e^{\int M(x)dx}$, and the above equation readily reduces to

$$D_1(x)\,D(x) = R(x).$$

Since $D(x)$ is rational, we see that a necessary condition that the equation in $M(x)$ have a rational solution, is that the normal equation (3) have a rational solution. A consideration of the first and third of the set of three equations given above will lead to the same conclusion unless $L(x)$ and $N(x)$ are both zero.

We have then the following theorem:

If the equation

$$\phi_1(x) = \frac{R(x)}{\phi(x)}$$

has no algebraic solution, all its solutions are transcendentally transcendental.

BLOOMINGTON, INDIANA.

*Binary Conditions for Double and Triple Points on a Cubic.**

By Leroy A. Howland.

The cubic plane curve has been exhaustively studied by use of the theory of ternary forms. Conditions necessary and sufficient for the existence of one or more double points, of a triple point, etc., have been derived in terms of ternary invariants, chiefly by Aronhold, Gordan, Clebsch and Gundelfinger.†

The theory of binary forms, however, may also be applied to the study of the cubic. Compare, for example, the treatment of the inflexions of a cubic by Clebsch ‡ and by the writer. § Conditions that a cubic degenerate into a conic and a straight line or into three straight lines (non-current), *i. e.*, conditions for two or three double points, were derived by Brioschi, expressed in terms of the simultaneous invariants of two binary forms. || In this paper conditions of the same nature will be derived for the case of one double point (node or cusp) and for the case of a triple point.

The argument will be based upon two properties of the polars of algebraic plane curves. Let P be a point of the curve C_n, and let Q be a point of the plane not lying on a tangent to C_n at P.

Property I. *A necessary and sufficient condition that P be an r-fold point for C_n is that the first $r-1$ polars of C_n with respect to Q pass through P.*

That it is necessary follows at once from the well-known

Theorem A. *An r-fold point of C_n is an $(r-s)$-fold point of the s-th polar $(s<r)$ of any other point.*

To prove it sufficient we will take the vertices III and I of the frame of reference to be the points P and Q respectively. The equation of the curve will then be

$$x_3^{n-1} u_1 + x_3^{n-2} u_2 + \dots + x_3 u_{n-1} + u_n = 0,$$

and the equations of the polars will be

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* Read before the American Mathematical Society, April 27, 1912.

† Cf. Clebsch-Lindemann, *Vorlesungen über Geometrie*, Bd. I, p. 580 ff.

‡ "Theorie der Binären Formen," pp. 234–254.

§ Dissertation, München, 1908.

|| "Sulle condizioni per la decomposizione di una forma cubica ternaria in tre fattori lineari," *Annali di Matematica*, (2) VII (1875), pp. 189–192. "Sulle condizioni per la decomposizione di una cubica in una conica ed in una retta," *Acc. R. d. L.*, (2) III (1876), pp. 89–90.

$$x_3^{n-1}\frac{\partial u_1}{\partial x_1}+x_3^{n-2}\frac{\partial u_2}{\partial x_1}+\ldots\ldots+\frac{\partial u_n}{\partial x_1}=0,$$

$$x_3^{n-2}\frac{\partial^2 u_2}{\partial x_1^2}+x_3^{n-3}\frac{\partial^2 u_3}{\partial x_1^2}+\ldots\ldots+\frac{\partial^2 u_n}{\partial x_1^2}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_3^{n-r+1}\frac{\partial^{r-1} u_{r-1}}{\partial x_1^{r-1}}+\ldots\ldots+\frac{\partial^{r-1} u_n}{\partial x_1^{r-1}}=0.$$

Since by hypothesis all these polars pass through III, we have

$$\frac{\partial u_1}{\partial x_1}=\frac{\partial^2 u_2}{\partial x_1^2}=\frac{\partial^3 u_3}{\partial x_1^3}=\ldots\ldots=\frac{\partial^{r-1} u_{r-1}}{\partial x_1^{r-1}}=0.$$

This means that u_k is divisible by x_2 for $k=1,2,\ldots,r-1$. But the first non-vanishing u may not be divisible by x_2, otherwise I would be on a tangent to C_n at III, contrary to hypothesis. Consequently,

$$u_1\equiv u_2\equiv u_3\equiv\ldots\ldots\equiv u_{r-1}\equiv 0,$$

and C_n has an r-fold point.

PROPERTY II. *A necessary and sufficient condition that s of the r branches of C_n through P be tangent at P to a line t, is that C_n and its first $s-1$ polars with respect to Q be tangent to t at P.*

To prove the condition sufficient we choose the frame of reference as before, taking, in addition, the side I to be the line t. The equations of the curve and its polars are then

$$x_3^{n-r}\,u_r+x_3^{n-r-1}\,u_{r+1}+\ldots\ldots+u_n=0,$$

$$x_3^{n-r}\frac{\partial u_r}{\partial x_1}+\ldots\ldots+\frac{\partial u_n}{\partial x_1}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_3^{n-r}\frac{\partial^{s-1} u_r}{\partial x_1^{s-1}}+\ldots\ldots+\frac{\partial^{s-1} u_n}{\partial x_1^{s-1}}=0.$$

Assume

$$u_r=a_0x_1^r+r\,a_1x_1^{r-1}x_2+\ldots\ldots+r\,a_{r-1}x_1x_2^{r-1}+a_rx_2^r.$$

Then

$$\frac{\partial^k u_r}{\partial x_1^k}=r(r-1)\ldots\ldots(r-k+1)\,[a_0\,x_1^{r-k}+(r-k)\,a_1\,x_1^{r-k-1}x_2+\ldots\ldots+a_{r-k}\,x_2^{r-k}].$$

By hypothesis this must be divisible by x_1 for $k=0,1,\ldots,s-1$. That is, $a_{r-k}=0$ for $k=0,1,2,\ldots,s-1$.

Hence,

$$u_r = a_0 x_1^r + r\, a_1 x_1^{r-1} x_2 + \ldots + \frac{r(r-1)\ldots(s+1)}{(r-s)!} a_{r-s} x_1^s x_2^{r-s},$$

which proves the theorem.

That the condition is necessary is well known, and is indeed obvious, if we write down the equation of C_n referred to the frame of reference chosen above.

Consider the cubic

$$y_3^3 + U_1 y_3^2 + U_2 y_3 + U_3 = 0.$$

The transformation with determinant equal to 1,

$$x_1 = y_1, \qquad x_2 = y_2, \qquad x_3 = \tfrac{1}{3} U_1 + y_3,$$

carries this over into

$$x_3^3 - 3 f x_3 + 2\phi = 0, \tag{1}$$

where f and ϕ are forms of the second and third degree respectively in x_1 and x_2. If the line

$$x_3 = \xi_x \tag{2}$$

cuts the curve at P in three coincident points, we have

$$\xi_x^3 - 3 f \xi_x + 2\phi \equiv \eta_x^3. \tag{3}$$

A necessary and sufficient condition that P be a double point is (Property I) that the polar of III pass through P; *i. e.*, that

$$\xi_x^2 - f \text{ be divisible by } \eta_x, \tag{4}$$

or, representing f and ϕ symbolically by a_x^2 and α_x^3, that

$$(\xi\eta)^2 - (a\eta)^2 = 0. \tag{5}$$

We shall use the following simultaneous invariants of f and ϕ, the notation being that of Clebsch:

$D = (ab)^2$, discriminant of f;

$\Delta = (\alpha\beta)^2 \alpha_x \beta_x$, Hessian of ϕ;

$R = (\Delta\Delta')^2$, discriminant of Δ and of ϕ;

$P = (a\alpha)^2 \alpha_x$;

$E = (a\Delta)^2$;

$F = (ap)^2$;

$F - 2DE$ is the resultant of f and ϕ.

We shall also use the following abbreviations:

$$K = (a\xi)^2, \quad L = (a\xi)(a\eta), \quad M = (a\eta)^2, \quad N = (\xi\eta)^2.$$

Using a well-known symbolic identity, we find

$$Nf = K\eta^2 - 2L\xi\eta - M\xi^2. \tag{6}$$

Combining (3) and (6), we have

$$2N\phi = N\eta^3 + 3K\xi\eta^2 - 6L\xi^2\eta + (3M - N)\xi^3. \tag{7}$$

Building the invariants of f and ϕ by means of (6) and (7), we find

$$\left.\begin{aligned}
DN &= 2(KM - L^2), \\
2p &= (2D - K)\xi + M\eta, \\
4F &= K^3 + M^3 - 2KLM + 4D(ML - K^2) + 4D^2K, \\
2E &= KD + LM - K^2 - LN, \\
8R &= 4K^3 + 8L^3 - 12KLM - 6DK^2 + 12KLN - 12DLN - 9M^2N \\
&\qquad + 6MN^2 - N^3.
\end{aligned}\right\} \tag{8}$$

The quantities

$$\left.\begin{aligned}
A &= 8E - D^2, \\
B &= 20DE - 8R - 16F + D^3
\end{aligned}\right\} \tag{9}$$

have the values

$$A = 4L(M - N) - (2K - D)^2, \tag{10}$$

$$B = 6L(M - N)(2K - D) - (4M - N)(M - N)^2 - (2K - D)^3,\text{*} \tag{11}$$

$$\begin{aligned}
A^3 + B^2 = (M - N)^2 [&64L^3(M - N) - 12L^2(2K - D)^2 + (4M - N)^2(M - N)^2 \\
&- 12L(4M - N)(M - N)(2K - D) + 2(4M - N)(2K - D)^3].
\end{aligned}$$

By use of the first equation (8), we can change this into

$$\begin{aligned}
A^3 + B^2 = (M - N)^3 [&16L(2D - K)(M - N) - 4L(2K - D)(4M - N) \\
&+ 16L^3 + (4M - N)^2(M - N) - 4(2K - D)^2(2D - K)].
\end{aligned} \tag{12}$$

A necessary condition for a double point is then

$$A^3 + B^2 = 0. \tag{13}$$

It does not appear to be sufficient, for when $A^3 + B^2$ is zero, it may happen that $M - N$ is not zero, but rather the quantity in brackets. (13) is not a sufficient condition that P be a double point; it is, however, sufficient that the curve have a double point, as we can show by using some results of Clebsch.

Clebsch shows that if $x_3 = \xi_x$ is a line which cuts the curve in three coincident points, the other lines having this property are

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* Equations (6)–(11) are taken, with some change of form and corrections for misprints, from Clebsch, *loc. cit.*, p. 241.

$$x_3 = \xi_x + z,$$

$$z = \frac{-3(\xi - m^2\eta) \pm \sqrt{9(\xi - m^2\eta)^2 - 12(1-m^3)(\xi^2 - m\eta^2 - f)}}{2(1-m^3)}, \qquad (14)$$

where m is a root of the equation

$$(M-N)\,m^4 - 2(2K-D)\,m^3 + 6Lm^2 - (4M-N)\,m + (K-2D) = 0.^* \qquad (15)$$

The two invariants of this biquadratic are

$$i = 3DN - 6(KM - L^2),$$

$$j = -\tfrac{3}{8}[16L(2D-K)(M-N) - 4L(2K-D)(4M-N) + 16L^3 + (M-N)(4M-N)^2 - 4(2K-D)^2(2D-K)].$$

$i = 0$ by reason of the first of equations (8). Hence, if j is zero, the equation (15) has a triple root, (14) gives only four values of z and there are but five lines which meet the curve in three coincident points. This can happen only when the curve has a double point. Therefore, (13) is a sufficient condition.

We have, however, overlooked the possibility that the curve may have a double point the tangent at which goes through III. Any line through III has the form

$$\zeta_x \equiv \zeta_1 x_1 + \zeta_2 x_2 = 0.$$

Both ζ_1 and ζ_2 can not be zero. Let us suppose $\zeta_1 \neq 0$. If this line cuts the curve in three coincident points, then

$$\zeta_1^3 x_3^3 - 3\zeta_1 (a\zeta)^2 x_2^2 x_3 + 2(\alpha\zeta)^3 x_2^3$$

must be a cube and hence its Hessian must vanish identically; *i. e.*,

$$\zeta_1^4 (a\zeta)^2 = 0, \qquad 2\zeta_1^3 (\alpha\zeta)^3 = 0,$$

or f and ϕ have a common factor, and $\zeta_x = 0$ cuts the curve where $x_3 = 0$. For a double point it is necessary and sufficient that the three polars

$$x_3^2 - f = 0,$$

$$-3x_3 \frac{\partial f}{\partial x_1} + 2\frac{\partial \phi}{\partial x_1} = 0,$$

$$-3x_3 \frac{\partial f}{\partial x_2} + 2\frac{\partial \phi}{\partial x_2} = 0$$

pass through it. Hence, ζ_x must also be a factor of $\frac{\partial \phi}{\partial x_1}$ and $\frac{\partial \phi}{\partial x_2}$, and therefore a double factor of ϕ. The conditions that f and ϕ have a common factor, which is a double factor of ϕ, are easily seen to be

$$R = E = F = 0.$$

In this case $A = -D^2$, $B = D^3$ and $A^3 + B^2 = 0$.

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* *Loc. cit.*, p. 240; or for the form used here, cf. Howland, Dissertation, p. 11 ff.

THEOREM. *The necessary and sufficient condition that the cubic* (1) *have a double point, is* $A^3 + B^2 = 0$.

For a cusp, the tangent at which does not go through III, we must have (Property II) $x_3 = \xi_x$ tangent to $x_3^2 - f = 0$, *i. e.*, $\xi_x^2 - f$ must be divisible by η_x^2. Hence, in addition to $M - N = 0$, the discriminant of $\xi_x^2 - f$ must vanish, or $2(a\xi)^2 - (ab)^2 = 2K - D = 0$.

In this case we see from equations (10) and (11) that $A = B = 0$. Conversely, if $A = B = 0$, we have a double point and hence $M - N = 0$. (10) shows that then $2K - D = 0$ also and the point is a cusp.

If the tangent passes through III, we have shown that the point must be the point $\zeta_2 : -\zeta_1 : 0$ and that $R = E = F = 0$.

The equation of the tangents at this point is

$$2\phi_{11}x_1^2 + 4\phi_{12}x_1x_2 + 2\phi_{22}x_2^2 - 6f_1x_1x_3 - 6f_2x_2x_3 = 0,$$

ζ_2, $-\zeta_1$ being substituted for x_1, x_2 in the coefficients.

For a cusp this must be a constant multiple of ζ_x^2, and hence

$$\phi_{11}\phi_{22} - \phi_{12}^2 = 0, \quad f_1 = 0, \quad f_2 = 0.$$

The first equation says that the Hessian of ϕ has the factor ζ_x. This gives nothing new, for, as is well known, a double factor of ϕ is also a double factor of its Hessian. The other two equations, however, tell us that f has a double factor ζ_x. Consequently, $D = 0$, and by equation (9), $A = B = 0$.

THEOREM. *The necessary and sufficient conditions that the cubic* (1) *have a cusp are* $A = B = 0$.

Instead of taking up next the case of a triple point of a cubic, we shall consider a more general problem, for it is possible by this method to handle the problem of an n-fold point on a curve of the n-th degree.

The equation of the curve being

$$y_3^n + v_1y_3^{n-1} + v_2y_3^{n-2} + \ldots\ldots + v_{n-1}y_3 + v_n = 0,$$

we can transform it by the transformation with determinant 1,

$$x_1 = y_1, \qquad x_2 = y_2, \qquad x_3 = \frac{1}{n}v_1 + y_3,$$

into

$$x_3^n + \frac{n(n-1)}{2!}u_2x_3^{n-2} + \frac{n(n-1)(n-2)}{3!}u_3x_3^{n-3} + \ldots\ldots + nu_{n-1}x_3 + u_n = 0. \quad (16)$$

The polars of this curve with respect to the vertex III are:

$$x_3^{n-1}+\frac{(n-1)(n-2)}{2!}u_2x_3^{n-3}+\ldots.+u_{n-1}=0,$$

$$x_3^{n-2}+\frac{(n-2)(n-3)}{2!}u_2x_3^{n-4}+\ldots.+u_{n-2}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_3^3+3\,u_2x_3+u_3=0,$$

$$x_3^2+u_2=0,$$

$$x_3=0.$$

If this curve has an n-fold point P, then, by Theorem A, this point is $(n-1)$-fold for the first polar, $(n-2)$-fold for the second, ..., a double point for the $(n-2)$-th polar and a simple point for the $(n-1)$-th. It follows directly from this that u_2, u_3, ..., u_{n-1}, u_n are constant multiples of the second, third, ..., $(n-1)$-th, n-th powers of the same linear form, $a_x=a_1x_1+a_2x_2$.

These conditions are evidently sufficient, for the equation then has the form

$$x_3^n+\alpha\,a_x^2x_3^{n-2}+\beta\,a_x^3x_3^{n-3}+\ldots.+\nu\,a_x^n=0, \tag{17}$$

and any line through the point $-a_2:a_1:0$,

$$x_3+\lambda\,a_x=0,$$

obviously has with this curve at P a multiplicity of intersection equal to n.

The equations of the lines into which (17) degenerates are

$$x_3=\lambda_i a_x, \qquad (i=1,2,\ldots.,n),$$

where

$$\Sigma\lambda_i=0,\quad \Sigma\lambda_i\lambda_j=\alpha,\;\ldots.,\;\lambda_1\lambda_2\ldots.\lambda_n=(-1)^n\nu,$$

or, in other words, the λ's are roots of the equation

$$\lambda^n+\alpha\lambda^{n-2}+\beta\lambda^{n-3}+\ldots.+\nu=0.$$

a_x may be found as the square root of u_2. In case u_2 vanishes identically, a_x may be found as the quotient of u_i by u_{i-1}, or, if there is no pair of consecutive non-vanishing u's, as the quotient of u_i by $\dfrac{\partial u_i}{\partial x_1}$.

$H(u_p)\equiv 0$ is a necessary and sufficient condition that u_p be a p-th power, $H(u_p)$ being the Hessian of u_p. $H(u_p)\equiv 0$, $H(u_q)\equiv 0$, $I(u_p, u_q)=0$ are conditions that u_p and u_q be powers of the same linear form, $I(u_p, u_q)$ being any simultaneous invariant of u_p and u_q.

THEOREM. *Necessary and sufficient conditions that a curve of the n-th degree have an n-fold point are (when the equation of the curve has been brought into the form* (16))

$$H(u_2)\equiv H(u_3)\equiv\ldots.\equiv H(u_n)\equiv 0,$$

together with a set obtained by equating to zero any simultaneous invariants of a non-vanishing u with each of the other u's.

In particular, then, the conditions for a triple point on the cubic

$$x_3^3 - 3 f x_3 + 2 \phi = 0$$

will be

$$D = 0, \quad \Delta \equiv 0, \quad F = 0.$$

The ternary condition for an n-fold point is the identical vanishing of the Hessian. This gives rise to $\frac{1}{2}(3n-5)(3n-4)$ equations among the coefficients. The set of conditions above give rise to but $(n-1)^2 + n - 2 = n^2 - n - 1$.*

The number of conditions for the conic, cubic, quartic and quintic in the two cases are: 1, 10, 28, 55, and 1, 5, 11, 19 respectively.

MIDDLETOWN, CONN., *April*, 1912.

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* It is not intended here to imply that this is a minimum set of conditions. Cf. O. E. Glenn, "On Semi-discriminants of Ternary Forms," *Transactions of the American Mathematical Society*, Vol. XII, No. 3 (1911), pp. 367–374.

Modular Invariants of Two Pairs of Cogredient Variables.

By William C. Krathwohl.

Introduction.

§ 1. By the term invariant will here be understood a polynomial I in x_1, y_1, x_2, y_2 with integral coefficients taken modulo p such that

$$I(x_1, y_1, x_2, y_2) \equiv (ad - bc)^{\mu} I(x_1', y_1', x_2', y_2'), \qquad (\text{mod. } p),$$

for every transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad \begin{matrix} x_1 = a x_1' + b y_1', & x_2 = a x_2' + b y_2', \\ y_1 = c x_1' + d y_1', & y_2 = c x_2' + d y_2', \end{matrix}$$

with integral coefficients.

The constant exponent μ is called the *index* of the invariant.

The main result of the investigation is the following

Theorem. *As a fundamental system of invariants we may take*

$$L_i = \begin{vmatrix} x_i^p & y_i^p \\ x_i & y_i \end{vmatrix}, \quad Q_i = \begin{vmatrix} x_i^{p^2} & y_i^{p^2} \\ x_i & y_i \end{vmatrix} / L_i, \qquad (i = 1, 2),$$

$$M = x_2 y_1 - y_2 x_1, \quad M_1 = x_2 y_1^p - y_2 x_1^p, \quad M_2 = x_2^p y_1 - y_2^p x_1,$$

$$N_s = \frac{M_2^{s+1} L_1^{p-s-1} + (-1)^s M_1^{p-s} L_2^s}{M^p}, \qquad (1 \leqq s \leqq p-2).$$

The absolute invariants Q_i* and N_s are actually integral functions of x_1, y_1, x_2, y_2.

Among the syzygies needed are

$$\begin{aligned} &(S_0) \quad L_1 L_2 + M_1 M_2 - M^{p+1} = 0, \\ &(S_1) \quad M_2 L_1^{p-1} + M_1^p - M^p Q_1 = 0, \\ &(S_2) \quad M_1 L_2^{p-1} + M_2^p - M^p Q_2 = 0. \end{aligned}$$

The invariants N_s can be shown to be integral as follows: Multiplying numerator and denominator of N_s by L_1^s, we get

$$N_s = \frac{M_2^{s+1} L_1^{p-1} + (-1)^s M_1^{p-s} L_1^s L_2^s}{M^p L_1^s}. \tag{1}$$

Multiplying syzygy (S_1) by M_2^s, we have

$$M_2^{s+1} L_1^{p-1} = M^p Q_1 M_2^s - M_1^p M_2^s. \tag{2}$$

* Dickson, *Transactions American Mathematical Society*, Vol. XII, pp. 1–12.

Solving syzygy (S_0) for $L_1 L_2$ and then raising $L_1 L_2$ to the s-th power, we get

$$(L_1 L_2)^s = \sum_{k=0}^{s} (-1)^k \binom{s}{k} (M^{p+1})^{s-k} (M_1 M_2)^k. \tag{3}$$

Substituting (2) and (3) in (1), we get

$$N_s = \frac{M^p Q_1 M_2^s + (-1)^s M_1^{p-s} \sum_{k=0}^{s-1} (-1)^k \binom{s}{k} (M^{p+1})^{s-k} (M_1 M_2)^k}{M^p L_1^s}.$$

Every term in the numerator of N_s now contains M^p as a factor. Since M^p is prime to L_1^s, the numerator of N_s is divisible by their product, and hence N_s is integral in x_1, y_1, x_2 and y_2.

Preliminary Theorems.

§ 2. THEOREM. *The sum of the exponents of either set of cogredient variables in any term of an invariant is congruent modulo $p-1$ to the index μ of the invariant.*

This is proved by applying one of the transformations

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix},$$

where a is a primitive root of p.

§ 3. *Definition.* We shall say that x_1, y_1 form one set of variables and x_2, y_2 the other set.

THEOREM. *The terms of an invariant which are homogeneous in each set of variables form an invariant.*

We write the invariant as a sum of polynomials each homogeneous in each set of variables, and such that the sum of no two of these polynomials has that property. Then, since the linear transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ leaves unchanged the degree in each set of variables, each polynomial is evidently an invariant.

These theorems show that it is sufficient to consider an invariant of the form

$$I = x_2^{\mu} \sum_s C_s^{(0)} y_1^e x_1^f + x_2^{\mu-1} y_2 \sum_s C_s^{(1)} y_1^{e-1} x_1^{f+1} + \dots,$$

where the C's are integers modulo p, $e = v - s(p-1)$, $f = w + s(p-1)$, s runs from zero to such a value in any sum that none of the exponents in that sum are negative, and $e \equiv f + \mu$ (mod. $p-1$).

§ 4. *Definition.* A semi-invariant is a polynomial I in x_1, y_1 with integral coefficients taken modulo p such that

$$I(x_1, y_1) \equiv b^{\mu} I(x_1', y_1'), \qquad (\text{mod. } p),$$

for every transformation $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$.

We will make considerable use of the semi-invariants y_1 and

$$\lambda = x_1^p - x_1 y_1^{p-1} = \mathop{\pi}_{h=0}^{p-1} (x_1 - h y_1).$$

We have

$$L_1 = y_1 \lambda, \quad Q_1 = \lambda^{p-1} + y_1^{p^2-p},$$
$$N_s = x_2^{sp} \lambda^{p-1-s} + y_2 (\ ).$$

Products of the Invariants.

§ 5. *Lemma.* If m and k are any integers for which $1 \leqq m \leqq p-2$ and $1 \leqq k \leqq m$, there exists a product $\pi(N)$ of powers of $N_1, \ldots, N_{p-2}$, the first term of whose expansion is $x_2^{mp} \lambda^{k(p-1)-m}$.

If $m = kn$, we may take

$$\pi(N) = N_n^k.$$

If m is not a multiple of k, let n be the integer for which

$$\frac{m}{n+1} < k < \frac{m}{n}.$$

Then we may take

$$\pi(N) = N_n^{(n+1)k-m} N_{n+1}^{m-nk}.$$

§ 6. *Lemma.* There exists a product $\pi(M)$ of powers of M, M_1, M_2 of the form $x_2^u (y_1^v) + \ldots$, where u and v are any given integers for which either

$$(1) \qquad v = u + k(p-1), \qquad (0 \leqq k \leqq u, \ \ v \geqq u > 0),$$

or

$$(2) \qquad u = v + k(p-1), \qquad (0 \leqq k \leqq v, \ \ u \geqq v > 0).$$

If (1) holds, we may take

$$\pi(M) = M^l M_1^{k+m} M_2^m, \quad l = u - k - m(p+1).$$

If (2) holds, we may take

$$\pi(M) = M^l M_1^m M_2^{k+m}, \quad l = v - k - m(p+1).$$

Fundamental Theorems.

§ 7. THEOREM. *Given any invariant* $I = \sum_{k=0}^{u} x_2^{u-k} y_2^k f_k(x_1, y_1)$, *determine the integer* s *such that* $s \equiv u \ (mod.\ p)$ *and* $0 \leqq s \leqq p-1$. *Then* $f_t(x_1, y_1)$ *has the factor* y_1^{s-t} *if* $t < s$.

The theorem is obvious for $s = 0$. If $s \neq 0$, we will apply the transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to I, and form $I(x+y, y) - I(x, y)$, which must vanish identically. Equating to zero the coefficients of $x_2^{u-k} y_2^k$ $(k=0, \ldots, s)$, we get the following equations:

$$f_0(x_1 + y_1, y_1) - f_0(x_1, y_1) = 0, \tag{1}$$

$$\sum_{k=0}^{r-1} \binom{s-k}{r-k} f_k(x_1 + y_1, y_1) + f_r(x_1 + y_1, y_1) - f_r(x_1, y_1) = 0, \tag{2}$$

where $\binom{s-k}{r-k} = \frac{(s-k)(s-k-1)\ldots.(s-r+1)}{(r-k)!}$, $1 \leqq r \leqq s$, and $\binom{s-k}{r-k} \not\equiv 0$ (mod. p) since $s < p$.

The proof is made by mathematical induction. First let us assume that $f_0, f_1, \ldots., f_m$ each have the factor y_1, where $1 \leqq m+1 \leqq s-1$. Putting for r the value $m+2$ in equations (2), we have

$$\binom{s}{m+2} f_0(x_1+y_1, y_1) + \ldots. + \binom{s-m-1}{1} f_{m+1}(x_1+y_1, y_1) + f_{m+2}(x_1+y_1, y_1) - f_{m+2}(x_1, y_1) = 0.$$

Putting $y_1 = 0$ gives us $f_{m+1}(x_1, 0) = 0$. Hence, f_{m+1} has the factor y_1. For $r = 1$, equations (2) give us

$$s f_0(x_1+y_1, y_1) + f_1(x_1+y_1, y_1) - f_1(x_1, y_1) = 0.$$

Putting $y_1 = 0$ gives us $f_0(x_1, 0) = 0$, and hence $f_0(x_1, y_1)$ has the factor y_1. Hence, $f_0, f_1, \ldots., f_{s-1}$ each have the factor y_1.

Let us next assume $f_0, f_1, \ldots., f_m$ each have the factor y_1^n; then we will prove that $f_0, f_1, \ldots., f_{m-1}$ each have the factor y_1^{n+1}.

Let $f_t(x_1, y_1) = y_1^n f_t'(x_1, y_1)$, where $0 \leqq t \leqq m$. From (2) we get, after dividing out y_1^n from the first m equations, m equations of the form of (2) in the preceding discussion, but with f_t' in place of f_t. Hence, $f_0', f_1', \ldots., f_{m-1}'$ each have the factor y_1; and hence $f_0, f_1, \ldots., f_{m-1}$ each have the factor y_1^{n+1}.

We have proved that $f_0, f_1, \ldots., f_{s-1}$ each have the factor y_1; hence $f_0, f_1, \ldots., f_{s-2}$ each have the factor y_1^2. Similarly, $f_0, f_1, \ldots., f_t$ each have the factor y_1^{s-t}.

§ 8. *Lemma.* The polynomial f_0 is a semi-invariant. This follows from equation (1) of § 7.

§ 9. THEOREM. *The highest power of x_1 that occurs in any semi-invariant is congruent to zero modulo p.*

As in the theorem of § 2, we can show that the exponents of the same variable in different terms differ by multiples of $p-1$. Hence, let the semi-invariant be

$$f(x_1, y_1) = C_0 x_1^u y_1^v + C_1 x_1^{u-(p-1)} y_1^{v+(p-1)} + \ldots..$$

Let us suppose that u is not congruent to zero modulo p. Then, since $f(x_1, y_1)$ is unaltered under the transformation $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $f(x_1+y_1, y_1) - f(x_1, y_1)$ is identically zero. This gives us the equation

$$u C_0 x_1^{u-1} y_1^{v+1} = 0.$$

Since u is not zero, $C_0 = 0$.

Classification of Invariants.

§ 10. *Definition.* An invariant is said to be of type $\{t\}$, if it is of the form $x_2^u f_0(x_1, y_1) + \ldots$, where $f_0 \not\equiv 0$, and each (§ 2) exponent of x_1 in f_0 is congruent to t modulo $p-1$, where $1 \leqq t \leqq p-2$.*

Definition. An invariant is said to be of type $\{0\}$, if it is of the form $x_2^u L_1^r g_0(x_1, y_0) + \ldots$, where $g_0 \not\equiv 0$, and each exponent of x_1 in g_0 is congruent to zero modulo $p-1$.

Definition. An invariant is said to be of type $\{0\}'$, if it is of type $\{0\}$ and the exponent of y_1 in $g_0(x_1, y_1)$ is less than $p-1$.†

Definition. An invariant is said to be reduced, if it can be written as the sum of invariants or if its semi-invariant leader can be written as an invariant multiplied by a semi-invariant of lower degree.

Definition. The grade of the semi-invariant of $I = x_2^u L_1^r g_0(x_1, y_1) + \ldots$ is the degree of $g_0(x_1, y_1)$.

Reduction of Invariants of Type $\{t\}$.

§ 11. *Lemma.* Any invariant of type $\{t\}$ can be reduced either to an invariant of type $\{0\}$, $\{0\}'$ or to one which contains x_2 as a factor. The grade of the semi-invariant of the reduced invariant is less by $pt + t$.

The general form of such an invariant is

$$I = x_2^a (C_0 y_1^{b+s} x_1^t + \ldots + C_m y_1^s x_1^{b+t}) + \ldots, \tag{1}$$

where $a = d(p^2 - p) + h(p-1) + r$, $b = g(p^2 - p) + k(p-1)$, $r \leqq p-2$, $s \leqq p-2$, $k \leqq p-1$, $h \leqq p-1$ and $1 \leqq t \leqq p-2$. The C's are integers modulo p such that at least one of them does not vanish.

Case 1. $t \leqq s$.

Since the semi-invariant leader has the factor x_1^t and y_1^t, it has the factor L_1^t, and hence equation (1) can be written

$$I = x_2^a L_1^t (C_0 y_1^{e+s-t} + \ldots) + \ldots, \tag{2}$$

where $e = g(p^2 - p) + (k-t)(p-1)$. This is an invariant of type $\{0\}$, the grade of whose semi-invariant is less by $pt + t$. If $e = 0$, I is of type $\{0\}'$. If $e < 0$, the semi-invariant of equation (2) is identically zero and I has the factor x_2.

Case 2. $t > s$ *and* $t - k \not\equiv 0$ (*mod.* p).

Since, in equation (1), $b + t \equiv t - k$ modulo p and this is not congruent to zero modulo p, it follows from § 9 that $C_m = 0$. Hence, as in case 1, the semi-invariant of I has the factor L_1^t and can be reduced to the form of equation (2).

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* If an invariant is of the form $I = x_2^u L_1^r g_0(x_1, y_1) + \ldots$, where each exponent of x_1 in $g_0(x_1, y_1)$ is congruent to t modulo $p-1$ and $1 \leqq t \leqq p-2$, it is assumed in the following discussion that $I = x_2^u f_0(x_1, y_1) + \ldots$, where $f_0(x_1, y_1) = L_1^r g_0(x_1, y_1)$.

† By § 2, $g_0(x_1, y_1)$ contains only one term which is a power of y_1.

Case 3. $t > s$ *and* $t - k \equiv 0$ (*mod.* p).

Since both t and k are less than p, it follows that $t = k$. By § 2, $a + t \equiv b + s$ modulo $p - 1$. Hence, $r \equiv s - t$ (mod. $p - 1$), which gives $r = p - 1 + s - t$. If $C_m = 0$, the reduction is effected as in case 1. If $C_m \neq 0$, there is a condition imposed on h by § 7. Since $a \equiv s - h - t - 1$ (mod. p), and since this is less than zero, the semi-invariant of equation (1) must contain y_1 either to the power $p + s - h - t - 1$ or, if this is still negative, to the power $2p + s - h - t - 1$. In the first instance, by § 7, $p + s - h - t - 1 \leqq s$. Hence, $h = p - 1 - t + n$, where $0 \leqq n \leqq s$. The exponent a of equation (1) can now be written in the form

$$[d(p^2 - p)] + [n(p - 1) + s] + [(p - 1 - t)p],$$

and the exponent $b + t$ of equation (1) can be written as

$$p[(g + 1)(p - 1) - (p - 1 - t)].$$

If $g + 1 \leqq p - 1 - t$, we will form

$$I' = I - C_m \pi(M) \pi(N) Q_2^d.$$

If $g + 1 = p - 1 - t + l$, where $l > 0$, we will form

$$I' = I - C_m \pi(M) \pi(N) Q_1^l Q_2^d.$$

Here, by § 6, $\pi(M) = x_2^{n(p-1)+s} y_1^s + \ldots.$ If $s = 0$, we will take $\pi(M) = 1$. By § 5, $\pi(N) = x_2^{(p-1-t)p} \lambda^{(g+1)(p-1)-(p-1-t)} + \ldots.$ Then the semi-invariant of I' has, as in case 1, the factor L_1^t, and hence has the form of equation (2).

If $g > 0$, I' is of type $\{0\}$ and the grade of the semi-invariant of I' is less than that of I by $pt + t$. Since the term involving the highest power of x_1 in the semi-invariant of I' is of the form $C'_{m-1} y_1^{p^2-p+s-t} x_1^{(g-1)(p^2-p)}$, if $g = 0$, the terms involving x_2^a vanish and I' has the factor x_2.

In the second instance a similar line of argument shows that $h = 2p - 1 - t + n$, where $0 \leqq n \leqq s$. This can then be treated like the case above by replacing Q_2^d by Q_2^{d+1}.

Forms of Certain Invariants.

§ 12. *Lemma.* If v is a number of the form $d(p^2 - p) + h(p - 1) + u$, where $0 \leqq u < h \leqq p - 1$, and if $f(x_1, y_1)$ is a function of x_1 and y_1 which does not contain the factor y_1, then there is no invariant of the form

$$I = x_2^v \{C_0 y_1^u L_1^r f(x_1, y_1)\} + \ldots,$$

unless $r \geqq p - h$ or else $C_0 = 0$.

Since L_1 contains y_1 to the first power only, and since $v \equiv p - h + u$ (mod. p), where $0 < p - h + u \leqq p - 1$, f_0 must contain, by § 7, the factor y_1^{p-h+u}; hence $p - h + u \leqq u + r$, and hence $r \geqq p - h$.

Remark. If $r < p - h$ and the coefficient of x_2^v is a function of x_1 and y_1 not a constant, then, by § 7, $C_0 = 0$. If the coefficient of x_2^v is a constant, then,

since an invariant in two variables is a special case of a semi-invariant, $C_0 = 0$ by § 9.

§ 13. *Lemma.* If v is a number of the form $d(p^2 - p) + h(p-1) + u$, where $0 \leqq u < h \leqq p-1$, then there is no invariant of either of the forms

$$(1) \quad I = x_2^v (C_0 y_1^u) + \ldots.,$$
$$(2) \quad I = x_2^u (C_0 y_1^v + \ldots.) + \ldots.,$$

where $C_0 \neq 0$.

The first case follows from § 12 by taking $r = 0$ and $f(x_1, y_1) = 1$. If we apply the substitution $(x_1, x_2)(y_1, y_2)$ to the invariant in case (2), we get an invariant of the form of that in case (1). Hence, C_0 in case (2) equals zero.

§ 14. *Lemma.* If the degree u in x_2, y_2 of an invariant is less than p, and if the coefficient of x_2^u is of the form $L_1^r g_0(x_1, y_1)$, where $r > 0$, then g_0 is not zero and the invariant has the factor L_1^r.

Let $I = x_2^u L_1^r g_0(x_1, y_1) + x_2^{u-1} y_2 f_1(x_1, y_1) + \ldots.$, where $u \leqq p-1$. If g_0 is zero, I has the factor L_2 at least to the first power. This is of degree $p+1$ in x_2 and y_2, which is greater than u, and hence I is identically zero.

If g_0 is not zero, then, by § 7, $f_0, \ldots., f_{u-1}$ each have the factor y_1. Since $f_0(y_1, x_1) = f_u(x_1, y_1)$ and since $f_0(x_1, y_1)$ has the factor x_1, we see that $f_u(x_1, y_1)$ has the factor y_1. Hence, I has the factor y_1 and hence the factor L_1. Let

$$I^{(1)} = \frac{I}{L_1} = x_2^u L_1^{r-1} g_0(x_1, y_1) + \ldots..$$

If $r \geqq 2$, $I^{(1)}$ can be shown to have the factor L_1, and eventually

$$I^{(r)} = \frac{I}{L_1^r} = x_2^u g_0(x_1, y_1) + \ldots..$$

Hence, I has the factor L_1^r.

§ 15. *Lemma.* There is no invariant of degree less than p in x_2, y_2 whose semi-invariant leader is an invariant of two variables.

Let L_1^r be the highest power of L_1 which is contained in the semi-invariant leader. Then

$$I = x_2^u L_1^r g_0(x_1, y_1) + \ldots.,$$

where $u < p$ and g_0 is an invariant which does not contain y_1 as a factor. By § 14, I has the factor L_1^r. Hence, let

$$I^{(r)} = \frac{I}{L_1^r} = x_2^u g_0(x_1, y_1) + \ldots..$$

By § 7, g_0 must contain y_1^u as a factor; hence, $g_0 = 0$. Then, by § 14, I is identically zero.

Reduction of Invariants of Type $\{0\}$.

§ 16. *Lemma.* Any invariant of the form $I = x_2^a (C_0 L_1^r y_1^u) + \ldots.$, where $a = d(p^2 - p) + h(p-1) + u$ and $0 \leqq u < h \leqq p-1$, can be reduced to an invariant containing x_2 as a factor.

We have shown in § 12 that r must equal or exceed $p-h$.* If $h \geqq 2$, the invariant

$$I' = I - C_0\, Q_2^d\, N_{h-1}\, M^{p-h+u}$$

is an invariant having x_2 as a factor. If $h=1$ and $d \geqq 1$, then $u=0$ and

$$I' = I - C_0\, Q_2^d\, Q_1\, M^{p-1} + C_0\, Q_2^{d-1}\, M^{p^2-1}$$

has the factor x_2. Here d can not equal zero, since, by § 15, there is no invariant of the form that I becomes if $d=u=0$ and $h=1$.

§ 17. *Lemma.* There is no invariant of the form

$$I = x_2^u\, L_1^r \{C_0\, y_1^{b+u} + \ldots\ldots\} + \ldots\ldots,$$

where $b = g(p^2-p) + k(p-1)$, $0 \leqq u < k \leqq p-1$ and $C_0 \neq 0$.

Since $u < p$, then, by § 14, I has the factor L_1^r. Let

$$I^{(r)} = \frac{I}{L_1^r} = x_2^u \{C_0\, y_1^{b+u} + \ldots\ldots\} + \ldots\ldots.$$

Then, by § 13, $C_0 = 0$.

§ 18. *Lemma.* Any invariant of type $\{0\}$ can be reduced either to one of type $\{0\}'$ or to one which contains x_2 as a factor. The grade of the semi-invariant of the reduced invariant is less by p^2-1.

For the sake of simplicity we will take the general form of an invariant of type $\{0\}$ to be

$$I = x_2^a\,(C_0\, y_1^{b+s} + \ldots\ldots + C_m\, y_1^s\, x_1^b) + \ldots\ldots, \tag{1}$$

where $a = d(p^2-p) + h(p-1) + s$ and $b = g(p^2-p) + k(p-1)$. Cases where the argument is different, when all the terms involving x_2^a contain L_1 explicitly as a factor, will be treated separately. It should be noted that a and b have the correct form for all cases, since, by § 2, $a \equiv b + s \pmod{p-1}$ and $a + pr \equiv b + s + r \pmod{p-1}$.

Case 1. $1 \leqq k \leqq p-1$.

Since k is not congruent to zero modulo p, it follows from § 9 that $C_m = 0$. Hence, the semi-invariant of any invariant of this form contains y_1 as a factor to the power $s + k(p-1)$.

If, in equation (1), $a \geqq b + s$, let $g(p^2-p) + k(p-1) + s = u$, and let $d(p^2-p) + h(p-1) + s = u + d'(p^2-p) + h'(p-1)$, where $0 \leqq h' \leqq p-1$. Since $k \geqq 1$, it follows that $u \neq 0$ and $h' \leqq u$; hence, there exists a

$$\pi(M) = x_2^{h'(p-1)+u}\,(C_0\, y_1^u) + \ldots\ldots.$$

We will let

$$I' = I - C_0\, Q_2^{d'}\, \pi(M).$$

* It is sufficient to use $r = p-h$. The remaining powers of L_1 can be carried along in the reduction.

If $a<b+s$, let $d(p^2-p)+h(p-1)+s=u$ and let $g(p^2-p)+k(p-1)+s=u+g'(p^2-p)+k'(p-1)$, where $0\leqq k'\leqq p-1$. If $k'>0$, then, by § 17, $C_0=0$. If $k'\leqq u$ and $u\neq 0$, there exists a $\pi(M)=x_2^u\,(C_0\,y_1^{k'(p-1)+u})+\ldots\ldots$ Then we will let

$$I'=I-C_0\,Q_1^{g'}\,\pi(M).$$

If $u=0$, I is a function of x_1 and y_1, and hence a function of L_1 and Q_1.*

In either case the semi-invariant of I' contains the factors x_1^{p-1} and y_1^{p-1}, unless the semi-invariant of I is zero. Hence, the semi-invariant of I' has the factor L_1^{p-1}, and we have

$$I'=x_2^a\,L_1^{p-1}\,(C_1'\,y_1^e+\ldots.)+\ldots.,$$

where $e=(g-1)(p^2-p)+(k-1)(p-1)+s$. This is an invariant of type $\}0\{$, the grade of whose semi-invariant is less by p^2-1. If $g<1$, I' has the factor x_2. If $g=1$ and $k=1$, I' is of type $\}0\{'$.

Case 2. $k=0$ *and* $h\leqq s$.

There are four subcases which we will denote by subscripts.

Case 2_1. $h\neq 0$ *and* $s\neq 0$.

Since $h\leqq s$ and $s\neq 0$, there exists a $\pi(M)=x_2^{h(p-1)+s}\,y_1^s+\ldots..$ We will first form

$$I'=I-C_m\,Q_2^d\,Q_1^g\,\pi(M)=x_2^a\,y_1^{s+p^2-p}\,(C_0'\,y_1^{(g-1)(p^2-p)}+\ldots.)+\ldots.. \tag{2}$$

We see that the semi-invariant of I' has the factor $y_1^{s+p^2-p}$. Since $h<p$, it follows that $h(p-1)+s<p(p-1)+s$; and since $h\neq 0$, it follows that $p-h<h(p-1)+s$. Hence, there exists a $\pi(M)=x_2^{h(p-1)+s}\,y_1^{s+p^2-p}+\ldots..$ Let us next form

$$I''=I'-C_0'\,Q_2^d\,Q_1^{g-1}\,\pi(M).$$

Then the semi-invariant of I'' has the factors x_1^{p-1} and y_1^{p-1}. Hence, I'' has the factor L_1^{p-1}, and hence

$$I''=x_1^a\,L_1^{p-1}\,y_1^c\,(C_1''\,y_1^e+\ldots.)+\ldots., \tag{3}$$

where $a=d(p^2-p)+h(p-1)+s$, $c=s+(p-1)^2$ and $e=(g-2)(p^2-p)$. Then I'' is an invariant of type $\}0\{$, the grade of whose semi-invariant is less by p^2-1. If $g\leqq 1$, I'' has the factor x_2.

Case 2_2. $h=0$ *and* $s=0$.

By taking the first $\pi(M)$ in case 2_1 equal to unity, we get equation (2). If we next form

$$I''=I'-C_0'\,Q_2^{d-1}\,Q_1^{g-1}\,M^{p^2-p},$$

we get equation (3) of case 2_1. If $d=0$, I is a function of L_1 and Q_1.* If $g=0$, I is a function of L_2 and Q_2.*

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* Dickson, *Ibid.*

Case 2_3. $h=0$, $s \neq 0$ *and* $d \neq 0$.

The reduction to equation (2) is the same as in the case 2_1. Let

$$I'' = I' - C_0' Q_2^{d-1} Q_1^{g-1} M^{p^2-p+s};$$

then I'' is in the form of equation (3) of case 2_1. If $g=0$, I' has the factor x_2.

Case 2_4. $h=0$, $s \neq 0$ *and* $d=0$.

Let us form

$$I' = I - C_0 Q_1^g M^s.$$

If $g=0$, I' has the factor x_2. If $g \neq 0$, the semi-invariant of I' has the factor x_1^s and y_1^s, hence the factor L_1^s. Since the exponent a of x_2 in this case equals s, and this is less than p, then, by § 14, I' itself has the factor L_1^s. If in the beginning

$$I = x_2^s L_1^r g_0(x_1, y_1) + \dots\dots,$$

then I' has the factor L_1^{r+s}. Let

$$I'' = \frac{I'}{L_1^{r+s}} = x_2^s (C_1'' y_1^c x_1^e + \dots\dots + C_m'' x_1^{c+e}) + \dots\dots,$$

where $e = p-1-s$ and $c = (g-1)(p^2-p) + e(p-1)$.

From § 7, the semi-invariant of I'' has the factor y_1^s, and hence $C_m''=0$. Hence, the semi-invariant of I'' has the factors x_1^e and y_1^e, and hence L_1^e. Since, by § 14, I'' itself has the factor L_1^e, if

$$I''' = \frac{I''}{L_1^e},$$

then I''' is of type $\{0\}$, but the grade of its semi-invariant is less than that of I by p^2-1. If $g=0$, I'' has the factor x_2. If $g=1$, I''' has the factor x_2.

Case 3. $k=0$ *and* $h > s$.

Since $h > s$, then $d(p^2-p) + h(p-1) + s \equiv p - h + s \pmod{p}$, where $0 < p-h+s < p$. By § 7, the semi-invariant of equation (1) has the factor y_1^{p-h+s}; and since $p-h+s > s$, $C_m = 0$. Hence, if $g \neq 0$, the semi-invariant of I has the factor $y_1^{p^2-p+s}$. If $g=0$, I has the factor x_2 as a consequence of § 13. Since $h<p$, it follows that $h(p-1)+s < p^2-p+s$. Hence, let $u = h(p-1)+s$. Then $p^2-p+s = u + (p-h)(p-1)$. Since $h \neq 0$, it follows that $u \neq 0$ and $p-h < h(p-1)+s$. Hence, there exists a $\pi(M) = x_2^{h(p-1)+s} y_1^{p^2-p+s} + \dots\dots$ If we form

$$I'' = I - C_0 Q_2^d Q_1^{g-1} \pi(M),$$

then I'' has the form of equation (3) of case 2_1.

It might happen that the semi-invariant of I was of the form $L_1^r g_0(x_1, y_1)$. By § 12, $r \geqq p-h$. It is sufficient to use $r = p-h$. If $h>1$, we will form

$$I' = I - C_m Q_2^d Q_1^g N_{h-1} M^{p-h+s}.$$

If $h=1$ and $d \neq 0$, we will form

$$I' = I - C_m Q_2^d Q_1^{g+1} M^{p-1} + C_m Q_2^{d-1} Q_1^g M^{p^2-1}.$$

If $h=1$ and $d=0$, then $s=0$ and

$$I = x_2^{p-1} L_1^r (C_0 y_1^{g(p^2-p)} + \dots + C_m x_1^{g(p^2-p)}) + \dots .$$

Since the exponent of x_2 is less than p, then, by § 14, I has the factor L_1^r. Let

$$I' = \frac{I}{L_1^r};$$

then, by § 7, I' has the factor y_1^{p-1}, and hence $C_m = 0$. Thus, in all three cases we have reduced the invariant I to the form where the semi-invariant lacks the highest power of x_1. This is essentially the form of I at the beginning of this case, and hence the reduction can be completed in the same manner.

In every case we saw that the reduction of an invariant of type $\{0\}$ always led to another invariant of type $\{0\}$. By continuing the reduction, we saw that we could reduce the grade of the semi-invariant each time by p^2-1, until its grade lay between zero and p^2-1. Let us suppose the degree of the semi-invariant is then congruent to s (mod. $p-1$). Then the exponent of y_1 has the form $n(p-1)+s$, where $0 \leqq n \leqq p+1$ and $0 \leqq s \leqq p-2$. This is the case where either $g=1$ and $k=1$, or $g=1$ and $k=0$, or else $g=0$. We saw that then I either was or could be reduced to an invariant of type $\{0\}'$, or else it could be reduced to an invariant having the factor x_2. This proves the lemma of this section.

§ 19. *Lemma.* Any invariant of type $\{0\}'$ can be reduced to an invariant which contains x_2 as a factor.

If I is of the type $\{0\}'$, let us suppose

$$I = x_2^a (C_0 y_1^s) + \dots ,$$

where $a = g(p^2-p) + h(p-1) + s$. If $h \leqq s$ and $s \neq 0$, there exists a $\pi(M) = x_2^{h(p-1)+s} y_1^s + \dots .$ If $s=0$, we will take $\pi(M)=1$. Then

$$I' = I - C_0 Q_2^d \pi(M)$$

has the factor x_2. If $h>0$, then either $C_0=0$ by § 13, and hence I has the factor x_2, or else the semi-invariant of I has the form $L_1^r g_0(x_1, y_1)$, where $r \geqq p-h$. In this instance, by § 16, I can then be reduced to an invariant having x_2 as a factor.

Reduction of an Invariant in the Degree of x_2 and y_2.

§ 20. We will let J be a general symbol for an invariant which is expressible in terms of the invariants of § 1. By means of the lemmas of § 11, § 18 and § 19, any invariant can be written as

$$I = J + x_2(\ ) = J + L_2^\tau I', \qquad (\tau \geqq 1),$$

where I' is an invariant not divisible by L_2. If I' involves x_2, we have similarly

$$I' = J' + L_2^{r'} I''.$$

This process can then be repeated, until we get

$$I = J'' + L_2^{r''} F(x_1, y_1).$$

Then $F(x_1, y_1)$ is an invariant and hence is a function of L_1 and Q_1.* This proves the theorem of § 1.

Invariants for Modulo $p = 2$.

§ 21. While there are no invariants of the form N_s for modulo $p = 2$, the reduction just given will hold for this modulus also, if we bear in mind that $s = 0$, and h and k can take only the values zero or unity. Since every invariant modulo 2 is of the type $\{0\}$, and since the lemma of § 16 involves only $h = 1$, the invariants for modulo 2 can be reduced without using the invariants N_s.

Independence of Invariants.

§ 22. THEOREM. *No one of the invariants* Q_i, L_i, M_i, M, N_s $(i = 1, 2;$ $s = 1, 2, \ldots, p-2)$ *is a rational integral function of the remaining ones.*

The theorem follows by noting the degrees in x_1, y_1 and in x_2, y_2, and the following additional facts:

Q_i can not be a function of L_i $(i = 1, 2)$. If it were, this function would contain L_i as a factor, and hence Q_i would contain x_i as a factor, which is impossible.

M_i $(i = 1, 2)$ can not be a function of M and L_i, for by comparing exponents we can see that M and L_i can occur in such a function only to the first degree, and also that the function can not contain the product ML_i. Hence, if such a function exists, it has the form

$$M_i = C_0 M + C_1 L_i.$$

Putting $x_j = y_j = 0$, where $j = 1$ if $i = 2$, and $j = 2$ if $i = 1$, gives $C_1 = 0$. Since M_i is of degree p in y_i and M of degree 1 in y_i, no such function exists.

If $N_s = x_2^{sp}(x_1^{p^2-p-s} + \ldots) + \ldots$ is a function of the invariants in § 1, such a function can not contain N_t, where $t \neq s$. For if $t > s$, then $tp > sp$, and if $t < s$, $p^2 - p - t > p^2 - p - s$. Furthermore, N_s can not be a function of the invariants L_1, L_2, M, M_1 and M_2, since these invariants all vanish if $y_1 = y_2 = 0$, and N_s then equals $x_2^{sp} x_1^{p^2-p-s}$.

CHICAGO, ILL., 1913.

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* Dickson, *Ibid.*

INDEX.